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► **To cite this version:**

Harita Pankajkumar Raval, Lisa Österling. Developing an analytical tool for researching geometry discourse among pre-service teachers from India and Sweden. Thirteenth Congress of the European Society for Research in Mathematics Education (CERME13), Alfréd Rényi Institute of Mathematics; Eötvös Loránd University of Budapest, Jul 2023, Budapest, Hungary. hal-04421641

**HAL Id: hal-04421641**

**<https://hal.science/hal-04421641>**

Submitted on 27 Jan 2024

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# Developing an analytical tool for researching geometry discourse among pre-service teachers from India and Sweden

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*In this pilot study, we examine the procedures, constructs and geometrical properties pre-service teachers (PSTs) invoke when solving a geometry problem. The aim is to develop an analytical tool to research PSTs' mathematical discourse, assuming that the available mathematical discourse informs PSTs' noticing of mathematics learning. As a theoretical base for mathematics discourse, we use commognition. The empirical data consists of recorded conversations of seven PSTs solving a geometry problem. The analytical tool provides a visual representation of the procedures, geometrical properties and constructs, considered as the precedent search space of the person solving the task. The analytical tool we propose visualizes the precedent search space PSTs draw on and therefore enables a culturally sensitive analysis of mathematics discourse.*

*Keywords: Commognition, mathematics discourse, noticing, geometry, pre-service teachers.*

## Introduction

It is known how teachers select different mathematical tasks and teaching approaches in classrooms in different countries (Stigler & Hiebert, 2009). Cultural differences have also been described in what teachers notice (Goodwin, 1994; Louie, 2018). Our assumption is that the local mathematics discourse has a bearing on what teachers notice in a particular context. In this paper, we propose a methodology for analyzing geometry discourses engaged by Pre-Service Teachers (PSTs) in two different countries: India and Sweden.

The TIMSS video study video-recorded classrooms in seven countries (TIMSS, 1999). One difference between countries is the use of open-ended tasks (Stigler & Hiebert, 2009). Neither India nor Sweden was part of the TIMSS video study. Among the included countries, Japanese mathematics teachers stood out for posing more problems on making connections, giving students opportunities to solve mathematical problems, share their solutions and analyze them (Neubrand, J., 2006). This reasons with how Leikin (2009) defines open-ended mathematical tasks as tasks which enable participants to use flexible reasoning and to create novel ideas. From a comparison of mathematicians' and undergraduate students' solutions to the same open-ended task, the proposed solutions to a problem were found to vary with the level of expertise in mathematics (Leikin, 2009). Providing flexibility in solutions hence invites participants to include and make use of both their educational and cultural experiences. Therefore, we assume open-ended tasks to be a fruitful setting for investigating cultural and contextual aspects of mathematics discourse.

Geometrical problems enable a rich and varied mathematics discourse. In geometrical open-ended tasks, creativity and alternative strategies are found to be a necessary part of the process (Leikin & Sriraman, 2022). Knowledge about central strategies, such as the construction of auxiliary lines and visualizations, enhance and enables individuals to be creative when solving geometrical problems (Levav-Waynberg & Leikin, 2012). For learning geometry, Duval (1995) describes four kinds of apprehensions: perceptual (the recognition of a shape), sequential (to construct a figure), discursive (to use mathematical properties) or operative (an insight into a problem solution when looking at a figure). In addition, Markkanen (2021) found that secondary students tend to use algebra and abandon geometrical reasoning in solving geometry problems in dynamic software, and primary students used spatial reasoning for solving geometrical problems (Dindyal, 2015). Geometry enables

flexibility in the properties, constructions or strategies engaged when solving an open-ended task, with flexibility restricted to the available geometrical discourses.

We align with Biza and Nardi (2023) and claim that classroom discourse is accessible for teachers to notice. Teachers' professional noticing has been described in previous research as expertise in attending to students' mathematical thinking and responding based on students' mathematical understanding (Guner & Akyuz, 2020; Sherin et al., 2011). However, as different to discourse, thinking needs to be inferred by teachers or researchers. Our assumption is that PSTs bring different mathematics discourses depending on country or context. This paper set out to develop a methodology for researching the mathematics discourse of PSTs. Since commognition focuses on the mathematics discourse in particular, it is a powerful theory for capturing local mathematics discourse in the two contexts, and the next section will outline the main constructs used in this study.

