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► To cite this version:

Zoltán Paulovics, Csaba Csapodi, Zoltán Lóránt Nagy. The impact of incorrect solutions on the attitude and problem-solving abilities of prospective mathematics teachers towards combinatorics. Thirteenth Congress of the European Society for Research in Mathematics Education (CERME13), Alfréd Rényi Institute of Mathematics; Eötvös Loránd University of Budapest, Jul 2023, Budapest, Hungary. hal-04421403

HAL Id: hal-04421403

<https://hal.science/hal-04421403>

Submitted on 27 Jan 2024

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The impact of incorrect solutions on the attitude and problem-solving abilities of prospective mathematics teachers towards combinatorics

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Secondary school mathematics teachers often need to understand and correct their students' incorrect solutions, so it is useful for them to develop their error analysis skills during their university studies. Many studies have been conducted on the positive role of error analysis in learning. In a combinatorics course for second-year prospective mathematics teachers, we investigated the impact of analysing solutions to combinatorial problems with faulty reasoning on their attitude towards combinatorics and their problem-solving skills research questions. In a quantitative study with a control group, we compared the two groups using tests at the beginning and end of the course. Paired samples t-test analysis yielded significant results for both questions: the effect of the emphasis on error analysis significantly increased both the liking of combinatorics and the mean score on the end-of-semester test compared to the control group.

Keywords: Error analysis, worked detailed sample solutions, prospective mathematics teachers, correct and incorrect sample solutions, mathematics teaching practices.

Introduction and motivation

Teaching combinatorics is highly important in K-12 and undergraduate mathematics curricula, both for its applications in computer science, and for its rich potential as a problem-solving context (Lockwood, 2013; NCTM, 2000). In their work, secondary school teachers often encounter incorrect solutions from students. This is particularly true in combinatorics, where it is usually more difficult to formulate procedures and methods that are easy for students to learn and practice (Batanero et al., 1997; English, 2005). Thus, during problem-solving, students make many mistakes that their teachers should correct. But many Hungarian university students studying teaching mathematics (in the following part referred to as prospective mathematics teachers) report that, during their secondary school studies, their teachers did not understand their non-standard solutions - regardless of its correctness - and judged it as correct or incorrect according to whether the final result matched their own (Klein et al., 2021). Over time, the experience of incomprehension may lead many students to equate combinatorics with finding the right solution - that is, the one thought up by the teacher - i.e. identifying the relevant formula and then substituting into it.

The teacher attitude they experience during secondary school studies can be ingrained, and overriding it is difficult for prospective teachers, but not impossible (Ernest, 1989). Our aim is to shape the training of mathematics teachers in Hungary in such a way that this attitude changes in a positive direction, so that future teachers look at combinatorics as a treasure trove for developing creativity and problem-solving skills. One of the stages of this project is our present research, where, among other things, we investigated how prospective teachers' attitudes to combinatorics change through the analysis of incorrect solutions.

In addition to the motivation outlined above, there are many arguments for teaching prospective teachers through the analysis of incorrect solutions:

1. Even if they recognise the sometimes inhibiting, negative attitude of their secondary school teachers towards non-standard solutions, it is not so easy to override this. To do so can be helped by identifying the shortcomings of the methods they see (for example, by comparing only the results). The presentation of correct and incorrect methods and knowledge can lead to cognitive conflict, which can lead the student to build coherent knowledge within him/herself (Schnotz et al., 1999).
2. Siegler (2002, p. 55) says that “the mechanisms through which explaining other people's reasoning exercises its effects include increasing the probability of trying to explain observed phenomena”, so that if prospective teachers are more effective at filtering out errors in their solution to a problem, even if they are given one, they are more likely to be able to solve it correctly.
3. Since errors are a natural part of mathematical problem solving, analysing them helps to build a more credible and realistic picture of mathematics (Schnotz, 2001).
4. As will be shown below, there are several findings which suggest that the analysis of incorrect solutions can be an effective way to improve the learning and understanding of the subject matter, and thus the professional skills of prospective mathematics teachers (Große & Renkl, 2007).

Theoretical background

The question of whether using a combination of correctly and incorrectly constructed sample solutions increases mathematical understanding has been investigated by several researchers.

Worked detailed sample solutions with self-explanation

Detailed sample solutions of a problem can be divided into three main parts: the problem statement, the solution steps and the final result. Correctly constructed sample solutions are an effective method for the initial stages of acquiring knowledge and procedural skills (e.g., Adams et al., 2014; Große & Renkl, 2007; Rushton, 2018), most often explained by Cognitive Load Theory (CLT) (Adams et al., 2014; Große & Renkl, 2007; Sweller, 1988). By correctly elaborating sample solutions, learners can focus on acquiring new knowledge and do not have to struggle with processing too much unfamiliar and relevant information by considering a multitude of important aspects (e.g., initial state and set of conditions, goal state, degrees of freedom of intermediate steps), which while overloading working memory limits learning ability (Adams et al., 2014; Große & Renkl, 2007).

