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# Technology as a support in the development of didactic activities for covariational reasoning

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*This article presents a project that promotes covariational reasoning with the support of technology in secondary school students (14-15 years old) in the case of Mexico. A didactic sequence was carried out in which exploration and guided learning sheets were designed with their respective virtual didactic environments to develop the concepts of variation and covariation from an interactive and qualitative approach to describe how one quantity varies about another. This sequence was organised through a hypothetical learning trajectory. The levels of covariation proposed by Carlson and colleagues were applied to assess the degree of development of variational thinking. The interactive virtual learning scenarios played a central role in the design of the activities and, in turn, fostered an environment of cooperation and enthusiasm in the students, allowing them to move from the context situation to the various representations; however, due to the lack of mathematical prerequisites, the students progressed only up to level 2 of covariational reasoning.*

*Keywords: Active learning, junior high school students, algebra, technology, covariational reasoning.*

## Introduction

Variational thinking has been conceptualised as a dynamic form of thinking, which includes the formulation of strategies and forms of covariational reasoning, seeking the development of cognitive structures that allow the identification, analysis, and interpretation of situations of variation (Thompson and Carlson, 2017). Covariation allows for qualitative descriptions, i.e., describing how one quantity varies about another and that the interpretation of tables of values, the search for and description of patterns of values, are behaviours that are associated with a covariational approach (Confrey & Smith, 1995).

Although these concepts of variation and covariation are present from elementary algebra to calculus, it is still a subject that is a source of difficulties for students because they fail to perceive and relate the patterns that arise between the quantities involved in the different mathematical situations presented to them. There are even studies, such as that of Aldon and Panero (2020), which have reported cases where the interpretation of the mathematical situation proposed when analysing the variables involved, students confuse a graph with the trajectory of a moving object, from which a lack of clarity in the meaning of the axes and the interpretation of the graphical representations can be inferred.

Students who are unclear about the covariation between variables may show a tendency to overgeneralise linearity in non-linear contexts. This idea of overgeneralising linearity has been addressed by other research, mainly by De Bock et al. (2015). Some studies, such as that of

Vasco (2006), recommend that the teaching of variation and covariation should take place in a diversity of contexts that represent real situational problems for the student, which are adapted to the proposed learning objectives; this coincides with the didactics of Cuevas and Pluvillage (2003). One of the purposes of this type of variational-covariational reasoning has been to articulate research and social practices through contexts related to change, bringing it to life in teaching systems.

The use of digital tools can support strategies and skills relevant to specific mathematical content and facilitate students' real-world problem-solving (Greefrath and Siller, 2017) and mathematics learning through interactive activities. In our study, we design activities that promote variational and covariate reasoning with the support of digital technology, using simulators of real contexts to address mental actions linked to recognising variables and the relationships between them. This is to apply and extend knowledge through exploratory and interactive learning. As mentioned by Hitt and Dufour (2021), the characteristics of the environment can contribute to the development of conceptual understanding and should include: teachers' approaches, the types of tasks given to learners and the use of various representations.

## **Theoretical framework**

Our theoretical framework contemplates three axes; the first corresponds to technology, which supports promoting situations of variation and covariation dynamically and interactively. The second corresponds to the Hypothetical Learning Trajectory (HLT) (Simón, 2020), which allows us to organise the activities that will be graded according to the Levels of Covariation proposed by Carlson et al. (2002). The third corresponds to the Didactics of Cuevas and Pluvillage (2003), which we will denote as C&P for the design of the activities.

### **Technology - simulators**

Technology has been involved in education in recent years to support teaching and learning processes. Clark et al., (2009) and Greefrath and Siller, (2017) mention that virtual reality simulation is an activity that consists of using a computational model representative of a natural or artificial phenomenon to take advantage of the user's spatial learning and perceptual systems to understand and predict the behaviour of such a phenomenon. In this study, we designed four simulators of a real context in GeoGebra so that the student can move between reality and mathematics through dynamic environments in each task. The simulators allow students to perform actions that would be difficult to do in real situations.

### **Hypothetical Learning Trajectory**

A HLT is conceived as a tool the teacher designs to plan and organise learning activities. The position we take on the HLT corresponds to that of Simon (2020), who proposes a mathematics teaching cycle where three essential components are considered: the learning objectives - which will define the direction of the research project; the learning activities - designed and organised based on hypotheses about the learning process; and the hypothetical learning process - a prediction of how students' thinking and understanding will evolve in the context of the proposed learning activities. It should be noted that the HLT is subject to change and may be modified during its application.

