



**HAL**  
open science

## **Actions behind mathematical concepts: A logical-historical analysis**

Lonneke Boels, Rosa Alberto, Anna Shvarts

► **To cite this version:**

Lonneke Boels, Rosa Alberto, Anna Shvarts. Actions behind mathematical concepts: A logical-historical analysis. Thirteenth Congress of the European Society for Research in Mathematics Education (CERME13), Alfréd Rényi Institute of Mathematics; Eötvös Loránd University of Budapest, Jul 2023, Budapest, Hungary. hal-04420753

**HAL Id: hal-04420753**

**<https://hal.science/hal-04420753v1>**

Submitted on 26 Jan 2024

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Actions behind mathematical concepts: A logical-historical analysis

Lonneke Boels<sup>1,2</sup>, Rosa Alberto<sup>1,2</sup> and Anna Shvarts<sup>2</sup>

<sup>1</sup>University of Applied Science Utrecht, the Netherlands; [lonneke.boels@hu.nl](mailto:lonneke.boels@hu.nl)

<sup>2</sup>Utrecht University, the Netherlands

*According to embodied cognition theories, mathematical knowledge is grounded in actions with artefacts. Educational designers are seeking to identify what actions with mathematical artefacts could support the learning of this knowledge. Through logical-historical reconstruction, we answer the question: What actions and artefacts have historically constituted the logic of mathematical concepts' development, and how might these support students' development of mathematical knowledge? We use an example of data and distribution in histograms and trigonometry to illustrate this reconstruction. An answer to our question is that this analysis revealed that we might need to unpack several automatically generated artefacts including dotplots and pre-given intervals (for histograms), and the (unit) circle and radians (for trigonometry).*

*Keywords: Secondary school mathematics, statistics education, embodied cognition and design, histograms, trigonometry.*

## Introduction

The idea of embodiment, namely the involvement of bodily processes in cognitive processes, grounds a wide family of diverse approaches in cognitive and educational science, including mathematics education. Despite the diversity of historical sources and ontological and epistemological presumptions of those approaches, many point to *actions* as the source of knowledge, including mathematical concepts. From an enactivist perspective, cognition emerges from sensory-motor patterns to serve action (Reid, 2014). According to the objectification theory (Radford, 2021) the source of any human development is a practice that is unfolded in society through collaborative actions. From the perspective of ecological dynamics, learning can be captured as moving in an environment in new ways and perceiving new action possibilities (Abrahamson & Sánchez-García, 2016). Even relatively conservative theories of grounded cognition, recently acknowledge the pivotal role of action (Walkington et al., 2022). The attention to actions as constitutive for knowing mathematics raises a fundamental question for educational designers of which actions are relevant for learners to grasp specific mathematical concepts.

## Theoretical framework: mathematics concepts as systems of reified actions

In a platonic view of mathematical concepts as idealised timeless immovable forms, both actions and gestures might be expected to convey *idealised forms* or *abstractions* that are congruent to the concepts (Walkington et al., 2022). However, embodied and material approaches call for reconsidering the nature of mathematical concepts and drawing attention to the material-historical process in mathematical concepts formation (de Freitas et al., 2017). As scholars within socio-cultural approaches elaborate, cultural artefacts crystallise in history and reify specific actions (Radford, 2021; Shvarts & Alberto, 2021). For example, a spoon reifies an action of scooping: the spoon takes the

form most appropriate to facilitate such action. The idea of artefacts as reifying actions and crystallised from them, might be applied to formulas, visuals, and other mathematical inscriptions.

This theoretical position suggests that educators need to focus on the relation between actions-to-be-elicited and mathematical artefacts. However, visualisations and formulas in books and most of the objects in technological learning environments are given as static entities, ready for students' inspection. When these artefacts are presented as *ready-made*, students tend to incorporate them into their actions without questioning their functionality and historically embedded practices with them (Alberto et al., 2019).

We hypothesise that, as educational designers, we need to question the static materiality of artefacts. This implies creating environments that promote such students' actions and provide them with opportunities to reinvent mathematical artefacts. We call this part of the design process *melting artefacts back into actions*. This approach resonates with the ideas of quasi-investigation (Davydov, 1988) and mathematics reinvention (Freudenthal, 1973) that prompt students to constitute mathematics that is meaningful to them through the process of discovery.

