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Developing elementary mathematics teachers' knowledge of mathematical reasoning processes

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This paper aims to identify the in-service primary teachers' knowledge of their students' mathematical reasoning processes. Data were collected in the context of a teacher education experiment through recording of the Zoom sessions regarding the autonomous work carried out by the groups and the whole group discussions. The results show that the activity of planning and carrying out tasks that promote reasoning, as well as joint analysis with peers and experts, has a significant potential for developing knowledge that is recognized by teachers themselves.

Keywords: Mathematical reasoning, generalising, justifying, teacher knowledge.

Introduction

Teacher education should give special attention to mathematical reasoning, considering both the ability to reason, and knowledge about the reasoning processes (Stylianides & Stylianides, 2006). This paper is part of the *Mathematical Reasoning and Teacher Education* (REASON) project, which aims to study the mathematical and didactical knowledge teachers need to carry out a practice that promotes students' mathematical reasoning and to study the ways to foster its development in prospective and practicing teachers of primary, middle, and secondary school. The Project included a teacher education experiment where the participant teachers had to carry out tasks that promote mathematical reasoning in their practice and present that experience to their colleagues. In this communication, we aim at identifying the in-service primary teachers' knowledge of their students' mathematical reasoning processes and the potential of the teaching education experiment.

Mathematical reasoning and its processes

The meaning of mathematical reasoning is not consensual in literature. Our perspective is set out below through the contributions of different authors who complement each other. We consider mathematical reasoning as “making inferences in a substantiated way, that is, starting from given information to obtain new information through a justified process” (Ponte et al., 2020). Jeannotte and Kieran (2017) consider two perspectives on mathematical reasoning, representing different ways to look for a given discourse: structural and process. At the first, they include the different types of reasoning: deductive, inductive, and abductive. The process perspective involves (i) processes related to the search of similarities and differences, including generalising, conjecturing, identifying a pattern, comparing, and classifying; and (ii) processes related to validating – justifying and proving. The essential processes of mathematical reasoning are conjecturing, generalising, and justifying, being conjecturing central on abductive reasoning, generalising, a key process of inductive and abductive reasoning, and justifying, an essential process of deductive reasoning (Ponte et al., 2020). Regarding the mathematical reasoning processes we adopted the following perspectives: The process of conjecturing involves the production of statements that are thought to be true but are not known to

be true and the generalising one occurs when an individual identifies commonalities across cases or when he extends his reasoning beyond the set in which he originally identified the commonalities (Jeannotte & Kieran, 2017; Lannin et al., 2011). When the conjectures are expressed in general ways, they are generalisations. Justifying involves doing logical arguments based on already-understood ideas and evaluating their validity (Lannin et al., 2011). Thus, we consider that the process of generalising is fundamental in mathematics when we intend to make general statements about procedures, properties or concepts and that justification is central to mathematically validating those statements (Mata-Pereira & Ponte, 2018). These two processes interact with each other. In many situations, the language used in justification needs to be general so that its applicability to the entire domain is clear. For Jeannotte and Kieran (2017), exemplifying is an auxiliary process of generalising and justifying, which allows inferring data about a problem by generating elements that support those processes.

Development of teachers' knowledge about mathematical reasoning

Elementary school teachers should have opportunities to develop their knowledge about mathematical reasoning (Lannin et al., 2011; Stylianides & Ball, 2008; Stylianides & Stylianides, 2006), increasing their specialised mathematics knowledge, that is their mathematics knowledge needed for teaching. This “involves the use of decompressed mathematical knowledge that might be taught directly to students as they develop understanding” (Ball et al., 2008, p. 400). Concerning the aspect of reasoning, specialised mathematics knowledge is the knowledge about the reasoning, namely what reasoning means, what kinds of reasoning there are and their processes and also the relationships between them (Rodrigues et al., 2021). Given the importance of achieving a high level of knowledge (Rodrigues et al., 2021), we elaborated a framework (Table 1) that allows to analyse teachers' level of understanding of the reasoning processes.

Table 1: Framework for knowledge of mathematical reasoning processes (Rodrigues et al., 2021)

Category	Subcategories
Knowledge of the reasoning process (generalising - G, justifying - J, exemplifying - E, comparing -C, classifying- Cl)	5. Knowledge of the process fits the definition presented, and includes its relationship with the other reasoning processes
	4. Knowledge of the process fits the definition presented, and is explicitly outlined by enunciating the properties of the process
	3. Knowledge of the process fits the definition presented, and is explicitly outlined through illustrative example(s)
	2. Recognising a reasoning process though considering only 'correct' processes
	1. Knowledge of the process takes on the meaning of the term in everyday language
	0. The process is confused with other processes

