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Experimental epistemology of mathematics revisited, in the light of inclusive materialism and agential realism

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We revisit Brousseau’s experimental epistemology of mathematics in the light of the recent materialist and relational turn in mathematics education, initiated by de Freitas and Sinclair. We discuss the role of metaphorising in this context, from an enactivist perspective, contrasting with Nemirovsky’s phenomenology of things. We present and discuss two paradigmatic examples, from our classrooms and from Brousseau’s work, related to the main themes of inclusive materialism and agential realism. We end by pointing out some open questions and suggestions for further research.

Keywords: Experimental epistemology, embodied cognition, inclusive materialism, agential realism, metaphor.

Introduction.

In this article we revisit the “experimental epistemology of mathematics”, as conceived by Brousseau in the sixties (Kidron et al. 2008; Dorier 2017), in light of the current materialist and relational turn in research in mathematics education. Recall that Brousseau’s original somewhat provocative terminology (especially in the French tradition) was later abandoned in favour of the more mainstream and now prevailing denomination “didactics of mathematics” (Brousseau, 2002). He encoded some seeds of his main ideas in his tantalising and cryptic booklet “Les mathématiques du cours préparatoire” (Brousseau, 1965), supposedly a textbook for primary school (grade 1), but never employed in the classroom (Guy Brousseau, personal communication, January 8, 2023)¹. We unpack below some of his pioneering insights, several of which resonate with current landmark developments in inclusive materialism, agential realism and the phenomenology of things in the learning and nature of mathematical concepts (de Freitas & Sinclair, 2014; Sinclair, 2021; Nemirovsky, in press). We elaborate on Brousseau’s experimental epistemology of mathematics and his theory of a-didactical situations, with links to metaphorising, and discuss two illustrative examples of mathematical activity related to de Freitas & Sinclair (2014, 2016)), raising open questions for further research.

An experimental epistemology of mathematics?

Brousseau introduced the term “experimental epistemology of mathematics” in the late 1960s (Kidron et al. 2008; Dorier 2017), while Maturana and Varela (1980) were founding their Experimental Epistemology Lab at the University of Chile. The term “*experimental epistemology*” was especially disruptive in mathematics education, but also in biology. Indeed, while in English, epistemology addresses the question: “how do we know?”, in French it refers to the study of the historical development and unfolding of ideas; so for French philosophers it is hard to imagine how we could carry out *experimental* research on past events. For Maturana and Varela, experimental epistemology

¹ Notice that this booklet was published by Dunod, a well-known French *scientific* publisher, not a textbook publisher.

involved for instance the study of avian vision (e.g. birds' chromatic space); for Brousseau, it involved studying the conditions under which a designed embodied *situation* could trigger the construction of a mathematical concept by a class of students working autonomously and collaboratively, i.e. the genesis and unfolding of mathematical ideas, here and now in a classroom. At the time Brousseau was strongly motivated by his awareness of the cognitive abuse entailed by the traditional, top-down, punitive, and segregationist way mathematics was taught at French primary school (G. Brousseau, personal communication, November 11, 2021). This kind of cognitive abuse, also dubbed *cognitive bullying*, has been recently acknowledged in the English literature (Watson, 2021). Brousseau planned his tantalising 1st grade mathematics booklet (Brousseau, 1965) as an antidote to this cognitive abuse, based on early mathematical *relational* and *diagrammatic* thinking, and “lessons without words”, making intensive use of arrows, which often stand for material movement. The booklet is however cryptic and includes some deliberate incoherencies, hoaxes² and examples of “what not to do”, which show how it was not intended as a manual for teachers, but rather as a series of proposals and open questions for researchers (G. Brousseau, personal communication, November 11, 2021). Brousseau, inspired by Sartre’s “Existence takes precedence over essence”, intended to make mathematical objects exist, before defining them or trying to teach their essence, so that their “raison d’être” becomes pellucid for the students. His approach is akin to focusing on “what the concept is doing, that is, on the story in which it is involved” (de Freitas & Sinclair (2014: p. 216). This was a starting point for his experimental epistemology of mathematics, in which he designed and dubbed situations *a-didactic*, because the teacher’s didactic intentions were to remain implicit: that is, she steps aside to let the students explore and discover on their own. A classic example was his fundamental a-didactic situation for the construction of numbers as equipotency classes: the well-known *cups and spoons problem* for 5-year olds (Brousseau, 2002). Nemirovsky (in press) strikes the same chord when he writes that “the understanding of a mathematical concept rather than having a definitional essence, spans diverse perceptuo-motor activities which become more or less significant depending on the circumstances [...] Learning a different approach for what appears to be the “same” idea, far from being redundant, often calls for recruiting entirely different perceptuo-motor resources.” Brousseau favoured a strongly constructivist and social approach to mathematics learning, stretching along various scales, inspired by the work of the pioneering Soviet Ukrainian educator Makarenko (1951). In contrast, notice that cognitive abuse is typically inflicted on pupils who are not allowed to interact with each other and are *individually* assessed and punished. The collective dynamics launched in a classroom à la Brousseau, could be metaphorised as an *egregore*, including not just the thoughts and feelings of the participants, but also their movement, gesturing, diagrams ... The *egregore* metaphor also echoes the circulating worldly streams of subjectivity and “affects” in mathematical making, in the sense of Nemirovsky (in press).