To conduct the *melting*, one needs to distinguish which artefacts are critical for understanding a target concept and which actions those artefacts have reified. Assuming mathematics to be a historically formed practice, we turn to the concepts' historical formation (Davydov, 1988) and conduct a *logical-historical analysis*. We suppose that such a “study of the historical development of a concept in relation to the phenomena that led to the genesis of that concept” (Bakker, 2004, p. 7) will support us in distinguishing which actions need to be elicited in students and inform the artefacts' melting in the design process.

We focus on specific mathematical concepts that have been the focus of two PhD studies of the first and second authors: (1) statistical data and distribution and (2) trigonometric functions. In both of these cases, *melting artefacts* was an important design heuristic (Shvarts & Alberto, 2021). We question: *What actions and artefacts have historically constituted the logic of mathematical concepts' development, and how might these support students' development of mathematical knowledge?*

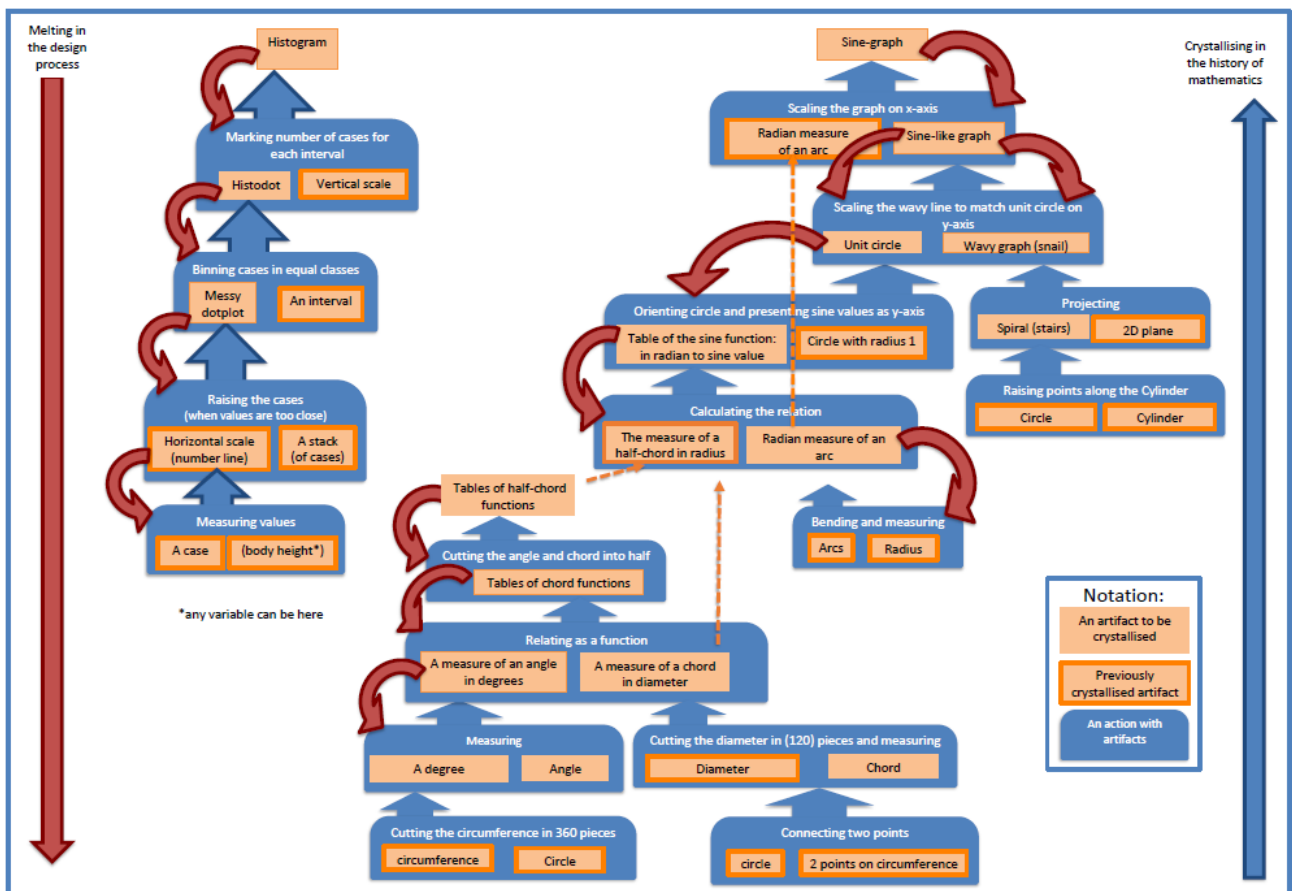
In this conceptual paper, we summarise our previous design attempts and reveal similar problems that students encounter across both topics. We further conduct logical-historical analyses for each topic and suggest design improvements based on this systematic reconstruction of historical development. Finally, we reflect on how logical-historical analysis might inform the practice of educational designers and support the melting of mathematical artefacts to foster their meaningful reinvention.

