

Prospective teachers' mathematical knowledge: Relating representations as design principle for developing knowledge of non-local mathematics for teaching in abstract algebra

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It is well known that pre-service teachers might experience a double discontinuity between school and university. They often see little or no connections of university mathematics to their future profession. We offer a theoretical contribution on how this second discontinuity can be addressed in teaching abstract algebra by considering design principles for developing non-local knowledge for teaching. Based on previous research, a logical structure for the principle of “Sequencing the learning arrangements by building-up, (re-)learning, stepping-down”, and the principle of “Relating local and non-local representations” by contrasting and comparing are elaborated. Exemplary realizations for the contents of solving equations and arithmetic properties are presented.

Keywords: Design principles, teaching practices, relating representations, professional development, local and non-local knowledge.

Introduction

Pre-service teachers (PST) often see little relevance in their mathematical content courses for their mathematics teaching (Ticknor, 2012 for the case of algebra). In reaction to this issue, the teaching of content knowledge in mathematics courses for pre-service teachers has seen some design innovations over the last years at German teacher education universities. Approaches for making connections more explicit between university mathematics and school mathematics, or between content and pedagogical content knowledge have been two (out of many other) ways (collection of examples Isaev et al., 2022). However, research settings that develop and experiment with respective teaching designs in design-based research approaches are still rather rare, especially in the field of abstract algebra. This results in a lack of design principles for teacher educators when it comes to their questions of how to structure and design learning arrangements.

This paper theoretically discusses and suggests design principles fruitful for integrated teaching designs that may strengthen connections to school algebra in the field of abstract algebra. For this we draw on the framework of non-local mathematical knowledge for teaching (Wasserman, 2018a). As for the design, we base our work on the research on an instruction model for sequencing learning arrangements (Wasserman et al., 2019) and the design principle of relating representations (Prediger & Wessel, 2013). We adapt and specify these considerations for the development of non-local knowledge for teaching and present exemplary realizations.

Theoretical background on teacher professionalization

According to Blömeke et al. (2015), teachers' professional competence can be conceptualized along a continuum. They see cognitive and affective dispositions as a foundation that impact teachers' skills (perception, interpretation, and decision-making) when dealing with situation-specific tasks which then becomes visible in classroom behaviour. The cognitive resources play an important role in

teacher education programs since they serve as necessary conditions for working on and dealing with situation-specific tasks on the job (Silverman & Thompson, 2008). In the cognitive facet, ‘content knowledge’ and ‘pedagogical content knowledge’ are differentiated, and in recent years research has been published to further differentiate these conceptualizations (overview in Neubrand, 2018). With respect to advanced mathematics, the Michigan group’s framework of *Mathematical Knowledge for Teaching* (Ball et al., 2008) introduces the construct of ‘horizon content knowledge’ as a facet of ‘content knowledge’ which is often drawn upon for conceptualizing the school mathematics from a higher standpoint perspective. Wasserman (2018a) builds on the idea of knowledge about the mathematical horizon and proposes the theoretical constructs of local and non-local mathematics and the notion of *non-local mathematical knowledge for teaching*. While local knowledge refers to the mathematics a (future) teacher is going to teach in class, the non-local mathematics is metaphorically “further away” in the mathematical horizon in terms of the time when the respective content is learned and the degree of interdependence. This might be mathematics, which is part of the school mathematics curriculum, or refer to university mathematics. For example, commutativity and associativity of the multiplication of integers is part of the local mathematical content in grade 5 in Germany. This is connected to the non-local content of (non-)commutativity and associativity of matrix multiplication in grade 12 and the (non-)commutativity and (non-)associativity of algebraic structures in an abstract algebra course at university. For example, cubes in geometry in grade 5 is also non-local content but we focus on non-local content that is further away in the curriculum spiral. We would like to emphasise, that the classification as non-local of the examples is made in relation to the grade 5 example. The non-local content knowledge is hypothetically meant to support reshaping the understanding of the local content from a higher standpoint and can possibly influence teaching practices (Wasserman, 2018a). Reshaping the local content can be operationalized by means of transformative processes (Lee, 2018). One transformative process as a facet of reshaping is *unifying* seemingly unrelated mathematical objects so that they are grasped more coherently under an overarching mathematical idea (Lee, 2018). These overarching mathematical ideas can be accounted for by fundamental ideas (also called Big ideas) which are the pillars in Bruner’s (1996) curriculum spiral as the centre of his teaching principle. We use an extended spiral to university mathematics for depicting the relations between the different facets of content knowledge (see Figure 1).