## Levels of covariation

The levels of covariation with which we graded the activities are those proposed by Carlson et al., (2002): Level 1(L1) Coordination, Level 2(L2) Direction and Level 3(L3) Quantitative Coordination, Level 4(L4) Average Ratio and Level 5(L5) Instantaneous Ratio. Due to the educational level targeted in this study, we use only the first three levels; these levels are presented in terms of mental actions (MA), which serve to classify students' behaviours when faced with covariation situations.

## Didactics of Cuevas and Pluvinage

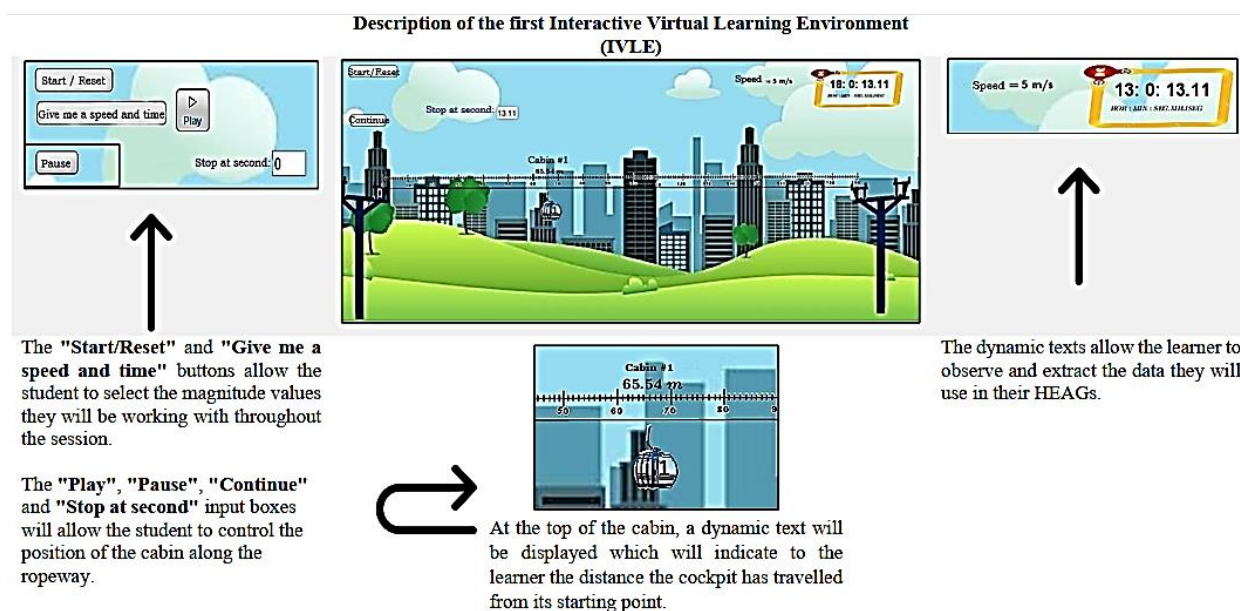
C&P Didactics proposes principles to guide the design of activities in a didactic intervention. For the purposes of this study, only the following principles were used: developing an action plan, starting from a context problem, including the inverse operation, grading the problem by dividing it into subproblems and using different representation registers. With the design and development of our THA and our didactic activities, the aim is to identify what level of covariation students achieve when interacting with mathematical models that allow them to interpret, predict, describes, and explain situations of functions, both linear and non-linear, with the support of technology.

## Method

This study was developed through design-based research (DBT), where we considered the three phases proposed by (Bakker, 2018): preparation and design phase, teaching experiment and retrospective analysis. *In the preparation and design phase*, the role of technology in this study is a means of supporting the simulation of our context of variation and covariation. The context consists of simulating the route of a ropeway (see Figure 1) and asking the student to analyse the movement of the cabins in different sections of the route. The context was divided into four sections of the ropeway, designed in Interactive Virtual Learning Environments (IVLE). The scenarios were graded from less to more complex, thus allowing us to observe the evolution of the students concerning the behaviours they develop in their mental actions when interacting with the IVLE and carrying out the activities on the Exploration and Guided Learning Sheets (EGLS). The first IVLE involves the analysis of the horizontal movement of a cabin on the ropeway (See Figures 1a and 2). The second IVLE involves the analysis of the horizontal movement of two cabins on the ropeway, where the student compares the magnitudes for each cabin (See figure 1b). The third IVLE involves the analysis of the movement of a single cabin when it is going uphill, i.e. its behaviour is not linear (see Figure 1c). The fourth IVLE involves the analysis of the motion of a cabin, but now when it is going downhill, i.e. its behaviour is not linear either. (See Figure 1d).