However, learning by correctly worked sample solutions does not necessarily lead to good results. One reason for this is that worked sample solutions usually do not contain all the important details needed to understand the steps of the solution (Chi et al., 1989; Gog et al., 2019). Some learners then explain the step to themselves, while others are superficial or passive explainers (Renkl, 1997). This highlights the importance of teaching self-explanation and improving its effectiveness in both oral and written forms.

The organisation of teaching by worked sample solutions is very important, and different aspects of this are summarised by Atkinson et al. (2000). Since a common mistake in solving combinatorial problems is that students use a learned but irrelevant solution based on superficial features of the task text, we will highlight the type of solution that promotes deeper understanding and internal structure.

More worked sample solutions with different surface features (for example cover stories) but with the same internal structure are shown to students, who develop their ability to distinguish different problem types by studying them intensively (Große & Renkl, 2007; Scheiter et al., 2019).

In summary, learning through worked sample solutions is effective, provided that students are motivated to practise self-explanation and that the sample solutions are constructed and structured.

Studying correct vs. incorrect sample solutions

Incorrectly worked sample solutions differ from correctly worked sample solutions only in that they contain errors at least in one step stage. In analysing the incorrectly worked sample solutions (referred to as error analysis), students find and explain the errors and then correct them, which of course involves giving reasons for their own solutions. Thus, error analysis is not only used to improve accuracy, but also to help students gain a deeper understanding of the problem (Rushton, 2018).

It can be expected that the application of the procedure will improve some parameter of learning (e.g. efficiency, depth, time-constancy). According to VanLehn's (1999) CASCADE theory, the perception of errors or uncertainty in the initial stages of skill acquisition is key to understanding, as reflections (e.g. self-explanations) in response to these (deadlocks) help to deepen understanding.

This is in line with, for example, the importance of including non-examples alongside examples in the conceptualisation process, i.e. of presenting examples that do not belong to the concept, although they may be related to it (Skemp, 1987). Well-chosen pairs of examples and non-examples, or correctly and incorrectly worked pairs of sample solutions, can help students to understand the critical elements of the concept or problem more clearly. Furthermore, according to Siegler (2002), they also aid learning, as they increase the probability of choosing the correct solution by decreasing the probability of choosing the incorrect solution. Curry (2004) found in his research that self-explanation of correct and incorrect solutions leads to better learning outcomes than self-explanation of correct solutions.

Research questions

There has been a lot of research on the analysis of incorrectly worked detailed sample solutions, but to our knowledge, there has been no research on the impact of it on the change in prospective teachers' learning and teaching attitudes. The aim of our study is to explore the impact of analysing incorrectly worked detailed sample solutions on prospective teachers' attitudes towards combinatorics. We sought to answer the following questions in our research:

RQ1) Does the exposure to incorrect sample solutions lead to changing the attitude of prospective mathematics teachers towards combinatorics?

RQ2) Does the exposure to incorrect sample solutions lead to prospective teachers developing combinatorial problem-solving skills?

Methods and design

A second-year class of prospective teachers (between the ages of 19 and 23) at Eötvös Loránd University in Hungary participated in the experiment as part of a compulsory Elementary Mathematics 2 problem solving seminar, which builds on the theory-focused combinatorics course completed in the first semester. The aim of the (third semester) course is to revitalise and consolidate

the combinatorial and graph-theoretic knowledge that is prominent in the secondary school curriculum, and to develop the skills needed to solve the problems of these topics to the level required by teachers. Given the focus of secondary school teaching, graph theory is only introduced in a few lectures during the course, but the fundamental topics of enumerative combinatorics are discussed in great detail.

The students were free to choose one of the three groups at different times at the beginning of the semester, after which they had no possibility to change it. Two of the three groups were held by one of the authors of the article (these groups of students represent the treatment group) and one by another lecturer (this group of students represents the control group). The lecturers had the same experience in teaching the course, and the topics covered during the semester were the same, and in addition, many of the exercises in combinatorics were the same. The test was prepared jointly by the lecturers of the course. This type of exam was regularly used to assess student performance. In all three groups, students were given the opportunity to solve problems independently, there was always a correctly worked sample solution presented by the lecturer, and occasionally there was joint problem solving and group analysis of solutions.