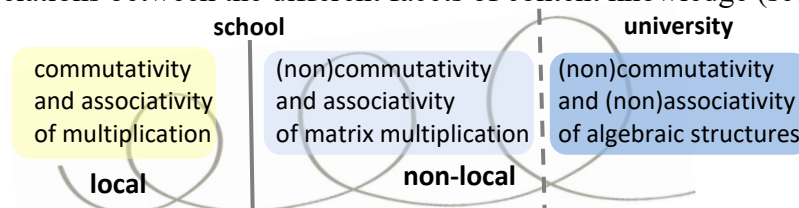


Figure 1: Exemplary extended spiral for commutativity and associativity

That teachers should be aware of these connections in the spiral is identified in empirical comparisons of US-American and Chinese elementary mathematics teachers: Ma’s (2010) criterium of teachers’ awareness and knowledge of “longitudinal coherence” is one of four key criteria that characterize teachers with profound understanding of fundamental mathematics (PUFM): “With PUFM, teachers are ready at any time to exploit an opportunity to review crucial concepts that students have studied previously. They also know what students are going to learn later, and take opportunities to lay the proper foundation for it” (Ma, 2010, p. 104). Transferring this to local and non-local mathematics,

(future) teachers should have a solid understanding of how mathematical concepts, e.g., commutativity and associativity, are present and related in local and non-local mathematics (see Figure 1). While Ma's (2010) empirical research refers to primary school teaching, a qualitative reconstruction in the context of an in-service teacher professionalization study shows that it seems to be very challenging for lower secondary mathematics teachers to perceive curricular connections between the mathematical content of different school levels (Rösike, 2021).

As pointed out, there is a need but also a challenge for (future) teachers to notice and properly understand relations between local and non-local mathematics. It remains to be specified what content connections are fruitful for non-local knowledge for teaching (which includes reshaping processes) in the field of abstract algebra and how (future) teachers can be supported in perceiving these connections and in the necessary reshaping processes.

Connections for potential reshaping processes in the field of abstract algebra

According to Blömeke et al. (2015) teachers' professional competence can be conceptualized along a continuum. They see cognitive and affective dispositions as a foundation that impact teachers' skill. Focusing on advanced mathematics in relation to secondary mathematics in university mathematics teaching, Wasserman (2018b) suggests a continuum of possible connections. The continuum ranges from "content connections" to "disciplinary practice connections", "classroom teaching connections", and "modeled instruction connections". Content connections refer to a meaningful relationship between a concept discussed in the advanced course (e.g., groups in abstract algebra) and some secondary content (e.g., inverse functions) (Wasserman, 2018b). For identifying fruitful content and teaching connections, abstract algebra curriculum designers can rely on a study by Wasserman (2016) who specifies how ideas in abstract algebra can be beneficial for secondary algebra (and early algebra) teaching. The school algebra curriculum only implicitly draws on the structures of groups, fields, and rings (largely fields and rings). This is why the study aimed at identifying common content areas that provide a synthesis of the types of school mathematics potentially informed by knowledge of abstract algebra. As a result, the four content areas "arithmetic properties", "inverses", "structure of sets", and "solving equations" have been shown as most relevant (Wasserman, 2016).

A growing body of research has been published rather recently for enhancing connections in abstract algebra university teaching (e.g., Ticknor, 2012; Wasserman 2018b). In these investigations, it is often the content connection perspective that is taken (Wasserman, 2018b). Our design incorporates content and teaching connections between abstract algebra and school algebra in the content areas of "associativity and commutativity", "solving equations" and "inverses". How this is accounted for in the design will be elaborated in the following.

