Spatial imaginaries and spherology: Exploring the mathematical shape of space

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We inherit from centuries of geometric thought, colonial conquest, imperial englobing of the earth, and stargazing, a tendency to imagine worlds as spheres. The sphere seems to be the very ideal of ideas and it appeals to our desire for perfect symmetry. Does this tendency in spherical thinking inhibit our ability to imagine new modes of connectivity, proximity, dependency relations, border crossing and mobility? This paper situates the embodied mathematical imagination within larger cross-cultural habits of spherical thinking, extending Peter Sloterdijk's project of cultural spherology. I discuss the figure of the hypersphere and spherical thinking in higher dimensions. Tactics for visualizing the hypersphere are discussed - analogical thinking, boundary gluing, contracting infinity, projection, rotation, looping and smoothing. These material practices of assembling spherical forms are both mathematical and embodied, and link mathematical and cultural imaginaries.

Keywords: Imagination, geometry, sphere, modelling, hyperspace.

The shape of space

Spherical modeling is a cross-cultural tendency and wide-spread practice in spatial imagination, characterized by Peter Sloterdijk (2011, 2014, 2016) in terms of *spherology*. We inherit from centuries of geometric thought, colonial conquest, imperial englobing of the earth, and star-gazing, a tendency to imagine worlds as spheres.[1] Spheres are a dominant spatial imaginary, a shape which we often impose on dark uncharted territories which we cannot navigate. For Sloterdijk, this habit also reflects our psychological tendency to imagine ourselves at the center of a sphere – perhaps a sphere of influence, of desire, of family, etc. Our very embodiment and perceptual physiology fuels our tendency to frame everything in terms of spherical englobing. But spheres accommodate distinctive spatial practices related to distinctive geometries, and therefore this tendency influences the way we experience proximity, situatedness, dependency relations, connectivity, borders, and mobility. These spatial concepts are increasingly in question, as we navigate new digital proximities in current computational cultures and witness new kinds of geo-spatial climate in/justice.

As a geometric concept, the sphere itself has constant curvature, and *performs* rounded, continuous, complete, perfectly symmetric and *traversable knowledge*. The ideal form of a sphere appeals to our desire for perfect symmetry. According to Latour (2017), spheres appeal to idealists because "a sphere has no history, no beginning, no end, no holes, no discontinuities of any sort. It is not merely an idea but the very ideal of ideas." (p. 136). These smooth qualities of "the very

[1] Although this paper stays tuned to Euro-Anglo spherology as a dominant paradigm, spherical cosmology can also be found in ancient Sanskrit models of the Brahamanda egg composed of 21 spheres, among other traditions.

ideal of ideas" mask the actual material practices of assembling spherical forms, practices which are both mathematical and embodied, and which link mathematical and cultural imaginaries (Nemirovsky, 2017). This paper explores those linked imaginaries, by focusing on mathematical visualizations of four-dimensional spheres. I discuss tactics for exploring familiar mathematical forms - like the sphere - in higher dimensions. These tactics are analogical thinking, boundary gluing, contracting infinity, projection, rotation, looping, and smoothing. By looking closely at such tactics, we can see how the geometric imagination is linked to spherical world making.

Hyperspheres

The term hypersphere was introduced in 1914 by the mathematician Duncan Sommerville, in his text *The elements of non-Euclidean geometry*, to describe a generalized sphere in higher dimensions. The generalized concept of the sphere is a set of n-tuples that are equidistant from a given point in higher (or lower) dimensions. A 3-sphere is a kind of hypersphere that exists in four dimensions. The general notion of a unit sphere can be captured algebraically to describe the 3-sphere, using the set of real numbers, extended to the case of four variables: $x^2 + y^2 + z^2 + w^2 = 1$. One can also characterize this object using the set of complex numbers, as: $w \cdot \bar{w} + v \cdot \bar{v} = 1$, where *w* and *v* are complex numbers. One can also represent the 3-sphere using quaternions, which are an expanded version of complex numbers from two dimensions to four dimensions, using the notation $i^2 = -1, j^2 = -1, k^2 = -1$ with non-commutative multiplication, where the 3-sphere becomes: $S^3 = \{ q: |q| = \sqrt{\bar{q}\, q} = 1\}$ and $q = a + bi + cj + dk$.