Metaphorising

With reference to experimental epistemology, we are interested in *metaphorising*, that is, in the *action* of “looking” at something and “seeing” something else (metaphorically speaking). We query the

²See for instance the exercise involving the fishbowls on page 3.

extent to which metaphor emerges in the (Ouroboric³) intra-action of a cognitive subject with “something”, rather than *metaphor* as that which over-codes the feigned independence of *object* and observer. We see metaphorising as *controlled hallucinating* in the sense of Seth (2021). Metaphors are not just rhetorical devices, but powerful cognitive mechanisms that enable us to construct new concepts. They are ubiquitous in mathematics (Manin, 2007; Thurston, 1994;) and mathematics education (Abrahamson et al., 2012; English, 1997; Sfard, 2009; Soto-Andrade, 2018, 2020). Lakoff and Núñez (2000) metaphorise (*conceptual*) *metaphors* as inference-preserving *mappings* from a more concrete source domain into a more abstract target domain, which enable us to fathom the latter in terms of the former. Nemirovsky (in press) points out that this is an “atomistic” view which sees metaphors as “building blocks” of our cognition. He proposes instead to attend to the dynamics of affects as they circulate in a certain region of the world inhabited, ephemerally, by living beings, akin to Ingold’s (2007) emphasis on lines instead of nodes in a mesh. In our enactivist view, metaphorising is just “the tip of the iceberg” of human cognition, a fundamental *biological process* related to structural coupling (Maturana & Varela, 1980; Varela et al., 1991). It arises in the learning of mathematics when trying to cope with open-ended situations (Brousseau, 2022) or to fathom emerging mathematical notions. We are interested in the *poietic* role of metaphor, as *generator* of mathematical concepts and theories, not as a representation of given mathematical concepts. For instance, Darwin’s *theory of evolution* functions as an unfolding of the “tree of life” metaphor. “overcoding” (Sinclair & de Freitas, 2019). The challenge is to seek the generativity of metaphors in mathematical discourse. Indulging in some kind of “overcoding” (Sinclair & de Freitas, 2019), we highlight below some metaphors in de Freitas & Sinclair (2014).

Besides “Multiplication as dilation” or as “change of units” (de Freitas & Sinclair, 2014, p. 222), we have the embodied metaphor “multiplication is shadow casting” (Dimmel et al., 2021) and “multiplication is area”, which triggers Grassmann’s invention of the (exterior) product of two vectors. We also have “multiplication is concatenation” (Soto-Andrade, 2020), an unfolding metaphor for *category theory*, a most relevant development of post-Bourbaki *relational mathematics* where “The relations are primary, not the individuals” (Freitas & Sinclair, 2014, pp. 210-211; Rotman 2012). Brousseau (1965) complies, when he starts with relations and correspondences embodied by arrows, instead of number recitation or the basic 2D figures, as in traditional curricula. At another scale of learning, we agree with the authors that the “learning trajectory” metaphor (de Freitas and Sinclair, 2014: p. 215) should be replaced by the metaphor of learning as a biased random walk, a notion quite close to Ingold’s (2007) wayfaring. Recall also Brousseau’s “kid’s dream” whereby concepts are attractor basins for the students’ random walk in an a-didactic situation.