Design on macro and micro levels for non-local knowledge for teaching

This section distinguishes between theoretical foundations for the structural design of learning arrangements (macro level) and those for a more concrete design at the task level (micro level). We concentrate on design principles for developing non-local knowledge for teaching, especially the initiation of unifying processes. Consistently, one would combine these with principles that focus on the development of conceptual understanding of non-local mathematics such as guided reinvention (Larsen, 2013), but we will not go into further detail since our focus is on an integrated perspective.

Sequencing Principle for initiating development of non-local knowledge for teaching

For developing knowledge of non-local mathematics for teaching, Wasserman et al. (2019) propose an instructional model composed of three phases. In the *building-up* phase, a situation or question from a (school) mathematics teaching context is presented and discussed. In the *learning* phase, relevant and helpful non-local mathematical content for dealing with the teaching-related situation is addressed. Finally, in the *stepping-down* phase, the PSTs return to the teaching situation. As criteria for selection, Wasserman et al. (2019) demand that the teaching situations should be authentic. This design intends enhancing content connections between local and non-local mathematics as well as establishing connections to the teaching of local mathematics (Wasserman et al, 2019). With their instructional model, the research group realizes considerations of Silverman and Thompson (2008) for the development of mathematical knowledge for teaching, in which the aim is to build a fundamental understanding of the school mathematics content as well as initiate respective reshaping processes.

Learning arrangements sequenced according to the principle were tested for real analysis and algebra teaching. Empirical data suggests that the instructional model seems to have helped pre-service teachers find connections between real analysis and their teaching (Wasserman et al., 2019). Furthermore, the job of “teaching with attention to scope” was investigated, i.e. the explication of areas of validity of mathematical statements in a secondary class, in regard to the effects of the module on real analysis (Wasserman et al, 2019). It appears that changes in the pre-service teaching practices were observable i.e. the PST repeatedly attended to scope more explicitly in their teaching. “Sequencing learning arrangements” according to the instructional model suggests being fruitful for the development of *non-local mathematical knowledge for teaching*. However, more concrete guidance on the task-design level that helps implementing the instructional model is needed.

Relating representations for conceptual understanding and non-local knowledge for teaching

Due to the specific ontological nature of mathematical objects as abstract and mostly relational entities, a mathematical concept cannot be understood just by referring to real objects. Therefore, the importance of relating multiple representations is emphasized (Duval, 2006). By transitioning between *verbal*, *symbolic*, *graphical*, and *concrete* representations, students construct the mental objects and relations to which a mathematical concept refers (Duval, 2006). Prediger and Wessel (2013) extended the design principle by unfolding the verbal representation into the relevant registers that students encounter (as learning medium and learning goals) in mathematics teaching. In this version, the principle of “Relating registers and representations” was proven as a fruitful design strategy for integrating language and mathematics learning that has been picked up in further design research studies with a focus on mathematics and language integrated learning. It suggests that activities should be designed for flexibly and deliberately relating the representations and registers.

Lifting the principle from school to university mathematics education, relating representations is empirically investigated as an activity for developing conceptual understanding, but compared to school level research, there are still only few studies. Moreno-Arotzeno and colleagues (2021) draw on Duval’s theoretical constructs (2006) of treatments und conversions: while relating within one semiotic system (e.g., within the graphical representation level) is defined as *treatment*, relating

representations from different semiotic systems (e.g., changing from verbal to graphical representation) is defined as *conversion*. The latter are often more challenging for students. Moreno-Arotzeno et al. (2021) conclude that a more fine-grained specification of the graphical representations especially for more abstract university mathematics is needed for identifying students' relating representations abilities. In teacher education, we see the need to extend this differentiation further to the local and non-local mathematical content and do this for all three levels of representations (symbolic, graphical, verbal, depicted in Table 1 with an algebra example).

Table 1: Exemplary specification of representations crossed linked with local and non-local dimension

	local	non-local
symbolic	$x \cdot \frac{2}{5} = \frac{5}{3}$	$d_{120} \circ x = s_1$
graphical		
verbal	Find the number that, when multiplied with two fifths, makes five thirds.	Find the mapping that, when composed with the rotation by 120°, is equivalent to the reflection s_1 .