These algebraic equations are useful, but they mask some of the peculiarities of the 3-sphere. In 1985 mathematician Jeffrey Weeks (2020/1985) published a wonderful book called *The shape of space*. Intended for a wide audience, the book was an informal introduction to topological ways of thinking about space. Weeks aimed to cultivate creative spatial imaginaries while exploring the shape of the universe, playfully diagramming non-standard options and higher dimensions. He focused on the experiential aspects of space and movement, using thought experiments and diagrams to motivate the mathematical investigation of spatial relationships. Chapters explore key topological and metrological ideas, such as connectivity; the distinction between simply and multiply connected manifolds is a powerful conceptual device for investigating the different mobilities that any given space affords. Notably, the sphere is *simply connected* because any loop on the sphere can be contracted to a point. Unlike the sphere, holey spaces don't allow loops to contract to points, and are multiply connected. The difference between a simply connected space and a multiply connected space can be used to explore alternative spatial configurations of our world, different relational ontologies, and new geo-metric mixtures.

In topology, a sphere is a closed or compact manifold with constant curvature, and the metric is immanent to the manifold rather than imposed from the ambient space. The 3-sphere is a compact, connected three-dimensional manifold without boundary, existing in four-dimensional space. The challenge is to imagine the spatial qualities of such an object as being somehow 'sphere-like'. Topologists distinguish between balls and spheres, where sphere refers only to the surface boundary of the ball (one is in some sense 3-dimensional, the other 2-dimensional). Can we use analogy, and say something similar about 3-spheres around 4-dimensional balls? One way of

imagining how one might construct a 3-sphere involves 'gluing' together (mapping) the surface of two different spherical 3-dimensional balls, point-by-point (Richeson, 2008). The 3-sphere would then include the two 3-dimensional balls, but with their boundary surfaces glued together. In gluing the boundaries together, we are effectively merging the two balls, and undoing what previously separated the space into separate regions.

How can one glue the surface of one ball to the surface of another, so that all the points on one have their match on the other, while conserving continuity? The intuition meets its match at the impossibility of the *infinite* stretch that is needed to complete that gluing. A fourth dimension is needed to achieve this effect. But how might we come to know such a thing? How do we sense or imagine the 3-sphere as an embodied object? We can use color to help visualize this object, where a green-white-blue transition characterizes movement along a given axis, and away from the center of the two merged balls. The gluing then maps one white surface of ball to the other white surface. When imagining oneself inside such a space, colors would modulate as one moved towards and away from white, never reaching it. Along with modulations of color, we can also use motion to explore higher dimensions.

Approaches to cultivating mathematical imaginations through animating higher-dimensional objects have a century-long history. There are three traditional tactics for doing so. Following Abbott (2002/1884) and Hinton (1980/1880), as well as Burger (1957), the process of cultivating spatial imaginaries in higher dimensions involves (1) generalizing inductively from one, two, and three dimensions, onward into higher ones; (2) imagining a light source and projecting shadows of an n-dimensional object onto an n-1 dimensional space, and (3) tracking the impact or trace of n-dimensional objects as they move through n-1 dimensional space. The latter reliance on motion to make sense of dimension is essential for Abbott and Hinton: they imagine an event where an observer witnesses the trace of a higher-dimensional object as it passes through their perceptual world. For instance, two-dimensional beings living on a two-dimensional surface witness a 2-sphere ($x^2 + y^2 + z^2 = 1$) passing through their two-dimensional planar world, seeing a point growing into a line, eventually reaching the diameter of the sphere, and then shrinking as it continues the passage. In fact, Abbott's two-dimensional creatures in Flatland would not "see " the circle, because their vision is confined to the plane, but they could *infer* that there was an object with a circular shape, by either moving around it and feeling it with their hands, noting that they return to their starting place without having felt any corners, or by measuring and noting the *different* distances from a given point on the plane to various points on the circumference traced by the passing sphere.

Using this tactic, we can generalize to four-dimensional space and imagine the trace of a 3-sphere passing through our three-dimensional world; what would appear to us would be a ball growing from a point, until it reaches the "equator" magnitude of the 3-sphere, and then we see it shrinking again, as it passes out of our world. But this is only the trace of the motion of the 3-sphere. In this thought experiment, we are again exploring higher dimensional space using motion – in other words, we are studying the mathematical object as part of a dynamic event in order to identify some of its qualities (de Freitas, 2012, 2017). Notably, our methods for exploring higher dimensional space assume that time and movement will be continuous across different dimensions.

Wrapping an infinite world onto a sphere

With the introduction of new media (film, video), we can train the mathematical imagination to do new things. As Steingart (2014) argues, we can think of computer animation "as a powerful tool with which mathematicians train their geometrical imaginations" (p.172). Mathematician Thomas Banchoff used computer graphics and created the first film of a four-dimensional torus in 1974, stating: "We knew what was going to happen, but we didn't know what it would look like . . . People made plaster models in the late nineteenth century. They knew this thing stretches out to infinity . . . but this isn't made out of plaster, it really moves. The first time anyone's ever seen it move. Literally". This use of graphics as a training instrument suggests that a mathematical imagination is not innate, as the word "intuition" might convey, but rather needs to be learned, practiced and cultivated.