Illustrative examples of generative metaphors

The frog’s random walk: a metaphorical construction of probability and beyond...

Our humanities university students were exposed during their first semester mathematics course to *the random walk of a “fair frog”* jumping equally likely right or left on a row of stones in a pond.

³ Ouroboros : the snake eating its own tail, an ancient metaphor for circularity and self-reference (cf. Maturana & Varela, 1980, the cover of whose 1972 Chilean edition depicts an Ouroboros)

Trying to answer the emerging “impossible question”: *Where will the frog be after n jumps?* (given its starting stone) some students “hallucinated” (Seth, 2021) and saw the frog splitting into two equal halves going right and left (a “*Solomonic metaphor*”); others saw the frog becoming double (a “*Borgesian metaphor*”⁴), one frog jumping right, and the other left, as in two parallel universes we are seeing at the same time (Soto-Andrade, Diaz-Rojas & Reyes-Santander, 2018). Their “metaphoric sleight of hand” turned the frog’s random walk into a deterministic splitting process, where they just needed to keep track of the frog’s splitting into pieces, or doubling, to answer the “impossible question”. Indeed, the probability of finding the frog at a given stone after n jumps is constructed either as the frog’s portion landing there after n splittings or as the ratio of Doppelgänger frogs crouching there, after n doublings (Soto-Andrade, 2018). Interestingly, our humanities students excelled in this enactive metaphorical setting as much as the students majoring in mathematics.

Working with in-service primary teachers and with dancers and dancer students in an open dance & math workshop, we enacted a choreographic version of the Borgesian metaphor, for a 3-jump random walk (Soto-Andrade & Shulman, 2021). For a group of eight, we posed the “choreographic challenge” that they each enact one of the eight possible 3-jump paths of the frog: an embodiment of the “unfolding” (we could even say “blossoming”) of the frog’s random walk. They developed two different procedures. First, a hierarchical one: to elect a *Magister Ludi* who would hand out a different script to each dancer. Second, a non-hierarchical and intercorporeal one: the dancers stand on a row one behind another, at the starting stone, as if they were just one frog. Then they split evenly into two groups (of four), then each group of four does the same, etc. They end up distributed according to this pattern 1,0,3,0,3,0,1, on the stones, where the central 0 is the number of people on the *starting* stone, after having each enacted, one of the eight possible 3-jump paths of the frog.

A second problem was posed to mathematics-physics secondary teachers, the analogous problem of the “same” frog jumping now on a (regular) polygon of stones. For 3 stones, calculating a few iterated splittings they became convinced that the frog will be equi-distributed in the long run. But they also found a more powerful method, working in random groups of four, to wit: decomposing the *initial state* of the frog (which is for sure squatting on some stone) as the superposition of its stationary state ($\frac{1}{3}$ at each stone) plus the “remainder”. Then, to compute the state of the frog after n jumps, they just let both components “jump” and superposed the resulting states (they noticed that the transition from one frog’s state to the next one, is linear). The stationary state remains put; the remainder changes under “jumping”, but transparently: each jump amounts to multiplying the previous state by $(-\frac{1}{2})$. So, in the long run, the contribution of the remainder tends to 0; so that the frog tends to the equi-distributed “thermal death” state. We encouraged students who did not feel at ease with splitting jumping frogs to metaphorise the process otherwise. One friendlier metaphor for many students was a polygon of friends, sitting at a round table, sharing one litre grapefruit juice.

The harder case of five or more stones needs a clever way to solve it, which connects neatly with the comments of de Freitas and Sinclair (2014) on the nature of 0 in mathematics, drawing on Châtelet. Indeed, 0 is not just what arises after eating the last cookie⁵, but it is also a starting or bifurcation

⁴ after the well-known Argentinian philosophical and fantasy short story writer Jorge Luis Borges.