## **A theoretical background to commognitive concepts**

We turn to commognition for a theoretical conceptualization of mathematics discourse (Sfard, 2008). Based on commognition, Wang (2016) distinguishes characteristics of geometrical discourse, such as direct recognition, recall, substantiation or construction. The concepts from commognitive theory used for this study are the *classroom precedent search space* (CPSS), and the *precedent search space* (PSS). In order to provide comprehensible definitions, we first introduce mathematical routines.

Routines can be described as the activity the performer feels bound to enact in a particular situation. In mathematics, routines have been defined as the task, as understood by the performer, together with the procedure to solve the task (Lavie et al., 2019). In geometry, comparing, reasoning, conjecturing or verifying were found to be routines students use when 'dragging' items in dynamic geometry software (Ng, 2017). The collection of all task-procedure pairs, or routines, available in a particular situation, has been referred to as the CPSS (Nachlieli & Tabach, 2022). For this paper, we are interested in how CPSS can be distinguished for PSTs from different countries. The task-procedure pair available for an individual has been referred to as the PSS (Nachlieli & Tabach, 2022). Therefore, mapping the procedures used by individual PSTs gives insights into what can be the CPSS in context.

Geometrical open-ended tasks offer flexibility in what procedure to choose in the given task situation. For the purpose of this study, we use PSS to describe the procedures and properties mentioned or written by participants solving an open-ended geometry task, and CPSS to describe the set of available procedures for the same task. The underlying assumption is that to notice the PSS of students, teachers need access to a large CPSS. The aim is, therefore, to develop a theoretically informed and empirically based method for the analysis of the PSS of the PSTs. What is the CPSS for the selected geometrical problem? How can the operationalization of PSS in a visual analytic tool enable the analysis of paths to find a solution?

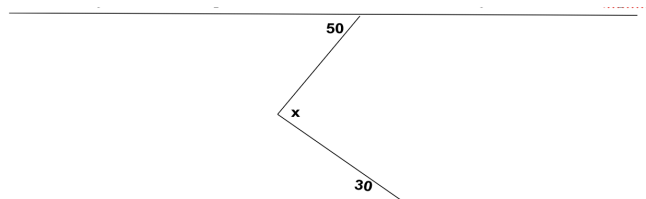
## **Method**

The study was conducted as part of an exploration to understand the discourse among PSTs of a secondary mathematics teacher education program. The pilot study involved participants from Sweden and India. The PSTs were asked to solve a geometry problem with multiple solutions. In the following section, we first describe the context and design of the study and, thereafter, how the conceptualizations of noticing and PSS informs the analytic tool: the PSS-tree.

### **Setting and data collection**

Open-ended mathematical tasks (Leikin, 2009) are used as part of this study since the flexibility provided invites participants to include and make use of their educational or cultural experiences. Therefore, we assume open-ended tasks to be a fruitful setting for investigating cultural aspects of

mathematics teaching. Neubrand (2006) distinguishes how open-ended mathematical tasks in algebra typically require procedural knowledge, whereas open-ended mathematical tasks in geometry are primarily conceptual problems with intra-mathematical applications. In the TIMSS-video study (1999), Japanese teaching was found to put students in the position to develop, share, and analyze solutions on their own. This was the basis for us to select one of the Japanese open-ended mathematical tasks in Geometry<sup>1</sup>, see below.



**Figure 1: The task—finding the value of an angle (TIMMS video study<sup>1</sup>)**

The study was conducted with two focus groups of PSTs, one in India and one in Sweden. We asked the PSTs to solve the given problem (see Figure 1) in as many ways as possible, and thereafter explain how they found the answer. The task was video recorded, and all the participants informedly signed the consent form for participating and video recording of the intervention. First, PSTs solved the task individually and tried to find different solutions. Thereafter, they were asked to explain their solutions. The first author, who was present for the interviews, posed follow-up questions about the geometrical properties or constructions they used to solve the task. The meetings lasted 30- 40 min.

A total of seven PSTs participated, who were in their final stages of the secondary mathematics teacher education program. These PSTs have completed their college mathematics and Pedagogy of Mathematics courses therefore they are expected to have a fair knowledge of the mathematics discourse required to solve the problem. Amaira, Benny, and Charles are pseudonyms of the PSTs from Sweden, and Deena, Elias, Farouqi and Gulu are pseudonyms of the PSTs from India.