Several authors (for example, Francisco & Maher, 2011; Loong et al., 2017) refer to the need to create opportunities for teachers to learn about how to develop mathematical reasoning in students. In the same sense, Stylianides and Ball (2008) defend the need to develop in teachers the ability to plan and implement tasks that promote the development of reasoning in their students. Melhuish et al. (2019) state that we must provide teachers with opportunities to analyse evidence of the students' reasoning during their education. Those opportunities will also help them to expand their didactical knowledge on how to promote and encourage work with reasoning in their classes (NCTM, 2000). They need to consider a variety of factors such as: identifying the potential of the task to develop mathematical reasoning; knowing what actions they should take to facilitate the development of reasoning; and being aware of students' knowledge on the subject (Davidson et al., 2018). When teachers select tasks that promote reasoning processes, implement them in the classroom and later reflect on the students' productions with other teachers, they may expand and deepen their didactical knowledge on how to develop their students' mathematical reasoning (Herbert & Bragg, 2021).

Methods

The present study followed a qualitative-interpretative approach (Patton, 2002), focusing on participants' processes and meanings. It was developed in the context of a teacher education experiment, corresponding to cycle 2 of Design-Based Research (Cobb et al., 2003), with 19 in-service teachers of grades 1 to 6. This experiment ended up being carried out online due to the pandemic, in the second half of 2020/2021. It was conducted by all authors of this article and took place during eight sessions, along four months, each session lasting two hours and 30 min. The experiment involved exploration and discussion of tasks focusing on the reasoning processes, on the teacher's actions to develop the pupils' reasoning and on the principles of task design. The tasks encompassed analysis of (i) professional journal articles, (ii) mathematical tasks for primary school pupils, and (iii) episodes from primary school lessons. All tasks were initially explored autonomously by the teachers, organised into four groups, and were later discussed by the whole group. Sessions 1 to 3 addressed the meaning of mathematical reasoning and its processes articulating theory with empirical data from classroom episodes. By the 4th session, the in-service teachers had to suggest, prepare, and carry out tasks with their students (*Put into practice I*) and for the 5th session the teachers had to present a critical reflection on the work carried out by their students, focusing on their reasoning processes, and on their teaching practices. The 6th session was dedicated to the teacher's actions to develop the pupils' reasoning and on the principles of task design. Finally, sessions 7 and 8 repeated the approach used in sessions 4 and 5 (*Put into practice II*).

The data of this paper regards 5th session's presentation and discussion in the whole group of the work carried out in *Put into practice I* by a pair of teachers of grades 5 and 6 (Alice and Fernanda) who proposed the same task (Figure 1), as well as a dialogue between the teachers from session 4.

<p>Regard the following products.</p> $\begin{array}{ll} 37 \times 3 = 111 & 37 \times 12 = 444 \\ 37 \times 6 = 222 & 37 \times 15 = 555 \\ 37 \times 9 = 333 & \end{array}$	<ol style="list-style-type: none"> 1. Do you find any regularity in these products? Justify your answer. 2. Based on the regularity observed in the previous products, determine the value of 37×27. Show how you arrived at your answer.
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Figure 1: Task proposed to 5th and 6th graders

Data were collected through recording by Zoom the whole group discussions and subsequent transcription, as well as the presentation slides. The teachers had more than twenty years of experience and were well considered by the community. In the collective discussion other teachers participated. All names are fictitious. The analysis was performed through content analysis of the collected data (Bardin, 2010), using the framework we elaborated before (Rodrigues et al., 2021) concerned with knowledge of reasoning processes (Table 1), where the categories are relative to the knowledge of the reasoning processes worked in the teacher education experiment, although we will focus only on the generalising and justifying processes. Each of these categories was divided into subcategories corresponding to six levels of specialised mathematical knowledge of the content, presented in hierarchical form. In terms of its use, it's like we have six analogous frameworks, each for a reasoning process. Thus, when a teacher discusses a process (for example, based on a student's answer) a level is attributed according to the description of the subcategory that best corresponds. This means that for different processes may be possible to attribute different levels.

Results

The task proposed by the group of teachers implies the formulation of a generalisation and, eventually, its justification, with or without the support of examples. However, regarding this process, the request in question 1 may indicate some confusion: What does it mean to justify the identification of a regularity? Students may explain the regularity, may tell what led them to notice it and may explain why it occurs. These are three different kinds of answers, but only the latter can constitute a justification. In this case, it is of an ambitious agenda for 5th grade students, but is this the intention of the teachers? Also, in question 2, students are asked to show how they arrived at 37×27 . It's a request of a communicative act – making explicit to others how the question was resolved. In doing so, students may or may not justify their answer. But wouldn't this be the time to ask them to do so?