## **Histograms**

Data and their distribution can be represented in graphs, including histograms. Given the persistence of histograms misinterpretations (e.g., Boels et al., 2019), it is worthwhile to reconstruct the logical history of these concepts embedded in the histogram artefact from a didactical viewpoint (Figure 1). Histograms are graphs of univariate data (e.g., students' body height) in which each bar presents data from multiple cases (students) within a (body height) interval. In a histogram, the area of each bar presents the number of data points in each bar. When bins are equal, the height of each bar presents the number of cases. Paying attention in our design to the actions that could have constituted the

concepts of data and distribution in histograms, led to a successful reinvention of the histogram artefact by students (Boels, 2023). In most statistics education software, horizontal scales and bars' heights (vertical scale) in histograms are pre-given, hence, never questioned by the students.

We now present a summary of our logical-historical reconstruction of histograms. In the 17th and 18th centuries graphs were rare and “overshadowed by statistical tables” (Beniger & Robyn, 1978, p. 2). Bar graphs appeared long before histograms. Although William Playfair is often credited for introducing value bar graphs (time plots) in 1786 (Ioannidis, 2003), the oldest known bar graph is from Oresme (around 1350; e.g., Friendly, 2008). *Histograms*<sup>1</sup> emerged in 1833 when Guerry created graphs of male suicides (Friendly 2008). Nowadays, histograms are used to present large amounts of data (Ioannidis, 2003).



On the left: Logical-historical reconstruction of histogram as a component of data and distribution concept.

On the right: Logical-historical reconstruction of unit circle and sine graph as components of the trigonometric function concept. The direction from bottom to top (blue arrows) represents the logic of historical development: crystallised artefacts and actions that have constituted the artifacts. The direction from top to bottom (red curved arrows) represents the educational designers' work as they design environments that foster actions needed for a reinvention of the artefacts.

**Figure 1: Logical-historical reconstruction of the target mathematical concepts**

<sup>1</sup> In this and the next section, all **actions** are bold and *artefacts* are in italics to facilitate the connection with Figure 1.

*Dotplots* can be considered as intermediate artefacts in the evolution from tables to histograms. Dotplots were already in use in 1884 (Wilkinson, 1999), when data points were **raised above** the *horizontal scale* to visually present their amount, similar to value bar graphs. Pearson described graphs with dots as stigmograms. **Binning** data into classes (or: bins, buckets, intervals) is necessary for both constructing *histograms* and *stacked dotplots* (also called: (number)line plots, Bakker, 2004). Stacked dotplots and histograms share that data are binned into classes that correspond to *intervals* on a numerical scale. First histograms, like the one from Guerry, seem to have had *equal intervals*.

Trying to understand the logical history of the dotplot artefact further, we come to the artefact *horizontal scale* or number line that allow **presenting measured values** using equidistant points for numbers. Although the concept of the number line emerged in the 20<sup>th</sup> century (Sinkevich, 2015), the idea of a coordinate system is much older and an equidistant (Cartesian) coordinate system was reintroduced in mathematics in 1637 by Descartes (e.g., Friendly, 2008). Moreover, equidistant number lines were in use already in 1644 by Langren.

Given our logical-historical analysis, when designing for teaching histograms, we need to start with cases (our data) and scales and have students reinvent histograms from these. First, we asked students to place measured values (of cases) on a *horizontal scale* (**measuring values**). The software automatically generates a *dotplot* by **raising** the cases when values are close to each other (we programmed the tasks in Numworx, see <https://embodieddesign.sites.uu.nl/>). In the next task, students were asked to overlay a dotplot with a histogram by **raising** the bars' heights' (**marking the number of cases for each interval**, the intervals were pre-given). Our logical-historical reconstruction led us to have students question these artefacts. In addition, it made us realise that our design did not make students question the equal intervals artefact (**binning cases**) nor **raise the cases** when these are too close, as these actions were outsourced to the software. It is likely that students also need to reinvent the equal intervals to allow them to understand the idea of presenting data when intervals are unequal in density histograms.