Figure 1: Scenarios used in each activity (Source: Own elaboration)



**Figure 2: Description of the components of the IVLE ropeway (Source: Own elaboration)**

Since a graded HTL was implemented according to the Levels of Covariation proposed by Carlson et al., (2002), the following table illustrates the articulation between them, describing within the hypothetical learning processes, the expected behaviours of covariational reasoning by the students when interacting with the different scenarios and variation-covariation activities, thus allowing us to observe and compare these processes with the results obtained by the students and to see their evolution throughout the activities.

**Table 1: The HLT of the levels of covariation, adapted from Simon (2020) and Carlson et al. (2002)**

HLT/ Level of Covariation	Learning Objective (LO) of each Mental Action (MA)	Learning activities	Hypothetical Processes
HLT_L1 (The Coordination)	<p>OL_MA1: Identify the types of variation in the quantities of distance, time, velocity, acceleration and height.</p> <p>OL_MA1.1: They establish functional relationships between the quantities of distance, time, velocity and acceleration.</p>	The coordination of the value of one variable with changes in the other variable is considered.	The learner identifies different quantities involved in the context of the ropeway (time-hours, time-seconds, distance, speed, height), classifying them as constant, parameter or variable. Once the learner identifies which quantities vary and which remain constant, the learner establishes a functional relationship.

<p>HLT_L2 (La Direction)</p>	<p>OL_MA2: Identify the types of variation in the quantities of: distance, time, velocity, acceleration and height.</p>	<p>In this activity, it will be observed whether the student can identify the direction of change, whether it is a growth or a decrease in the output value, while considering the changes in the input value.</p>	<p>In addition to the student displaying L1, he/she should identify the direction of change; whether there is a growth or decrease in the output value (distance) while considering the changes in the input value (time). When analysing these variables he/she should then identify which graphical representation is growing or decreasing faster compared to the other, and justify this behaviour.</p>
<p>HLT_L3 (Quantitative)</p>	<p>OL_MA3: Interpret the behaviour of one quantity from the changes of another quantity in a given interval.</p>	<p>The coordination of the amount of change in the independent variable with the amount of change in the dependent variable is carried out. In this case, the student must identify accelerating (increasing) or decelerating (decreasing) variational behaviour.</p>	<p>The student identifies that the value of the distance decreases or increases (considering the difference between the distances) for each increment of time, by intervals. Considering which quantities cause them to behave in this way and how they behave (in our case they consider what is happening with acceleration and velocity at this moment).</p>

The second phase of the teaching experiment describes that the research was carried out with 56 students in two groups (14-15 years old) in Mexico; however, only 21 students completed all the proposed activities. Each student was provided with a computer, printed exploration, and guided learning sheets (EGLS) and interactive virtual learning scenarios (IVLE) that simulated a real context. The teaching experiment was conducted in four 50-minute sessions. Data from our activities were obtained through exploration and guided learning sheets and photographs.

In the third phase of results and retrospective analysis, the results of four students are presented, the choice of this number of students is to show a sample of the different results that were found in the study, where two of them (students D\_7 and D\_15) reached level 2 of covariational reasoning and of the other two, one (student D\_6) managed to reach level 1, and the other (student D\_1) only managed to achieve some of the behaviours of level 1. Also, in this phase, we will focus on describing some of the results obtained from the comparison between the hypothetical learning processes of the HTL and the behaviours of the first three mental actions achieved by the student. For the retrospective data analysis, we consulted the student's responses to the exploration and guided learning sheets during their interaction with the IVLE, in addition to describing some categories describing the corresponding use of the digital tool within our didactic proposal. Behaviours associated with MA1: As the activities were sequential, it was possible to observe the evolution of the four students; an example is about the behaviour of "identifying and classifying magnitudes according to their type of variation into constant, variable and parameter". Student D\_7 identified and classified the difference

between the variable and constant magnitudes; however, he presents difficulties in the parameter magnitudes when identifying and classifying them (See Figures 3a and 3b).