American mathematician Clayton Shonkwiler has created a set of fifteen online dynamic visualizations of the hypersphere (Shonkwiler, 2015). Some of these are particularly challenging as "pictures of the hypersphere", including suspension models, open books, and mappings between two distorted tori. Some are more informational than visual, such as the Heegaard diagrams in which the 3-sphere is characterized by a set of closed curves. Here I focus on the more conventional visualization strategies. Arguing from analogy, Shonkwiler first considers how a stereographic projection takes the regular 2-sphere and maps every point on the sphere onto the plane, using the point at the north pole as a fixed point. For all other points on the sphere, one forms a chord with the north pole, which is then extended onto the plane to make the image point. Projection allows us to move back and forth between the plane and the 2-sphere. This works well for every point but the point at the north pole, for which the chord would be parallel to the plane. Reversing the mapping and visualizing how the infinite plane can be mapped to every point on the sphere, we begin to see how the visualization breaks down when points 'at infinity' in all directions must come together somehow and arrive at the north pole. Shonkwiler gestures first with two index fingers pointing off in different directions, and then bringing them together above his head, and then with bent wrists and hands hanging and cupped, to show how the points move towards the north pole, and must occupy the same location, holding one hanging fist above the other: "suddenly you are sort of smooshing everything at the north pole, at the last instant."

Extending this approach to four dimensions, he suggests that one can wrap the three-dimensional world onto or over the hypersphere: "if you want to build the 3-sphere, you take three-dimensional space, add points at infinity, and everything sort of smooshes up in some crazy way …".

Figure 1: Gesturing from top left clockwise of: (a) the hypersphere, (b) within 4-dimensional space, (c) performing the projection, (d) onto the x,y,z space where w=0, (e) the reverse mapping involves a "smooshing" of all the points at infinity onto the north pole.

Alternatively, he states: “3-dimensional space, all of it, is like the hypersphere minus the point at infinity.” Shonkwiler uses gestures and verbs (“smooshing”) to capture the physicality of these shapes, while also working analogically from lower to higher dimensions. Note how Figure 1b captures a gesture where the two hands are asymmetric, and 3-space or perhaps the 3-sphere seems to be held in the left palm, while the right hand hangs above, introducing the fourth dimension. He uses the terms “line” and “plane” when speaking of the projection in four dimensions (Figure 1d), but these terms are used colloquially to assist the analogy, and he performs a quick cutting gesture three times as he says “x,y,z space” to convey the fact that the projection is *onto three dimensional space*.

This approach to imagining higher dimensions involves mapping the entire world onto a sphere. All of infinite three-dimensional space is packaged and encapsulated into the 3-sphere. This is spherology at work! And yet rotations of the projections of the hypersphere reveal all sorts of animated images that bear no resemblance to our intuitions about rotation or the symmetry of these forms. Shonkwiler places the sphere inside a cube, projecting the edges of the cube onto the sphere. The straight edges appear curved on the sphere. These are then stereo projected onto a plane and create a kind of petal effect. He then does the same with the 3-sphere. Rotating around particular planes (x-w for instance), which involves one of our familiar dimensions (x) and the new fourth dimension (w), creates the trace of a movement that is distinctly un-rotation-like. My point is that these rotational movements are crucial in helping us understand how spheres may behave quite differently when moving in and out of four dimensions. This reveals the ways in which are imaginations bring forth conventional spherical expectations, much like flatlanders and spherelanders were challenged (Abbott, 2002/1884; Burger, 1965). Indeed, Shonkwiler (2015) suggests that vision on the 3-sphere entails entirely non-human perception, including the fact that objects moving away from an observer will begin to appear larger rather than smaller as they approach the antipode pole of the 3-sphere.

Because the imagination is pivotal to our spatial mapping of the fourth dimension, we can use this example to see the tendencies and fabulations that fuel spherical thinking more generally: (1) Analogical thinking is crucial (inductively generalizing from lower dimensions, or extrapolating), while both language and gesture carry the weight of this analogy, sometimes in a contradictory manner; (2) Boundary gluing and englobing infinity creates a kind of paradoxical double-bubble; (3) Projecting surface curves allows the mathematician to make visible some of the perspectival structure of the object; (4) Rotation and other movement creates traces of that structure in our three dimensional space. Various smoothing and simplifying devices can also be used, to see whether an object is topologically homeomorphic to the sphere. These belong more to analytic topology, where the tools of analysis are mobilized. To illustrate these in the next section of the paper, I will give a rough picture of Ricci flow differentiations, which are used to smooth manifolds and limit curvature.