The significant difference between the treatment group and the control group was that each lesson held for the treatment group included several (usually 2-5) incorrectly worked sample solutions, whereas in the control group, error analysis was only carried out when a student gave an incorrect solution. The experiment lasted for the whole semester. The incorrectly worked sample solutions came from previous teaching experiences of the authors. The structure of the incorrectly worked sample solutions was identical: Winnie the Pooh solved a counting problem, let's decide if he thought correctly. During the semester, it was very rare that Pooh solved a problem correctly – although not in the usual way – but it was basically clear to the students that the Pooh problems in the lesson were to be looked for mistakes. It is important to emphasise that the errors were not arithmetical errors, but were of a conceptual, logical nature, although sometimes with “smaller” errors, sometimes with “larger” ones. There was no aim to correct the wrong solution, it was enough to point out the mistake, but of course, understanding the wrong part usually led to the correction. Undercounting or overcounting was a frequent logical error, moreover, sometimes the solution seemed totally wrong due to small mistakes made in an atypical but correct solution.

One worked detailed sample solutions with short analysis

How many 8-digit numbers are there with at least one digit 5? Which of Winnie the Pooh's reasoning is correct?

a) Let's fix a digit 5 somewhere, it's 8 possibilities. The other places can be any number. So, the answer is $8 \cdot 10^7$.

b) If there is a digit 5 in the 1st place then the other places can be any digit. So, the answer is 10^7 . If there is a digit 5 in the k th place ($2 \leq k \leq 8$), then the other places can be filled with any number except the 1st place where 0 is forbidden. So, the answer is $10^7 + 7 \cdot 9 \cdot 10^6$.

Even with a little routine, the students quickly find an error in part a) and indicate that in this way Winnie the Pooh counted cases where the first digit of an 8-digit number is 0. Most of them are usually satisfied with this answer, since they found an error, so Winnie the Pooh's reasoning was

wrong. That's why part b) corrects this error, but not the overcalculation that is hidden in both parts. These incorrectly worked sample solutions show that there can be more than one error in a solution.

To explore the effect of error analysis on attitude formation and problem-solving skill improvement and to establish its statistical significance, we designed an experimental intervention study with a two-group pre-posttest design, and used SPSS for statistical analysis of the data (Creswell, 2014). RQ1 was answered using an attitude test that was completed in both the first and last classes of the semester (Szitányi & Csíkos, 2015), and RQ2 was tested with an input and an output test (the output test also provided an assessment of their performance during the semester). The two experiments¹, which are closely related but technically separate, are presented below. In the cohort of 47 students, there were students who, for personal reasons, were not present in the first class, the test, or the last class: hence the difference in the number of students participating in the study.

Experiment 1

On the very first and last sessions of the course, the students filled in the same attitude test. The aim of our study was to assess how students' attitudes towards combinatorics had changed during the semester. A total of 24 students in the two practical courses that made up the treatment group and 9 students in the control group participated in both sessions. The test consisted of two parts.

In the first part, the students answered 14 questions about the teaching of combinatorics, each scored on a scale of 1 to 5, where '1' means 'strongly disagree, not at all true for me' and '5' means 'strongly agree, absolutely true for me'. The 14 questions were divided into 4 categories, but the students were not aware of this; the categories can be briefly described as (1) the student's attitude towards solving combinatorial problems, (2) the student's general opinion on the difficulty of combinatorial problems, (3) the student's opinion on the importance of knowing and using the right formulas for combinatorial problems, and (4) the student's willingness to deal with solving difficult mathematical problems.

The second part consisted of a 16-item list of sub-areas of high school mathematics, including combinatorics. Students were asked to score on a scale of 1 to 5 how much they liked solving problems in that area, where '1' means 'very unwelcome' and '5' means 'very welcome'.

Experiment 2

The students took the pretest¹ on the very first day of the course and then the post-test¹ at a common mid-term test at the third quarter of the semester. The aim of our study was to assess how students' problem-solving skills in combinatorics had changed. A total of 31 students from the two practices that made up the treatment group and 10 students from the control group took the tests on both sessions. The tests included counting tasks and claims, and the pretest included two and the post-test one error analysis exercises. Both tests consisted of 5 problems, each problem worth 10 points. The pretest was clearly easier than the post-test, since one of the aims of the course is precisely to achieve significant skill development.

¹ The materials of experiments can be found here:

Results

Experiment 1

When comparing the 4 question groups of the 14 questions, a significant difference ($p < 0.05$) was found for the question groups 1 and 3. Concerning the student's attitude in question group 1, the within-group change (post-test – pretest) based on the paired samples t-test did not increase significantly in the control group ($M = .3667$, $SD = .6481$, $t = 1.697$, $p = .141$), but changed significantly in the treatment group ($M = .425$, $SD = .6415$, $t = 3.246$, $p = .002$). The analysis of the data for question group 1 also shows that the treatment group ($M = 3.467$) preferred combinatorics at the beginning of the course more than the control group ($M = 3.056$), and when asked directly about combinatorics, both means increased (treatment 3.79, control 3.11) and the difference also changed significantly.