### Steps in developing the PSS-tree

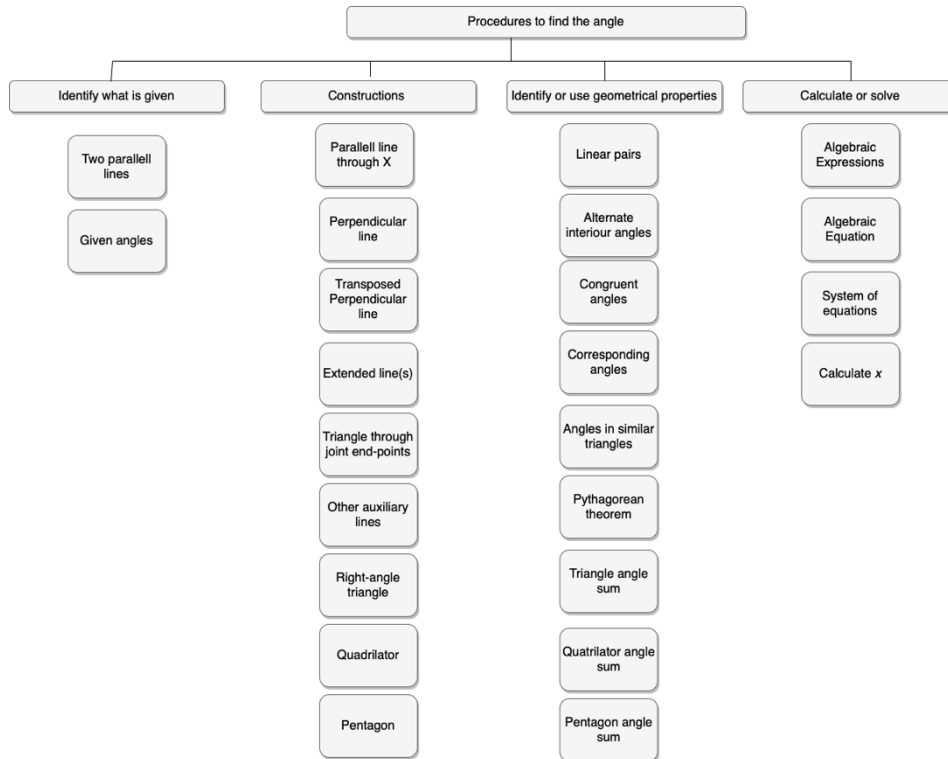
The development of the said tool was an iterative process between data and theories. The first theoretical step was to develop analytic distinctions for PSS by expanding the procedures also to include geometrical constructs or properties. For finding the CPSS, all the different constructions, properties or procedures used were summarized, starting from the solutions in the Japanese classroom. Thereafter, we turned to the written solutions from the seven participating PSTs, and, finally, to the videos. This resulted in a list of procedures, constructs or properties.

This list was thereafter used to categorize which procedures were similar. The first step was similar for most PSTs, to identify parallel lines and the given angles. Thereafter, the constructions made by PSTs emerged, for example; perpendicular lines or extensions of given line segments. Thereafter, PSTs identified or used geometrical properties to find the angle X, and in the process, some use algebraic expressions or system of equations, or some just wrote the answer. Thereafter, we looked at the video of when PSTs explained their solutions, and added new properties or procedures.

Inspired by Weingarden et al. (2019), we used a tree diagram to visualize our categories. In the tree, each frame is one particular property, construct or procedure, from the categories above. The branches

<sup>1</sup> <http://www.timssvideo.com/jp1-finding-the-value-of-an-angle>

in the tree were formed through an iterative process between the thematization of the previous categories and by the perceptual, sequential, discursive and operative apprehensions (Duval, 1995).



**Figure 1: The CPSS-tree for the task of Finding the angle**

We use “Identify what is given” as a marker for the perceptual and “Constructions” as corresponding to the sequential construction of new figures and the new figures formed by construction. We use “Identify or use geometrical properties” rather than discourse. Finally, we use “Calculate or solve” as a category to describe how answers were represented. The frames in the CPSS-tree are all possible to observe in discourse and used to form the tree-diagram branches.

### Operationalizing the PSS-tree

In this paper, we use the case of Amaira to demonstrate how the CPSS-tree was used for analysis. Thereafter, we describe the variation we saw among the seven PSTs, as a way of exploring the tool. In the diagram, we coloured the frames that were part of PSTs solutions in light grey and the ones named in dark grey. Next, we added arrows to indicate the path and the connections in solutions.

In the first solution (see Figure 3), we could identify the constructions and properties used, interpreted as indicating the PSS for solving the problem. The corresponding frames were shaded in the Tree-diagram (see Figure 5 below). We also saw how Amaira wrote “parallel”, thus naming some important properties. Named properties were coloured in dark grey.