The Poincaré conjecture

In the year 2000, the Clay Mathematics Institute set seven millennial problems, and offered a million-dollar prize for solutions. One of those problems was the Poincaré conjecture, initially formulated in 1904 by Henri Poincaré, which states that all compact and simply connected 3-manifolds are topologically equivalent to the 3-sphere. In addition, Poincaré, and others, believed

that the shape of the universe might be that of the 3-sphere. His conjecture was linked to cosmological theories about space-time relations of the universe, many of which are still open questions (Roukema et al, 2008). In some sense, his conjecture dares to affirm the fundamental ontology of the sphere, that being a compact and simply connected shape with constant positive curvature.

For the purposes of this paper, it's notable that Poincaré's topology papers total about 10 out of nearly 500 papers he wrote, many of them concerned with cosmology, geography, physics, and so on (O'Shea, 2007). And it's also important to mention that Poincaré was a key member of the Bureau of Longitude in Paris, and "spearheaded the quest to deliver synchronized time to the world." (O'Shea, 2007, p. 142). Poincaré was intimately involved in the mapping of the Earth through telecommunications, while also rethinking space-time relativity, and writing popular books such as *Science and hypothesis* (1902) which influenced Valery, Picasso and Einstein (Galison, 2004). His contributions to topology were produced at the very moment when global telecommunications was reshaping the earth. I wonder how, in our current digital remapping of the earth, we are similarly involved in collectively making new geometric and spatial models, to capture the new intensive distances and distributed proximities we have instated.

Inroads into proving the Poincaré conjecture occurred in 1960s and 1970s and related claims were proved for cases of the n-sphere when n is larger than 3. But the conjecture remained unsolved in 2000 when the Clay Institute announced the prize. Then in 2003 a paper appeared online at the open access site arxiv.org by mathematician Grigori Perelman, which claimed to have proved a related conjecture. After a few years the mathematics community accepted Perelman's proof, although he never claimed the prize money. The proof uses another important instrument in differential geometry – related to theoretical models for heat diffusion – that being the *Ricci flow*. It is this operation or tactic – the Ricci flow – that is used so effectively in Perelman's proof. Ricci flow is a process that takes a complex folded and highly curvaceous manifold, and begins to smooth it out, in small local ways, operating at each point on the manifold. The Ricci flow is a process of smoothing and in some sense *forcing* the manifold to become, in this case, sphere-like. "Ricci flow is an excellent tool for simplifying the structure of a manifold - generally speaking, it compresses all the positive-curvature parts of the manifold into nothingness, while expanding the negative-curvature parts of the manifold until they become very homogeneous (the manifold begins to look much the same no matter which one vantage point in the manifold one selects)."[2] (Tao, 2008, p.279)

The Ricci flow was pivotal for proving the famous Poincaré conjecture, where it became another tactic for pursuing spherology within mathematics. This kind of sphere-making takes any asymmetric or distorted and crumply manifold – those shapes that have irregular curvature and problematic singularities – and smooths it, unfolding all the secret hiding places, the creases where oddness lives, making a space that is regular and ready for control and regulation. A smothering

[2] The definition of Ricci flow is $\frac{dg}{dt} = -2Ric \; where \; g \; is \; the \; metric \; on \; the \; manifold$ & Ric is a measure of curvature. See Tao (2008) for a short and intuitive discussion.

sphereology at work again! Obviously, I am not contesting the validity of the mathematics, but drawing attention to tactics that serve a spherical spatial imagination. As long as we are in a finite world that appears simply connected, then we can force this world to become a world of constant curvature.

Conclusion

By reflecting on our technical methods for investigating higher dimensional spheres, we learn about human spatializing tactics and gain insight into our shared investment in spherical thinking writ large, revealing the historical links between topology, geography and cosmology. Sloderdijk, for instance, argues that a particular kind of spherology became dominant when Platonist mathematics and Christian imagery merged to resolve the contradiction between geocentric and theocentric spheres in Western culture. This merging of two spheres created a paradoxical double-bubble, which married ontology to geometry. According to Sloterdijk, Western colonial globalism grows out of this merging of two spheres, in a paradoxical spatial configuration that sounds like the strange gluing tactic discussed above, where the surfaces of two balls are mapped to make the 3-sphere. Indeed Peterson (1980) and Shonkwiler (2015) suggest that there is an affinity between the topology of the 3-sphere and Dante's complex spherical arrangements of the universe.

Empirical measurements of the shape of the universe advanced significantly in the late 20th century, when new methods emerged that traced the patterned residue of the big bang (microwave background radiation) using cosmic crystallography. Today, physical cosmologists distinguish between the observable and non-observable universe and continue to debate the latter's spatial qualities: the key unresolved questions concern boundedness, curvature, and connectivity. The exact shape of the universe and its topology remain unknown – it may be flat, spherical, hyperbolic, dodecahedral, torus-like, or perhaps some other utterly unfamiliar shape. The scientific consensus has shifted over the last decades in this highly speculative area of physico-mathematics.

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