The paired samples t-test showed that for question group 3 on the student's attitude the within-group change (post-test – pretest) in the control group ($M = -0.3056$, $SD = .6468$, $t = -1.417$, $p = .097$) did not change significantly, but changed significantly in the treatment group ($M = -0.2188$, $SD = .4503$, $t = -2.380$, $p = .013$). When testing for preference for combinatorics in the second part of the attitude test, the paired samples t-test showed that the within-group change (post-test – pretest) did not significantly change in the control group ($M = .333$, $SD = .866$, $t = 1.115$, $p = .141$) but significantly increased in the treatment group ($M = .375$, $SD = .711$, $t = 2.584$, $p = .008$).

Experiment 2

With the independent samples t tests, the Levene's test revealed no significant difference in pretest scores (sig. 0.386, $t = .102$, $df = 39$, $p = .459$) between the control and treatment groups. The paired samples t test showed that the decrement within the control group (post-test – pretest) was significant ($M = -7.90$, $SD = 7.156$, $t = -3.491$, $p = .003$) compared to that of the treatment group ($M = -1.129$, $SD = 12.795$, $t = -0.491$, $p = .313$).

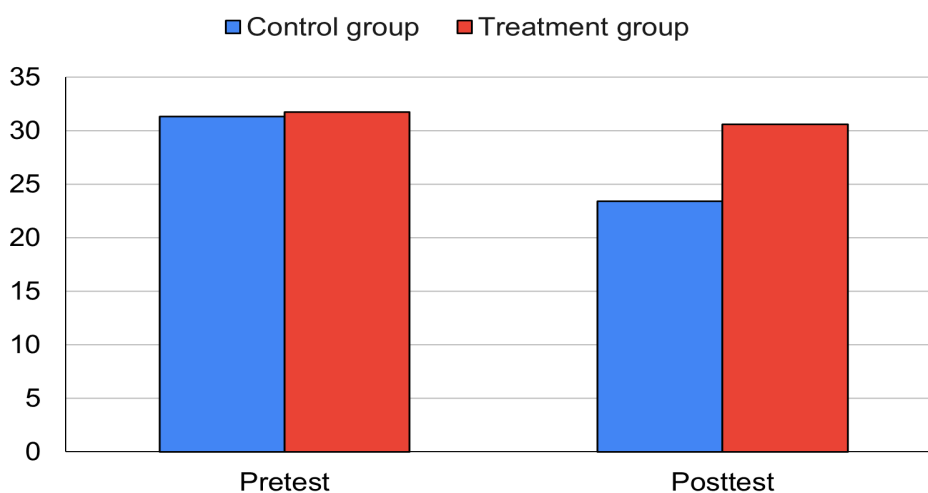


Figure 1: Mean scores of pretest and post-test of Experiment 2

Discussion, limitations and avenues for future research

Based on the results, in relation to RQ1, we can conclude that there is a positive effect on the attitude of the prospective teachers of regular, thorough error analysis. A further positive finding was the

significant decrease in question group 3, i.e. that as a result of the error analysis, teachers perceived less that the combinatorics was about learning and applying combinatorial formulas. With regard to RQ2, we can conclude that the treatment group performed significantly better on the post-test compared to the control group. Thus, the response here is also positive. From the results, we can conclude that the method (using error analysis in teaching) seems to be effective concerning both research questions.

As lecturers, we clearly noticed the change in the attitude of the students during the semester. In the first half of the course, many prospective mathematics teachers did not or rarely took a position regarding a Winnie the Pooh solution, and they easily made mistakes in solving the other exercises. Over time, more and more prospective mathematics teachers discovered the errors in Winnie the Pooh's reasoning, and their thinking became clearer, and they made mistakes more rarely when solving problems independently. Over time, they got used to the unusual situation (they rarely get an error analysis problem in other courses) and were happy to analyse Winnie the Pooh's arguments, because they realised that it would be very useful during their teaching careers. They have repeatedly expressed that it also helps them solve their own tasks, because they recognize typical incorrect solutions themselves.

Our study has limitations. Unfortunately, institutional constraints did not allow for all courses to be delivered by one instructor. Even if we have tried to avoid significant differences, it would be worthwhile to investigate in a future study (preferably with larger and nearly equal groups) how the results vary with the same instructor.

In this stage of the shape of Hungarian mathematics teacher training, we examined (only specifically in the field of combinatorics) the effect of the analysis of incorrectly worked sample solutions on prospective mathematics teachers. The obtained results - as a good starting point - are sufficient to gain a deeper understanding of the factors that lead to the research results.

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