In the field of abstract algebra, Larsen (2013) conducted design research and implemented graphical representations in their learning trajectories for developing conceptual understanding of algebraic concepts such as associativity, commutativity and group axioms (Larsen, 2013). The graphical representations in their teaching modules support the development of mental models and structure sense which is initiated by students comparing algebraic structures and patterns across contexts. Students are asked to systematically analyze similarities and differences when more abstract (for students often very challenging) concepts like isomorphisms are developed. Such cognitive activities of analyzing similarities and differences initiated by prompts of comparing and contrasting are theoretically backed by cognitive learning theories: They show high potential for students identifying characteristic properties of a term, a concept or an operation and are shown to have positive effects on learning (Alfieri et al., 2013).

For school and university mathematics teaching we see the potential of the “Relating representations” principle, particularly when aims of integrating learning goals are pursued. Drawing on the principle in teacher education opens additional directions of relating representations when adding the local and non-local dimension (see arrows in Table 1). We conject that such relating activities function as means to initiate reshaping processes and can thus develop *non-local mathematical* knowledge for teaching. These connections and the effects and conditions of success of the extended design principle are yet not systematically investigated empirically, nor specified theoretically.

Research question

As the literature review revealed, “Sequencing” on a macro level and “Relating representations” on a micro level seem fruitful for connecting school and university mathematics. However, further refinement of the design principles with regard to local and non-local mathematics and possible reshaping processes is needed. We therefore ask: What are the design principles’ logical structures and how can they be realized for exemplary abstract algebra content learning?

Design principles' logical structures and exemplary implementations

Methodologically, the theoretical considerations of this paper are situated in a design research project (Prediger, 2019). In general, design principles have a logical if-then structure that is grounded in a theoretical argument and a (mostly empirically backed) qualifier as conditions of success.

Logical structure and exemplary implementation: Sequencing principle

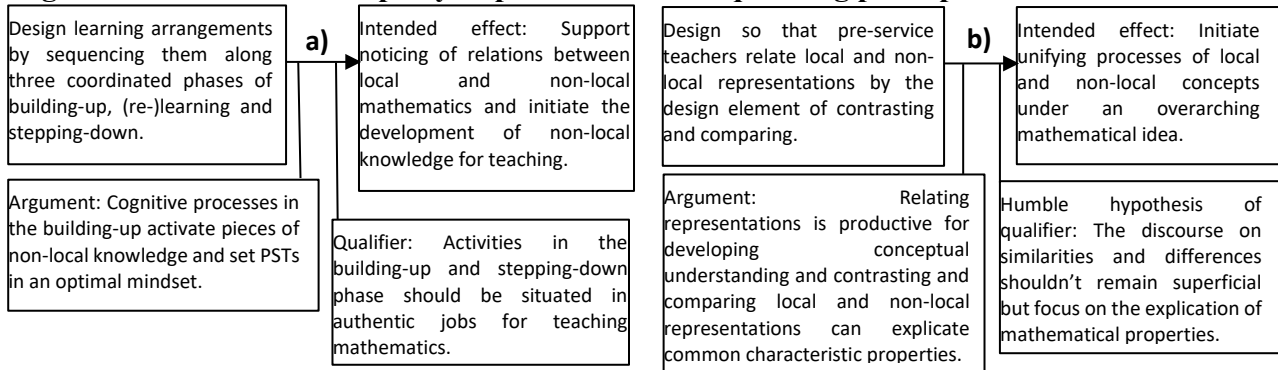


Figure 2: Logical structure of the design principles “Sequencing” (a), “Relating representations” (b)

In Germany, PST typically first encounter groups in their first semester linear algebra lecture and again in their later studies when they (have the option to) take an abstract algebra course. As the designed learning arrangement is intended to be implemented in an abstract algebra course, it seems appropriate to modify Wasserman et al.’s (2019) model as “Sequencing learning arrangements in the three phases building-up, (re-)learning and stepping-down”. The theoretical considerations and empirical studies as elaborated above provide an argument and structure as summarized in Figure 2a.