- Find the angle in the bend with any method. (5-7 min)

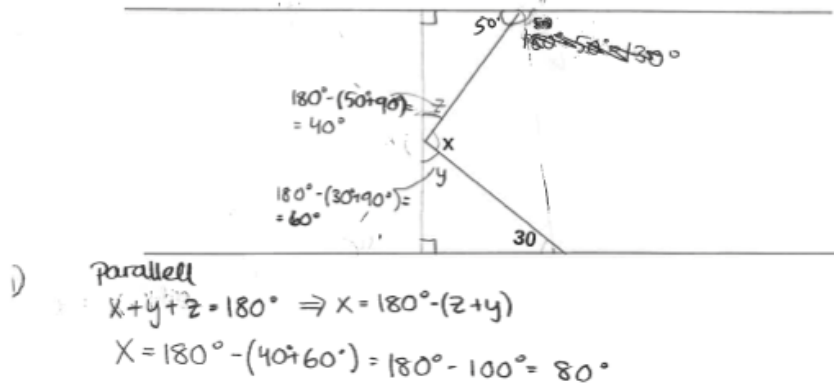


Figure 2: Amaira's first solution to the problems

As a second step, we consulted the video recordings from when Amaira explained her solution to confirm our interpretations and add new procedures or properties used. This is the transcript from when she explains solution 1:

- Amaira: And I began with drawing one line here. [Points to the perpendicular line] That was when we could find the angle, find the angle to these two parallel lines. And at first, I was unsure if I can do that. But then, after a bit of conversation, we figured it out that because it's crossing this point here [points]. So it's just crossing this in one point, then we can for sure, draw a line that is perpendicular, perpendicular?
- Interviewer: Perpendicular.
- Amaira: Yeah, so then we know that these two angles are 90 degrees [points]. And I also already had 30, and 50 here, [points] so we can find y and z, which is 60, and 40. And then I know that x plus plus y is 180. Because this is 180 degrees. And then I just solve for x by subtracting z, then y.

For Amaira, the transcripts confirmed what we had already seen in the written solution, so no procedures were added. But from the video, we could also follow the path she took to arrive at the answer, from how she identified the parallel lines and drew the transversal to how she used an algebraic equation to solve for x. Arrows between the frames indicate the path (figure 5).

In solution two (figure 4), Amaira identifies both the parallel lines and the given angles from the task. She constructs a transversal by extending the line segments and thereafter identifies alternate angles.

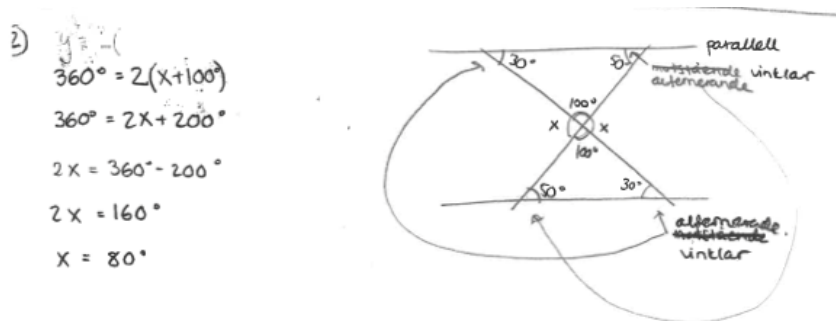
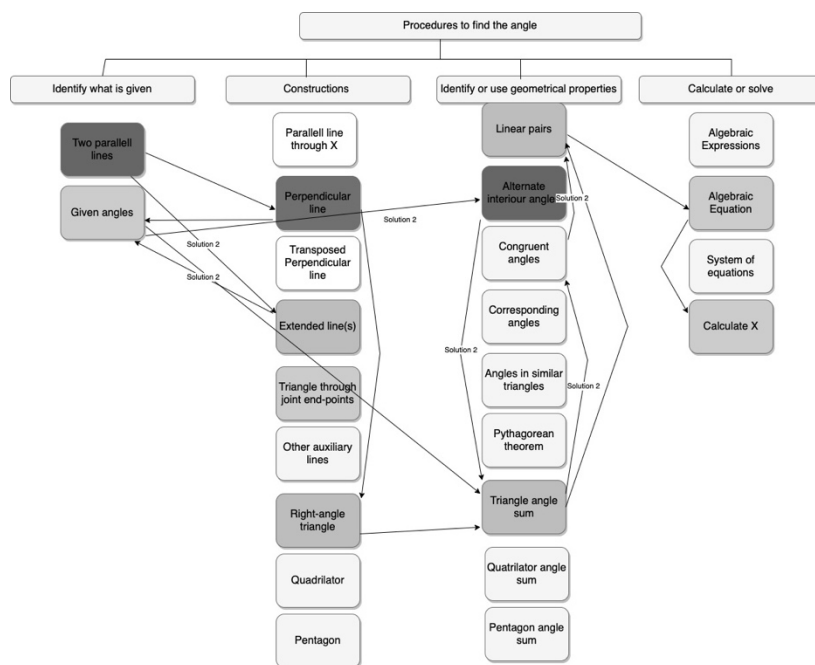