## **Unit circle and sine graphs**

In the natural sciences, surveying, abstract mathematics and other areas, the sine (and other trigonometric) functions and models play(ed) a significant role that influences human life. Trigonometric functions are modelled in various forms such as in the unit circle, triangles or the sine graph which essentially represent the ability to switch between measurement of angle and segment lengths (e.g., Van Brummelen, 2009). Given the widespread use of these functions, it is important that students understand them well.

Previous research on trigonometric functions however shows that despite educators' and publishers' best efforts, students seem to have trouble understanding and remembering these trigonometric concepts. Students may ask, "If the sine of this [90°] angle here is 1, how can a bigger angle be less?" (Presmeg, 2006, p. 27). Our studies on the link between the unit circle and sine graph revealed several challenges when the (measured) models were supplied in ready-made form (e.g., Alberto et al., 2019). That is, while students were familiar with the contour of the sine graph (the wavy pattern), they were not always aware of how this graph came to be - a result of equivalency with the unit circle point's angle and segment height. Students also struggled with the radian-based measurement system used

in the sine function: they seemed to lack conceptual understanding of how to use the radius as a unit of measurement for both angle and segment height and instead used number matching to solve trigonometric equations. These and other student insights inspired us to reconstruct the logical history of the sine function and its model artefacts.

Trigonometric functions (unlike linear functions) do not have an explicit formula for computing the output values for given input parameters, hence a variety of visual artefacts have been created, acted upon and transformed throughout history. As depicted in Figure 1, Hipparchus and his associates were the first to take a step toward trigonometric conversion in the second century BC to explain astronomical phenomena. They used so-called *chord functions* (e.g., Van Brummelen, 2009) which employed *angles* (formed by **intersecting** two lines) and *chords* (formed by **connecting two points on a circle's circumference**). The Greeks borrowed from the Babylonians the measurement system of *degrees*, **cutting** the circle's *circumference* into 360 equal sections and thus measuring the angles in degrees. They also used the *diameter cut* into 120 sections to **measure** the associated *chords*. The angle-to-chord conversions were calculated through various inscribed polygons by **relating** the *angles measured in degrees* and *chords measured in pieces of diameter* and stored in *chord tables* (e.g., Van Brummelen, 2009).

The *tables of chord function* artefact was transformed by Indian astronomers in the 3rd to 5th century AC (e.g., Van Brummelen, 2009). They started to favour *half-chords* in which the angles and chords were **cut in half** (see Figure 1). This resembles our modern sine measurements. A next key insight was that—since the (*half*)*chord function* connects two measurements (an angle and a chord)—it would be preferable to measure them both in the same unit. In their first attempt, they measured both in degrees, meaning that the measurement of half-chords (but not angles) was changed (Ramasubramanian, 2019). However, degrees (or minutes) cannot be utilized as a measurement in other contexts.

Therefore, the *radius* was introduced to **measure** the *angles* instead of a degree measure (*radian measure of an angle*). Although often credited to Roger Cotes in his paper *Logometria*, it was already in use by, for example, Euler. Through **relating** *radians* and *half-chords*, the tables of sine could be built. The use of radians for angular measures advanced calculus for example by easing derivative calculations. Thomas Muir and/or James Thompson coined the term “radian” (radial angle; ca. 1870).

As translated by Hairer and Wanner (2008), Euler explained: “We let the radius, or *total sine*, of a circle be equal to 1, then it is clear enough that the circumference of the circle cannot be expressed exactly as a rational number” (p. 101). For radius one, the circumference would be  $2\pi$  and an arc length like  $45^\circ$  becomes  $\pi/4$ . After **orienting** a circle with radius 1 in a particular way, a *unit circle* appears to be a comfortable way to present trigonometric relations visually. The sine graph plots the sine function from the *unit circle* into a Cartesian plane. The graph plots the angle as a distance on the *x*-axis, and the half-chord length as the *y*-coordinate in the plane. The origin of the sine *graph* might be attributed to Dürer who **projected** a 3D *spiral* of stairs on a 2D *plane* (Dürer, 1525). This *wavy graph* does not use radians as the *x*-axis and does not use a unit circle idea; consequently, it needs **scaling** along both axes. The first use of a modern sine graph is unknown.