Table 2: Exemplary learning trajectory for “associative and commutative properties”

Building-up	(Re-)learning	Stepping-down
<ul style="list-style-type: none"> • PST place textbook excerpts in the curriculum spiral and think of a corresponding fundamental idea. • PST retrace a students' solution of the task $64+3+97$. 	<ul style="list-style-type: none"> • PST comment on: „All commutative algebraic structures are also associative.“ • PST check the properties of different algebraic structures (local and non-local). • PST check if a manipulation like $64+3+97=3+97+64$ is valid in other algebraic structures. 	<ul style="list-style-type: none"> • PST evaluate and adapt a textbook excerpt on commutativity and associativity. • PST reflect the spiral and the fundamental idea.

For an overview of the *implementation* of the “Sequencing principle” in the learning arrangement for “commutativity and associativity” for PST in an abstract algebra course the activities are shown in Table 2. With this learning arrangement, PST are supported to deepen their understanding of both concepts and unify different local and non-local examples of checking for these properties. In the building-up phase PST get textbook excerpts from different grades. They are asked to place these in the curriculum spiral and formulate a fundamental idea which they associate with the excerpts. With this task, the job “accounting for longitudinal coherence” according to Ma (2010) is considered. The excerpts were chosen from current textbooks to be authentic as possible. The PST suggested fundamental ideas naturally lead to the mathematical concepts of commutativity and associativity that are subsequently studied in depth in the abstract context of algebraic structures. In the stepping-down phase, the extended spiral is revisited and one excerpt is discussed in more detail with special emphasis on the previous non-local mathematics content.

Logical structure and exemplary implementation: Relating representations

We draw on “Relating local and non-local representations” complementing the structural macro level of “Sequencing” as a principle on the micro task design level. Prompts to contrast and compare serve as a theoretically and empirically grounded design element to stimulate unifying processes (see Figure 2b). In Duval’s framework (2006), the principle initiates treatments within one semiotic system but different dimensions (local or non-local). As an example of implementation, in the learning arrangement on “solving equations” the local symbolic representation $x \cdot \frac{2}{5} = \frac{5}{3}$ is related to the non-local symbolic representation $d_{120} \circ x = s_1$ and in the (re)-learning phase, PST are explicitly asked to identify similarities and differences between these equations and the given solutions (see Figure 3). Subsequently, the findings are generalized for an equation in an algebraic structure $(M, *)$. In the stepping-down phase PST discuss and adapt a textbook excerpt on strategies for solving equations (Figure 3) with regard to non-local mathematics. As an intended effect, PST unify the steps for solving simple equations under the overarching mathematical concept of the group axioms.

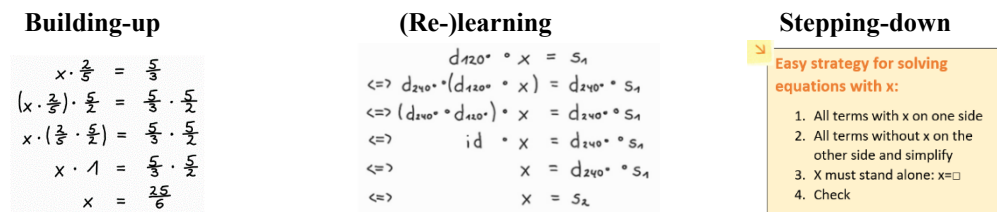


Figure 3: Insight into teaching material

Discussion and outlook

Overall, the design element of the extended spiral and fundamental ideas seem fruitful for realizing the design principle “Sequencing a learning arrangement”. In their combination, the job “accounting for longitudinal coherence” (Ma, 2010) can be addressed in the building-up and stepping-down phase. The design principle “Relating representations” is lifted from school mathematics to university mathematics integrating the teaching of the school mathematics perspective by the design element of contrasting and comparing local and non-local representations. It is still to be empirically researched to what extent the presented design principles contribute to developing non-local mathematical knowledge in the sense of Wasserman (2018a). For this we conducted design experiments with pairs of PST ($N=8$). The analysis is in progress and intends to reconstruct PST individual learning pathways with an emphasis on PST’s reshaping processes and resulting challenges. Furthermore, the design principles are to be empirically refined with regard to their effect and conditions of success.