For the presentation of the classroom experiences, Alice and Fernanda selected students' productions that "seemed important ... better and with different reasoning". The first group of productions concerns question 1 (Figure 2) and shows that three students found similar regularities but based on different ways of structuring the numbers.

Yes, the number 37, when it is multiplied by multiples of 3, increases 111. Therefore, when the multiplier is a multiple of 3, the digits of the product will all be the same up to 999.	David, 6th grade	Generalising	Justifying
I find two regularities, the first is that in the result all the digits always increases 111. The second is that the multiplier always increases 3.	Miguel, 6th grade	Generalising	Justifying
Yes, they are multiplying 3, 6, 9, 12, 15 (the 3 times table), after it would be $37 \times 18 = 666$. The result always increases 111.	Madalena, 6th grade	Generalising	Justifying

$$\begin{array}{r}
 37 \times 3 = 111 \quad +111 \\
 37 \times 6 = 222 \quad +111 \\
 37 \times 9 = 333 \quad +111 \\
 37 \times 12 = 444 \quad +111 \\
 37 \times 15 = 555 \quad +111 \\
 \hline
 \text{Always plus 3}
 \end{array}$$

Figure 2: English translation of the presentation slide of three students' productions of question 1

Let's see what the teachers say about the reasoning processes involved.

Fernanda: This boy [David] we thought he generalises, justifies in order to generalise. The other [Miguel] ... also justifies in order to generalise and here, aside, he exemplifies to justify what he says.

The speech reaffirms the identification of the processes of generalising and justifying presented in the slide. Regarding the first, we agree that the students generalise, either by saying that “the digits of the product will all be the same up to 999” (David), or when referring that the result always increases 111 (David, Miguel and Madalena). However, what is there in their answers that may be associated with the justifying process? Possibly, the fact that one of the factors corresponds successively to multiples of 3, a property that is fundamental to justify, although it would need further elaboration. However, Fernanda mentions more than once that the student “justifies in order to generalise”, indicating a misunderstanding. In fact, while the chosen productions of the students are adequate as evidence of the generalising process (level G3), they don’t correspond to the process of justifying, thus showing a confusion with generalising (level J0).

A look at the previous session in which Fernanda planned the task with her group may shed some light on this confusion:

- Fernanda: I am now being confronted with generalisations and then justifications, but all my life I justified the whys and then generalised. Today, here, I am being confronted with situations that are not what I thought. And that's why it's hard to get it.
- Antónia: Great, I said exactly that when she [the educator] entered our room and we were having an interesting discussion. For me, one of the things of this course, it is exactly the clarification of concepts.
- Fernanda: That's right! But this was a thing that thought: First, you justify, then you generalise.

In this group discussion, we confirm that, on the one hand, there was a confusion between the processes of generalising and justifying and that, on the other hand, the teachers were becoming aware of this confusion from the work sessions. Let us now analyse the teachers' perspective on three of their students' resolutions to question 2 (Figure 3):

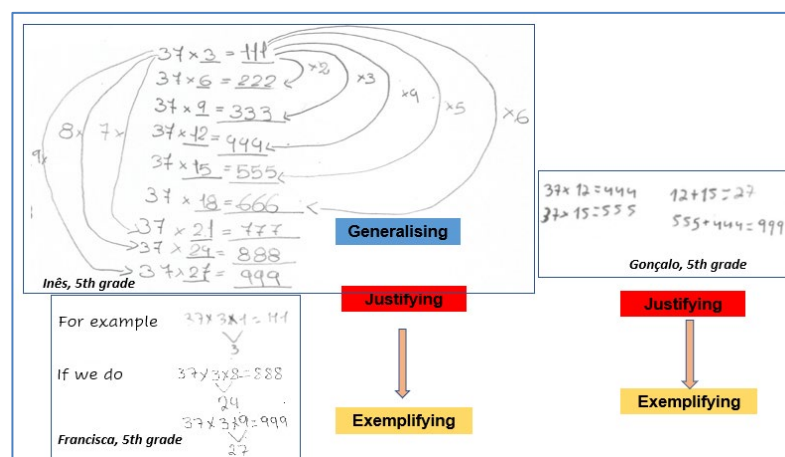


Figure 3: English translation of the presentation slide of three students' productions of question 2

In the case of Inês and Francisca, the group considers that they generalise and justify, while for Gonçalo they only identify the latter. In all situations, the teachers consider that the process of justifying is supported by exemplifying:

- Alice: Inês also verified that they were multiples of 3, didn't she? And so it goes, 3, 6, 9, 12, 15, 18, and then she says another pattern. She identified that in the case of the result, it will always be 111 to be multiplied by 2, by 3, by 4, by 5 and by 6 . . . she realised that when it's 21, she makes 3×7 , she puts the arrow there. . . and then here