The logical-historical reconstruction of the sine function models—and actions that created them—highlights that our modern artefacts emerged from a protracted series of transformations. From a didactical viewpoint, this research may provide us with ideas for design alternatives for presenting the sine function in its finished, ready-made form. For instance, trigonometry lessons could include environments that foster (1) cutting angles on a circle and measuring them in degrees and radians, (2) enacting angle-to-length conversions in variously sized circles, (3) reinventing the advantages of a unit circle, namely a circle with radius 1 and special orientation, and (4) hands-on activities on discovering the (advantageous) practice of using radius as a unit of measurement.

## Conclusion and discussion

We presented an attempt of a logical-historical reconstruction of mathematical concepts. We focused on the artefacts and reconstructed the actions that could have produced those artefacts (Figure 1) because we share the belief that mathematical concepts emerge in the historical evolution of material culture (de Freitas et al., 2017). We compared the historical analysis with actualized learning designs.

Some of those actions were evident to us before systematic logical-historical analysis and supported the design of our embodied activities (Boels, 2023). In these activities, we did not present ready-made artefacts to the students; instead, we created technological environments (*fields of promoted actions*), where students' actions were constrained or fostered by continuous feedback (Abrahamson & Trinic, 2015). As students would discover new (for them) forms of actions, they would reinvent the target artefacts. Empirical analysis suggests, that when artefacts appear as an outcome of such promoted actions, students easily grasp them and their position within a broader structure of mathematical knowledge.

At the same time, other actions we did not notice before conducting this logical-historical analysis. As a result, many aspects were still embedded into our technological environments as features of ready-made artefacts. Based on the conducted analysis we would now *melt down* more artefacts. For histograms this could mean unpacking automatically generated *dotplots* and pre-given *intervals*, and for trigonometry the (unit) circle and radians.

In historical evolution, there are inevitably some branches—actions and artefacts—that are no longer in use or have been supplemented by more advanced forms and notations. It is an open question which of these are needed for students' reinvention. For example, measuring both the angle and chord in degrees might not be needed, but perhaps students would benefit from inventing chord functions (easier to calculate!) before trigonometry functions.

Another open question is to which degree melting is valuable when designing for the students at a particular point in the school curriculum. For example, for introducing a circle, it seems valuable to melt it down to drawing with a string or compass, but for trigonometry, it is most likely redundant. At the same time, students' difficulties in understanding data in histograms reveal the need to come back to placing variables on a number line. Clear guidelines on when melting is sufficient is a future direction of research and is partly situated depending on teaching aims and students' preferences.

Reflecting on our design process, we note that for our first designs, reading historical literature served as inspiration. In combination with empirical evidence of students lacking understanding due to

facing ready-made artefacts, it made us question each mathematical artefact that we previously had been taking for granted. Conducting a *systematic* logical-historical analysis highlighted which crucial actions and artefacts might still be lacking in our designs. A logical-historical analysis is not easy as it requires (hard-to-find) historical sources and quite some mathematical content knowledge. However, it has the advantage of providing specific solutions to students' persistent difficulties and diminishing the number of design and try-out cycles that might alternatively be needed to uncover all those difficulties. To conclude, we hope that *melting artefacts back to actions* might be a useful metaphorical heuristic for transforming static mathematical environments into fields of promoted actions. *Logical-historical analysis* might be an instrument in revealing which artefacts need to be melted and which can be fostered so that the students constitute meaningful mathematical knowledge.

## Acknowledgment

The authors thank Paul Drijvers and Arthur Bakker for their contribution to the PhD research of Lonneke Boels and Rosa Alberto regarding histograms, unit circles and sine graphs, and Martin Kindt for his reconstruction of Durer's spiral graph.

## References

- Abrahamson, D., & Sánchez-García, R. (2016). Learning is moving in new ways: The ecological dynamics of mathematics education. *Journal of the Learning Sciences*, 25(2), 203–239. <https://doi.org/10.1080/10508406.2016.1143370>
- Abrahamson, D., & Trninic, D. (2015). Bringing forth mathematical concepts: Signifying sensorimotor enactment in fields of promoted action. *ZDM –Mathematics Education*, 47(2), 295–306. <https://doi.org/10.1007/s11858-014-0620-0>
- Alberto, R., Bakker, A., Walker-van Aalst, O., Boon, P., & Drijvers, P. (2019). Networking theories in design research: An embodied instrumentation case study in trigonometry. In U. T. Jankvist, M. van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Proceedings CERME11* (pp. 3088–3095). ERME and Utrecht University. <https://hal.science/hal-02418076v1/document>
- Bakker, A. (2004). *Design research in statistics education: On symbolizing and computer tools* [Doctoral dissertation, Utrecht University]. <https://dspace.library.uu.nl/handle/1874/893>
- Beniger, J. R., & Robyn, D. L. (1978). Quantitative graphics in statistics: A brief history. *The American Statistician*, 32(1), 1–11. <http://www.jstor.org/stable/2683467>
- Boels, L. (2023). *Histograms: An educational eye* [Doctoral dissertation, Utrecht University].
- Boels, L., Bakker, A., Van Dooren, W., & Drijvers, P. (2019). Conceptual difficulties when interpreting histograms: A review. *Educational Research Review*, 28, 100291. <https://doi.org/10.1016/j.edurev.2019.100291>
- Davydov, V. V. (1988). The concept of theoretical generalization and problems of educational psychology. *Studies in Soviet Thought*, 36(3), 169–202. <https://doi.org/10.1007/BF01043781>