References

- Alfieri, L., Nokes-Malach, T. J., & Schunn, C. D. (2013). Learning through case comparisons: A meta-analytic review. *Educational Psychologist*, 48(2), 87–113.
- Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, 59(5), 389–407.
- Blömeke, S., Gustafsson, J.-E., & Shavelson, R. J. (2015). Beyond dichotomies. *Zeitschrift für Psychologie*, 223(1), 3–13. <https://doi.org/10.1027/2151-2604/a000194>
- Bruner, J. S. (1996). *The culture of education*. Harvard Univ. Press.

- Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics*, 61(1), 103–131.
- Isaev, V., Eichler, A., & Loose, F. (Eds.). (2022). *Professionsorientierte Fachwissenschaft: Kohärenzstiftende Lerngelegenheiten für das Lehramtsstudium Mathematik* [Profession-oriented subject science: Coherence-creating learning opportunities for the teacher training programme in mathematics]. Springer Spektrum.
- Larsen, S. P. (2013). A local instructional theory for the guided reinvention of the group and isomorphism concepts. *The Journal of Mathematical Behavior*, 32(4), 712–725.
- Lee, Y. (2018). *University students' school mathematics understanding and its growth in their learning of collegiate mathematics*. [Doctoral dissertation, The Pennsylvania State University].
- Ma, L. (2010). *Knowing and teaching elementary mathematics. Teachers' understanding of fundamental mathematics in China and the United States*. Routledge.
- Moreno-Arotzena, O., Pombar, I., & Barragués, J. I. (2021). University student understanding of the gradient of a function of two variables: an approach from the perspective of the theory of semiotic representation registers. *Educational Studies in Mathematics*, 106(1), 65–89.
- Neubrand, M. (2018). Conceptualizations of professional knowledge for teachers of mathematics. *ZDM*, 50(4), 601–612. <https://doi.org/10.1007/s11858-017-0906-0>
- Prediger, S. (2019). Theorizing in design research: Methodological reflections on developing and connecting theory elements for language-responsive mathematics classrooms. *Avances De Investigación En Educación Matemática*, 15, 5–27. <https://doi.org/10.35763/aiem.v0i15.265>
- Prediger, S. & Wessel, L. (2013). Fostering German language learners' constructions of meanings for fractions - Design and effects of a language- and mathematics-integrated intervention. *Mathematics Education Research Journal*, 25(3), 435–456.
- Rösike, K.-A. (2021). *Expertise von Lehrkräften zur mathematischen Potenzialförderung* [Expertise of teachers in promoting mathematical potential]. Springer.
- Silverman, J., & Thompson, P. W. (2008). Toward a framework for the development of mathematical knowledge for teaching. *Journal of Mathematics Teacher Education*, 11(6), 499–511.
- Ticknor, C. S. (2012). Situated learning in an abstract algebra classroom. *Educational Studies in Mathematics*, 81(3), 307–323. <https://doi.org/10.1007/s10649-012-9405-y>
- Wasserman, N. H. (2016). Abstract algebra for algebra teaching: influencing school mathematics instruction. *Canadian Journal of Science, Mathematics and Technology Education*, 16(1), 28–47.
- Wasserman, N. H. (2018a). Knowledge of nonlocal mathematics for teaching. *The Journal of Mathematical Behavior*, 49, 116–128. <https://doi.org/10.1016/j.jmathb.2017.11.003>
- Wasserman, N. H. (2018b). *Connecting abstract algebra to secondary mathematics, for secondary mathematics teachers*. Springer. <https://doi.org/10.1007/978-3-319-99214-3>
- Wasserman, N. H., Weber, K., Fukawa-Connelly, T., & McGuffey, W. (2019). Designing advanced mathematics courses to influence secondary teaching: Fostering mathematics teachers' "attention to scope". *Journal of Mathematics Teacher Education*, 22(4), 379–406.