⁵ which is nonetheless more dynamic and suggestive than the dry definition: “zero is the neutral element for addition”.

point full of vibrant potentialities. Indeed, 0 is not just the neutralisation of A by $-A$ (ibid. p. 55), but also the resultant (in the sense of d'Alembert) of two “identical” pulls pointing in opposite directions. Going further than Châtelet, with an all-round view, we can fathom 0 as the resultant of m identical pulls pointing in divergent, albeit symmetrically distributed directions. This is embodied in the complex plane by the terse algebraic fact that the *sum of all m -th complex roots of unity vanishes*. This material behaviour of 0 is crucial to solve the frog’s random walk on a polygon (e.g., a pentagon with vertices named 1, 2, 3, 4, 5). The *states* of the frog are given by “iconic vectors” (a pentagon with non-negative real weights at each vertex, which add up to 1). The initial state (where the frog crouches at vertex 1), can be decomposed as the superposition of the uniform state ($1/5$ at each vertex) plus the *complex states* given by the four fifth complex roots of unity different from 1. Indeed, for each fifth root of unity w we have the state S_w given by the complex weight $w^k/5$ at vertex k . Notice that S_1 is the uniform state and that we must get 0 at every vertex different from 1 when we superpose our five complex states. This holds because 0 is the resultant of the five fifth roots of unity! When each “complex frog” S_w jumps, its “fate” is pellucid: each jump amounts to multiplying its complex 5-dimensional iconic state vector by a scale factor equal to the real part of w . This allows the students to easily solve the frog’s random walk, or the iterated grapefruit juice sharing.

Notice that to solve a real problem with a real solution we “escape” to the complex realm, work there, and then come back to the real realm. Indeed, the initial state of the real frog (given by a real iconic vector) whose jumping fate is opaque, is cleverly decomposed (thanks to 0 “unfolding” into the fifth roots of unity) as a superposition of ghost complex frogs, with a pellucid fate. We see that in Argand’s plane, “zero evokes new and as yet unscripted directions – new branches of mobility that might invent alternative symmetries of choice.” (de Freitas and Sinclair, p. 55). This is a baby example of *spectral analysis* (decomposing the frog into ghost frogs) and *spectral synthesis* (putting them back together after n leaps to get the n -th state of the frog). The concept of probability and spectral theory are seen to emerge from the entangled student – random walk – enacting story, as in the perspective of agential realism and inclusive materialism.

Vectors in four dimensions in Grade 1?

Returning to Brousseau (1965) and his grade one students, we find some common ground with new materialism; although he did not focus on 0, he introduces a thought experiment about “iconic vectors” (“gastronomic vectors”) in 3 and 4 dimensions and their addition (p.57). In his “existentialist” approach, he neither names nor defines vectors; they arise as the three clowns choose what treats to bring for the elephant: x peanut packages, y bananas, z sugar cubes (one clown brings 0 bananas). To answer the question: *What did the elephant eat, altogether?* we need to add the clowns’ treats in a “natural way” or “vernacular” way⁶. Accordingly, he also puts together the assets of the three clowns, given (at the time) by coins of 1, 2, 5, 10 francs. His embodied and material approach bypasses the quandary many teachers meet when teaching linear algebra, first geometrically in 2D and 3D, then going to abstract linear algebra in n dimensions. This leap baffles many students who

⁶ Brousseau would say “in a vernacular way” (<http://guy-brousseau.com/2950/premiere-mathematique-1972/>): “do not add (or compare) apples and oranges”, vernacular wisdom says.

missed the opportunity of playing around with economic, gastronomic, or iconic vectors in several dimensions (e.g., purchases at the supermarket). From the perspective of agential realism, we have an iconic situation with an entanglement of students, material objects and a story, creating the concept of vector addition through metaphorizing. Also, the multiplicity (existence) versus the essence of the *concept of vector* is emphasized (de Freitas & Sinclair, 2016).