- Dürer, A. (1525). *Underweysung der Messung mit dem Zirckel und Richtscheyt, in Linien, Ebenen unnd gantzen corporen*. [Course in the art of measurement with the compass and ruler, in lines, planes and whole bodies.] Nuremberg. <https://digital.slub-dresden.de/werkansicht/dlf/17139/1>
- de Freitas, E., Sinclair, N., & Coles, A. (2017). Introduction. In de Freitas, E., Sinclair, N., & Coles A. (Eds.), *What is a mathematical concept?* (pp. 1–16). Cambridge University Press. <https://doi.org/10.1017/9781316471128.001>
- Friendly, M. (2008). A brief history of data visualization. In C. H. Chen, W. Härdle, & A. Unwin (Eds.). *Handbook of data visualization*, 15–56. Springer. <https://doi.org/10.1007/978-3-540-33037-0>
- Freudenthal, H. (1973). *Mathematics as an educational task*. Springer. [https://doi.org/10.1007/978-94-010-2903-2\\_6](https://doi.org/10.1007/978-94-010-2903-2_6)
- Hairer, E., Wanner, G. (2008). Introduction to analysis of the infinite. In Hairer, E., Wanner, G. (Eds) *Analysis by its history. Undergraduate texts in mathematics*. Springer. [https://doi.org/10.1007/978-0-387-77036-9\\_1](https://doi.org/10.1007/978-0-387-77036-9_1)
- Ioannidis, Y. (2003). The history of histograms (abridged). In J.-C. Freytag et al. (Eds.), *29th International conference on very large databases* (pp. 19–30). <https://doi.org/10.1016/B978-012722442-8/50011-2>
- Presmeg, N. C. (2006). A semiotic view of the role of imagery and inscriptions in mathematics teaching and learning. In J. Novotná, H. Moraová, M. Krátká, & N. Stehliková (Eds.), *Proceedings of PME30, Vol. 1* (pp. 19–34). PME
- Radford, L. (2021). *The theory of objectification. A Vygotskian perspective on knowing and becoming in mathematics teaching and learning*. Brill. <https://doi.org/10.1163/9789004459663>
- Ramasubramanian, K. (2019). Indian values of the sinus totus. In K. Ramasubramanian (Ed.), *Gaṇitānanda. Selected works of Radha Charan Gupta on the history of mathematics* (pp. 397–415). Springer. <https://doi.org/10.1007/978-981-13-1229-8>
- Reid, D. A. (2014). The coherence of enactivism and mathematics education research: A case study. *AVANT*, V(2). <https://doi.org/10.12849/50202014.0109.0007>
- Shvarts, A., & Alberto, R. (2021). Melting cultural artefacts back to personal actions: Embodied design for a sine graph. In M. Inprasitha, N. Changsri, & N. Boonsena (Eds.), *Proceedings of PME44: Vol. 4* (pp. 38–46). PME.
- Sinkevich, G. (2015). *On the history of number line*. arXiv. <https://doi.org/10.48550/arXiv.1503.03117>
- Van Brummelen, G. (2009). *The mathematics of the heavens and the earth*. Princeton University Press. <https://doi.org/10.2307/j.ctv1pzk6f0>
- Walkington, C., Nathan, M. J., Wang, M., & Schenck, K. (2022). The effect of cognitive relevance of directed actions on mathematical reasoning. *Cognitive Science*, 46(9), e13180. <https://doi.org/10.1111/cogs.13180>
- Wilkinson, L. (1999). Dot plots. *The American Statistician*, 53(3), 276–281. <https://www.jstor.org/stable/2686111>