- times 8 . . . and then it gets here to 999, which would be 111 times 9. We also considered that [the processes] would be generalising and justifying.
- Fernanda: We have here the example of Francisca, also from the 5th grade, who says that 3 is 3 times 1, therefore it is 111. If it was 3 times 8, which was for 24, the 8 would mark the 8, 8, 8 and for 27, the 9 would mark the 9, 9, 9. They justify in both cases by giving examples.
- Alice: Exemplifying as a support to justifying.
- Fernanda: And this one [Gonçalo] has a very different reasoning.
- Alice: So, he thinks like this: he adds 12 to 15 and gets 27. And then he adds the results 555 plus 444 to arrive at 999. And then he doesn't explain further, nor he does make other examples. We considered a justification, using examples, without generalising, because he only does this example that is here.

One aspect that seems clear in the three cases is that the students do exactly what they are asked to do: calculate 37×27 and show how they arrived at the answer. In this question it is not supposed to make a generalisation because that was the purpose of the previous question. Thus, what would be expected in this question is that students, when showing how they calculated 37×27 , rely on generalisation, which in this case, becomes an argument for justifying. This change in status is a very sensitive aspect and creates difficulty in the analysis of reasoning, as we identified in the teachers' discourse. In the productions of Inês and Francisca, the teachers identify that there is a generalisation because it is evident in the explanation of how they arrive at the result, but in these cases, it works as a justification. With good reason, they only identify the process of justifying in Gonçalo's answer because, in fact, he used the distributive property of multiplication in relation to addition to obtain the product, instead of using the regularity. Indeed, this student only justifies decomposing 27 into 12 and 15 to be able to use the products presented in the task and not by enunciating the property, which is only implied (level J3).

So, considering the teachers' analysis and reflections regarding the answers of the six students, it is clear that they don't make a clear distinction between generalising and justifying, but they relate generalising and justifying with other characteristics that make some sense: on one hand, it seems that every time there is a pattern they identify the process of generalising; on the other hand, when students present how they thought, they identify the process of justifying, even though sometimes it may not be the case. Even so, these episodes show the relevance of discussing the processes of reasoning with peers and experts in articulation with classroom practice. Another example is the question raised by a teacher concerning this task proposed by her colleagues:

- Maria: It was very interesting to see their productions. I question if they are generalising beyond the domain, if they go beyond the domain they are working with. If it is a generalisation or if it is more a justification with examples. Because, if they go beyond the 27, this regularity will no longer happen.
- Fernanda: We considered generalising within the scope of the task, we did not think about generalising beyond, because it did not allow it.
- Maria: But isn't generalising extending beyond the reasoning of the problem?
- Educator: What is the domain they are working with here? This task only allows you to go up to 27 . . . I find a generalisation in the way they identify this regularity, because the domain presented only goes up to 5, 5, 5. When they manage to go beyond 555, we can consider that they are going beyond the domain of the examples presented.

In a way, it seems that Maria has the implicit idea that the generalisation should extend further, possibly unlimitedly. This perspective is partially in line with the literature, but this episode reveals

how Maria seeks to make sense of the reasoning process mobilising the definition presented (Lannin et al., 2011) in the analysis of real student productions, showing how this can be a powerful resource development of teacher knowledge.

Conclusion

The analyses of the episodes show how the identification of the reasoning processes may impose different levels of complexity due to the dynamism of the mathematical activity, thus imposing different challenges to the teachers. In the task proposed to the students, they should start by identifying a regularity and make a generalisation. However, in the next step, they had to use that generalisation in order to calculate, so the relation, which was a generalisation in the first place, became an argument to justifying. This different status may explain why the teachers identify the two processes in almost all of the student productions. In fact, the only case they did not relate to generalising is when a student used a property that was learned a long time ago.

The challenges that we exposed and were able to deal with are clearly associated with the kind of work we develop in the education experiment. In fact, as Melhuish et al. (2019) suggest, if we want teachers to promote their students' ability to generalise, we must provide them with opportunities to analyse evidence of these processes during their education. Our research shows that the activity *Put into practice*, involving planning, and carrying out tasks that promote reasoning, as well as joint analysis with peers and experts, has a significant potential for discussing the meaning of the reasoning processes, which is essential for developing teachers' knowledge – both didactical and specialised content knowledge. Even the way teachers formulate the questions in the mathematical task is dependent on and, at the same time, revealing of their knowledge about the reasoning processes. Moreover, the analysis of the way in which students think and solve the task brings out the dynamic nature of reasoning and the relationships between its processes, which favours a greater understanding of it. The relevance of the education experiment was also recognized by the teachers when they acknowledge the differences between their conceptions and the consensual meanings of the processes established in the literature, which remained stable until the experiment.

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