Discussion and open questions

We have observed that a (visual or embodied) *poietic* metaphor creates a concept or a theory, like the splitting or sharing metaphor for a random walk, which *creates* the notion of probability. Such splitting can be enacted, unlike a container metaphor meant to depict the same mathematical concept. Our use of metaphorising is akin to the use of a diagram to create a new mathematical concept (Sinclair, 2021). De Freitas and Sinclair (2014, p. 224) write: “Abstraction strips mathematical entities of their physical emergence and pretends that they have no experiential history”. Brousseau (1965) avoids this “pitfall of abstraction”, which is a major source of cognitive bullying in the teaching of mathematics (Watson 2021). We claim moreover that abstraction is a “pious lie”. Indeed, when we pretend to think abstractly, most of the time we are unwittingly activating material metaphors. Neuroscientific evidence by Dehaene and collaborators (e.g. Knops et al., 2009) support this. The computer screen interface metaphor might be pertinent here: our neat and clear-cut abstract mathematical objects (à la Bourbaki) are as much of an hallucination as the neat and clear-cut objects we see and manipulate in our computer screen. We are very good at telling ourselves stories, like the Brownian particle which tells us, when we interview it, that it sees a teleological direction in its random path backwards (Fried, 2001). Such storying is a distinctive capacity of humans, among all living beings! We share in this regard the metaphor of de Freitas and Sinclair (2014), of well known “universal” geometric figures, and “revealed” theorems, being “cut out” from a “quivering” *primaeval* mess.

We have shown that several pioneering ideas of Brousseau in the 1960s resonate with contemporary ideas on inclusive materialism and agential realism: his approach to early mathematics with relations and arrows; his embodied and material intent to make mathematical concepts emerge before naming them, before addressing their “essence” or defining them à la Bourbaki, He does not seem to question the ontological status of mathematical objects, but he sees them from an interactive and relational perspective. Recall that his ideas were tested and implemented in closely controlled experimental conditions for 25 years (1973-1998) at the COREM⁷. We remark that when the teacher enacts something without words (Brousseau, 1965, pp. 2-4), a flow of perplexity (Nemirovsky, in press) circulates first across the collection of pupils, then at the individual scale some students seem to understand something, and this propagates to the whole class, as in vernacular learning. In Brousseau’s experimental epistemology of mathematics, we find embodied, enactive and extended cognition too, as in the collective exploration, where student’s bodies interact with objects or concrete systems, implemented in his famous a-didactic situations for inferential statistics and proportionality (Brousseau, 2002). Students are not depicting or enacting their mental processes, their thinking *is* their collective acting, manipulating and bodily exploring. We appreciate “scales of mattering” in

⁷ Centre pour l’Observation et la Recherche sur l’Enseignement des Mathématiques, in Talence, France.

these examples: individual and collective explorations in the classroom, not just pilot experiments in a laboratory setting, but also the influence of culturally shared metaphors and their enactive encountering. Several participants in our mathematics and dance workshop, voiced their feeling of having for the first-time experienced mathematics as accessible, embodied and enjoyable, where body knowledge was also acknowledged: a democratised access to mathematics! In terms of mattering scales (Sinclair & de Freitas, 2019), workshops began with a dozen disconnected individual stories regarding mathematics, which became entangled (Ouroboric relations) in a shared experience of group coordinated movements, so that a sort of material-affective-mathematical embodied assemblage arose (Sinclair and de Freitas, 2019; Chronaki, 2019).

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References

- Abrahamson, D., Dutton, E. & Bakker, A. (2021) Towards an enactivist mathematics pedagogy. In S. A. Stolz (Ed.), *The body, embodiment, and education: An interdisciplinary approach* (pp. 156–182). Routledge.
- Brousseau, G. (1965). *Les Mathématiques du cours préparatoire. [Preparatory Mathematics.]* Dunod.
- Brousseau, G. (2002). *Theory of didactical situations in mathematics*. Kluwer
- Chronak, A. (2019). Affective bodying of mathematics, children and difference: choreographing ‘sad affects’ as affirmative politics in early mathematics teacher education. *ZDM*, 51, 319–330
- de Freitas, E. & Sinclair, N. (2014). *Mathematics and the body*. Cambridge University Press.
- de Freitas, E. & Sinclair, N. (2016). Concepts as generative devices. In E. de Freitas, N. Sinclair & A.Coles (Eds.), *What is a mathematical concept?* (pp. 76–89). Cambridge University Press.
- Dimmel, J., Pandiscio, E. & Candem, G. (2021) Multiplication by sunlight. *Journal of Mathematics Education*, 12(1), 9–16.
- Dorier, J.L. (2017) La didactique des mathématiques: une épistémologie expérimentale? [Mathematics teaching: an experimental epistemology?] In M. Bächtold, V. Durand-Guerrier & V. Munier (Eds.), *Epistémologie et Didactique* (pp. 33–44). PUFC.
- English L. (Ed.) (1997). *Mathematical reasoning: Analogies, metaphors, and images*. Lawrence Erlbaum
- Fried, M. (2001). Can mathematics education and history of mathematics coexist? *Science & Education*, 10(4), 391–408. <https://doi.org/10.1023/A:1011205014608>
- Hutto, D. & Abrahamson, D. (2022) Embodied, enactive education: Conservative versus radical approaches. In S L. Macrine, & J. M. B. Fugate (Eds.), *Movement matters* (pp. 39–52). MIT Press.
- Ingold, T. (2007). *Lines: A brief history*. Routledge.