Figure 4: Amaira's second solution to the problem

Finally, she uses an angle at any point is 360 degrees to formulate equations. She wrote “parallel” and “alternate angles” erasing the Swedish word (“motstående”). Thus, she names some important properties, which were coloured in dark grey (figure 5).

Amaira’s third solution is an unresolved attempt, and Amaira’s PSS is represented below (figure 5).



**Figure 5: Amaira’s PSS, together with the connections she made between procedures**

The tree presents a visual map of the PSS, with procedures as well as the naming and the connections made. Looking at the details shows how Amaira’s two first solutions take different paths. Following the path through solution 1, she constructs an auxiliary perpendicular line and identifies the right-angle triangles resulting from the construction. She combines and connects triangle angle-sums, linear pairs and alternate angles and names angles  $y$  and  $z$  to formulate equations for finding  $x$ . For the second solution, Amaira first constructs the auxiliary line by extending the line segments given in the figure, which will create two transversals intersecting at the angle which needs to be found. Then she uses the properties of transversal lines on parallel lines and with the alternate inter-angle to be equal (property) she finds the angles remaining angle of the triangle constructed by the transversals. Then she uses the angle sum property of a triangle and finds the 100 degrees, which leads to getting the opposite angles at the intersection to be equal. Then using the angle sum at a point is 360 degrees she algebraically solves the equation and finds the solution.

### An overview of the cases

Looking into the PSS-trees for all the PSTs reveals what was part of several solutions, but also some differences. All used the sum of angles in a triangle, and six used linear pairs. Five PSTs used different properties for transversals and parallel lines as alternate, corresponding or congruent angles. All PSTs constructed auxiliary or extended lines. All solutions contained a perpendicular line constructed through a point. Four PSTs constructed a transversal, whereas one transposed a transversal and constructed a quadrilateral. Only two constructed a parallel auxiliary line, as the Japanese teacher showed for his class in the TIMSS video. Two referred to the Pythagorean theorem without using it. Thus, the sum of angles for triangles was part of the PSS for all PSTs and properties of transversals to parallel lines for at least five of the participating PSTs.

Among the seven PSTs, six solved the task, and five wrote the value of  $x$ . Three formulated an equation or system of equations to solve the task, two among the PSTs from Sweden and one PSTs from India. All the others used geometrical reasoning and conjectures to arrive at the solution. Deena does not identify the given parallel lines and therefore does not have enough information to solve the task. Among the PSTs from Sweden, it was only Amaira who stated what was given in the task and named the properties. Among the PSTs from India, Elias only named several properties in his oral explanations, whereas the others used geometrical names both in their written and oral explanations.

### **Tentative findings**

Using more or less the same PSS as a base, PSTs took different paths through the solutions. As three main different strategies. Benny uses algebra, formulates four different equations, and works with substitutions of variables to find  $x$ . Charles combines different angle sums of triangles, whereas the other PSTs base their final calculations on linear pairs. From using the tool on solutions of all seven PSTs, two tendencies in this data can be relevant to pursue further. First, we observed fewer procedures, properties and connections used in the written solutions compared to the video-recorded explanations. The differences between written solutions and oral explanations could be interesting to explore further. Second, the Indian PSTs tended to use more formal language in their solutions. Such differences could be an indicator of aspects of mathematics discourse that are culturally sensitive.

### **Concluding discussion**

The affordance of the proposed tool is that it shows what properties or procedures PSTs draw on and how they are connected to the arrival of a solution. In addition, it shows the dead-ends, where PSTs express how they try a particular procedure, which does not lead further. Still, such dead ends contribute to the CPSS and inform teachers' noticing.

In this paper, we made a move from attending to mathematical thinking towards attending to mathematics discourse—a move, we envision can inform future research on noticing. The commognitive constructs of CPSS and PSS were helpful for a fine-grained analysis of the geometry discourse PSTs participants. This, in turn, can give insights into what is possible for PSTs to notice in learners' mathematics discourse. The PSS-tree enable the analysis of discourse across contexts.

### **Acknowledgement**

This study falls within the research school REMATH, funded by the Swedish Research Council, grant number 2021-00534.

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