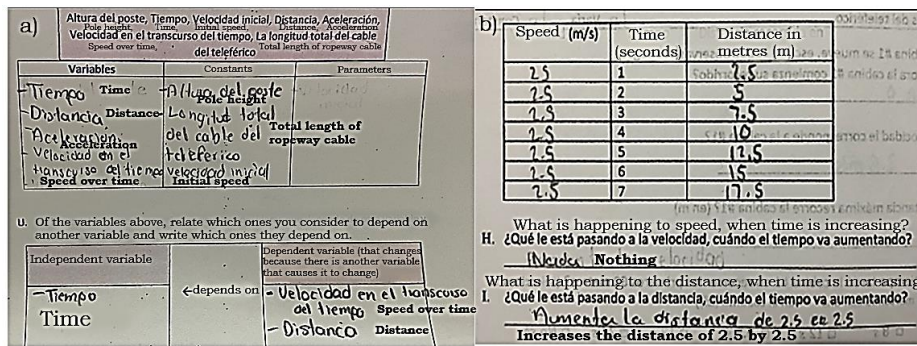


Figure 3. Student D\_7's response to activities #1 and #4

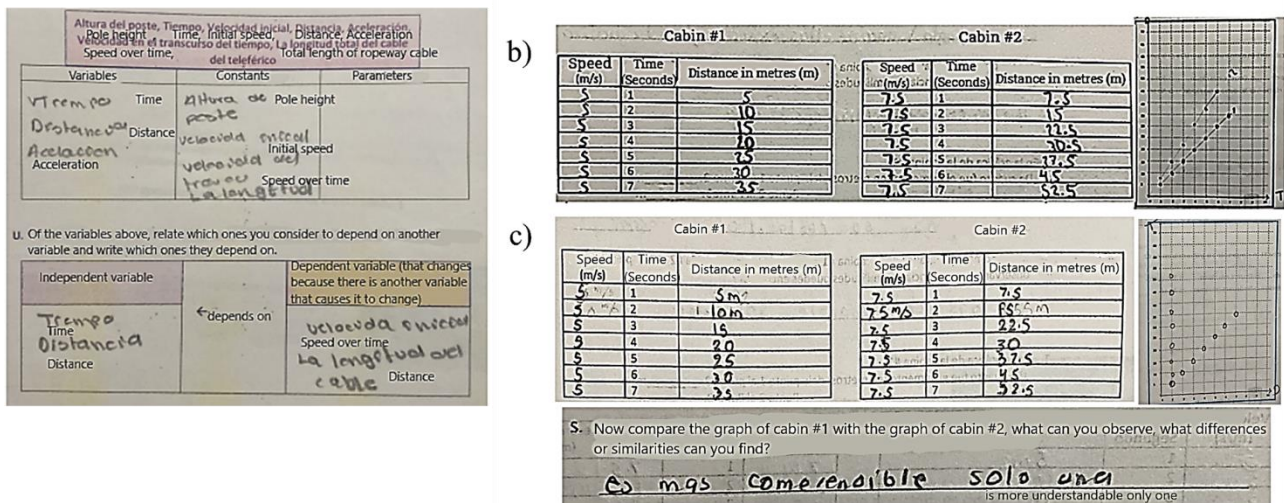


Figure 4. Students D\_1, D\_5 and D\_15's response to activities #1 and 2 - items L, Q and S.

On the other hand, some of the students did not manage to classify the magnitudes into variables, constants, and parameters in their entirety, as in the case of student D\_1, confusing magnitudes that are variables in constants and presenting problems in the identification of parameters (see figure 4a). Regarding another behaviour of mental action 1 (MA1), some students managed to establish functional relationships both in tabular and graphical form, for example, student D\_15 (see figure 4b); however, other students, despite not having problems in establishing functional relationships in the tabular form present problems when graphing these values, and even more if they have to graph two tables in the same plane, as in the case of student D\_5 (see figure 4c), who even expresses that it is more comprehensible to graph only one of them. *Use of technology in MA1: Experiment*, allowed to change parameters, conditions, and observe the effects; *visualise*, allowed to compare which magnitudes were varying and remaining constant; *measure*, allowed to find distances between points. *Behaviours associated to MA2*: In activity #2, some students do not present difficulties in finding the linear increasing pattern and in extrapolating values, i.e. they manage to find the values for magnitudes that are not visible in the graph; for example, student D\_7 (see figure 5a), however, some of the students who do identify the linear increasing pattern do not manage to find the values for magnitudes that are not visible in the graph. So, these students did not progress towards developing