- Kidron, I., Lenfant, A., Bikner-Ahsbabs, A., Artigue, M., & Dreyfus, T. (2008). Toward networking three theoretical approaches. *ZDM – Mathematics Education*, 40(2), 247–264
- Knops, A., Thirion, B., Hubbard, E., Michel, V., Dehaene, S. (2009) Recruitment of an area involved in eye movements during mental arithmetic. *Science* 324(5934), 1583–1585.
- Lakoff, G. & Núñez, R. E. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. Basic Books.
- Makarenko, A. S. (1951). *The road to life: An epic of education*. Progress Publishers.
- Manin, Y. I. (2007). *Mathematics as metaphor: Selected Essays*. American Mathematical Society
- Maturana, H. R. & Varela, F. J. (1980). *Autopoiesis and Cognition*. Springer.
- Nemirovsky, R. (in press). Bodies, Incorporeals, and the Birth of a Mathematical Diagram. In L. D. Edwards & C. M. Krause (Eds.), *The body in mathematics*. Brill.
- Rotman, B. (2012). Topology, algebra, diagrams. *Theory, Culture, Society*, 29, 247–260.
- Seth, A. (2021). *Being you: A new science of consciousness*. Faber.
- Sfard, A. (2009) Metaphors in education. In H. Daniels, H. Lauder H. & J. Porter (Eds.), *Educational theories, cultures and learning: A critical perspective*. (pp. 39–50). Routledge,
- Sinclair, N. (2021, July 11–18). Agential Realism [Presentation]. *ICME 14*, Shanghai, China.
- Sinclair, N., & de Freitas, E. & (2019). Body studies in mathematics education: diverse scales of mattering. *ZDM – Mathematics Education*, 51, 227–237. <https://doi.org/10.1007/s11858-019-01052-w>
- Soto-Andrade, J. (2018) Enactive metaphorising in the learning of mathematics. In G. Kaiser, H. Forgasz, M. Graven, A. Kuzniak, E. Simmt & B. Xu (Eds.), *Invited lectures from the 13th International Congress on Mathematical Education (ICME 13)* (pp. 619–637). Springer.
- Soto-Andrade, J. (2020) Metaphors in mathematics education. In S. Lerman S. (Ed.), *Encyclopedia of mathematics education*. Second Edition (pp. 447–453). Springer.
- Soto-Andrade, J., Díaz-Rojas, D. & Reyes-Santander, P. (2018). Random walks in the didactics of probability. In C. Batanero & E. Chernoff (Eds.), *Teaching and learning stochastics* (pp. 125–143). *Advances in Probability Education Research, ICME 13 Monographs*. Springer.
- Soto-Andrade, J., Shulman, A. (2021). A random walk in stochastic dance. *Link2021 Interdisciplinary Symposium on Art and Design*. <https://doi.org/10.24135/link2021.v2i1.71>
- Thurston, W. P. (1994). On proof and progress in mathematics. *Bull. Amer. Math. Soc.* 30(2), 161–177.
- Valdés-Zorrilla, A., Diaz-Rojas, D., Jiménez, L., Soto-Andrade, J. (2023). Random Walks as a Royal Road to E-STEAM in Math Education, *Constructivist Foundations*, 18(2), 259–294.
- Varela, F. J., Thompson, E. & Rosch, E. (1991). *The embodied mind*. MIT Press.
- Watson, A. (2021). *Care in mathematics education*. Palgrave Macmillan.