the behaviour of being able to find the intermediate values proposed in the graph and the values outside those points; they even limit themselves to observe and consider only those values that are visible graphically, for example, student D\_5 (see figure 5b).

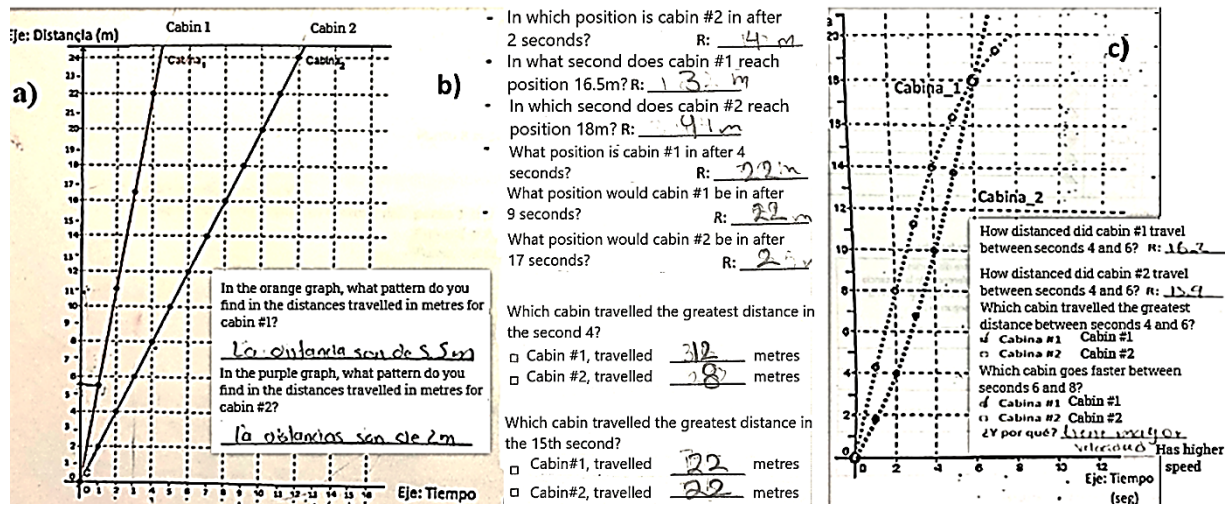


Figura 5. Student D\_5's response to activity #2-part 2 and activity #4.

**Behaviours associated with MA3:** The development that students presented in the behaviour of "identifying graphically the distance that exists from one point to another in different time intervals" of MA3, were difficulties with interpreting graphs, specifically when analysing and interpreting data in certain intervals. They also had difficulty finding an interval's length (see figure 5c). **Use of technology in MA3:** Experiment, allowed to change the parameters, conditions and observe the effects; visualisation, allowed to observe what happens to a magnitude when certain changes are made to another magnitude, what makes that magnitude behave in that way and how it behaves (in our case to consider what is happening with acceleration and velocity at this moment); measuring to find the distances between points or destinations.

## Conclusions

The development of the students during the didactic activities and their interaction with the interactive virtual didactic environments, i.e. the simulators of the cable car context, allowed us to observe the evolution of the students throughout the didactic intervention once their responses were compared with our HTL we concluded that some of the students developed sufficient behaviours to achieve a Level 2 of covariate reasoning. This same analysis was carried out with the rest of the students, so that of the 21 students, only 27% of the 71% of the students who achieved L1 were able to achieve L2, as we observed difficulties in the interpretation of graphs. Some students even failed to identify the covariation between two variables when the situation was not linear, and so were in L3. This analysis also allowed us to know where in our Hypothetical Learning Trajectory we should focus on redesigning our sequence of didactic activities and proposing technology-supported activities that address the difficulties encountered in our first design research cycle. It also allowed us to categorise and describe the corresponding use of technology within our didactic proposal.



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