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# Concrete 3D printed manipulatives in geometry learning and problem solving: usefulness of instructional 3D printed tools in the topic of polygons and tilings

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*The purpose of the presented study was to observe the role of concrete 3D printed manipulatives in solving geometry planar problems. 25 elementary school students and 34 secondary school students in the Czech Republic were given tasks on properties of polygons and their applications in tilings in the plane. Our aim was to examine how collaborative activities with concrete manipulatives from 3D printers help students to learn, to support, and to develop students' conceptual understanding of planar objects namely polygons. The experiment was divided into several phases and the aim was to explore students' success rate in solving geometric tasks with and without concrete manipulatives. The results showed that students knew the polygons but did not have any understanding how to use them in real applications such as tilings. After experiments students were able to find not only regular and semi-regular tilings but also prove their constructions.*

*Keywords: Polygons, 3D printed models, concrete manipulatives, geometry.*

## Introduction

There is a significant growth in using 3D printers in many fields such as construction industries, medicine, cultural heritage, art designs (Tsoulfas et al., 2020). In recent years, 3D printing has entered the education sector too. 3D printing technology brought possibilities for new teaching practices in a range of educational settings. According to some authors, it can promote understanding of STEM subjects by connecting engineering, technology, and applications of science subjects (Novak & Wisdom, 2018). In the scientific literature, there are efforts to summarize possible applications of 3D printing at schools and universities in order to present it to teachers and to give them the ideas where and how to use 3D printing technology and improve their teaching and students' learning. According to Ford and Minshall (2019) there are several categories of where and how 3D printing is being used in the education settings; these are: (1) teaching students about 3D printing; (2) teaching educators about 3D printing; (3) a support technology during teaching; (4) producing artefacts that aid learning; (5) creating assistive technologies; and (6) supporting outreach activities. The impact of the usage of 3D printing technology in education is the subject of international studies. Herrera et al. (2019) revealed a positive impact of 3D tools (besides 3D printing, they investigated also augmented reality and virtual environments) in developing spatial mathematical skills. Corum and Garofalo (2015) have shown that 3D printing support students' understanding in concrete mathematics topics such as surface area. Novak et al. (2021) identified in their systematic literature review 78 publications on learning with 3D printing technologies. They concluded that 3D printing technology has positive effects on students' learning and it even engages students in real world problem solving and creates opportunities for interdisciplinary researches. 3D printing technology and 3D printed models directly support

teaching process; students can actively use 3D printing technology and therefore, they are more enthusiastically participating in the classes and their spatial abilities are effectively stimulated (Szulżyk-Cieplak et al., 2014). According to Huleihil (2017), 3D printing requires higher levels of thinking, innovation and creativity.

It is evident that there are areas where 3D printing is used but the recommendations for the implementation into the educational process and the assessment of its usefulness still leaves the room for explorations.

## **Theoretical Framework**

### **Concrete and virtual manipulatives in mathematics education**

Manipulatives are instructional tools which are used as teaching and learning aids to engage students in learning, to help building their conceptual understanding and solving mathematical problems. Concrete manipulatives are physical objects; on the other hand, virtual manipulatives are dynamic interactive computer visualizations (Bouck et al., 2018; Clements, 1999; Hwang et al., 2009; Moyer et al., 2002). These dynamic visual representations can be manipulated in the same ways as a concrete manipulative can.

The effectiveness of manipulatives in mathematics instructions has been evaluated in various research studies. Sowell (1989) found using meta-analysis review that using manipulatives produces greater achievement at all grades and levels of elementary schools. However, the review did not identify the nature of situations in which manipulatives are most appropriate in particular situations. Students who can use manipulatives in mathematics instructions usually outperform those who do not (Clements, 1999). It holds in various grade levels, ability levels, and topics, if it makes sense to use manipulatives for that topic. The benefits of using manipulatives are seen as a bridge in concretizing abstract mathematical concepts (Kontaş, 2016). Manipulatives can be particularly effective in further developing conceptual understanding in mathematics because it can help students to connect concrete and abstract concepts especially in an inclusive setting (Witzel & Allsopp, 2007). According to Moch (2008) using of concrete manipulatives in mathematics instructions can arouse students' interest in mathematics and in learning in general. Also other studies showed an increase positive attitude towards mathematics lessons (Sowell, 1989).

The action oriented training methods that work with real physical models have always shown good results in the improvement of spatial ability. This can be based on the approach of embodied cognition which emphasizes that cognition involves a motor behavior (Schneegans & Schöner, 2008). Embodied mathematics thinking is related to the fact that mathematical cognition is deeply rooted in embodied interactions with environments and materials (Palatnik & Abrahamson, 2022; Ng & Ye, 2022).

On the other hand, there are also studies which concluded that manipulatives have no effect on students' achievements in mathematics (Clements, 1999). Always keep in mind that manipulatives do not guarantee success and meaningful learning. To use manipulatives only in rote learning has no effect on understanding mathematics concepts. Sometimes, teachers start using manipulatives just to vary and reform their teaching without reflecting on which mathematical abstract concepts

can be modeled using those manipulatives and without connecting representations of mathematical concepts and ideas.

### **Multiple representations**

In compliance with international studies on teaching mathematics and particularly geometry the importance of using different representations is emphasized which help students build conceptual understanding and develop their thinking. Hwang and Hu (2013) followed in their research five kinds of representations in mathematics education defined also by other researchers: (1) symbolic representation (such as numbers, letters, and symbols); (2) linguistic representation; (3) illustrative representation (such as figures and graphs); (4) manipulative representation (concrete and virtual manipulatives); (5) realistic representation. Gagatsis and Shiakalli (2004) highlighted the ability to pass from one representation to another because it was associated with students' success in problem solving and learning mathematics.

### **3D printing technology**

3D printing technology has become more and more accessible and nowadays it is quite common that we can work with 3D printers in the classrooms at elementary and secondary schools and universities. 3D printing technology is the construction of a physical object from a digital model where material is added together usually layer by layer.

The whole process of 3D printing combines the design, 3D computer modeling with CAD or other computer modeling software (using modeling with Constructive Solid Geometry or modeling with parametric curves and surfaces based on differential geometry), and the process of fabrication itself. A very detailed description of a process of designing a digital geometric model for 3D printing can be found in literature (Surynková, 2022).

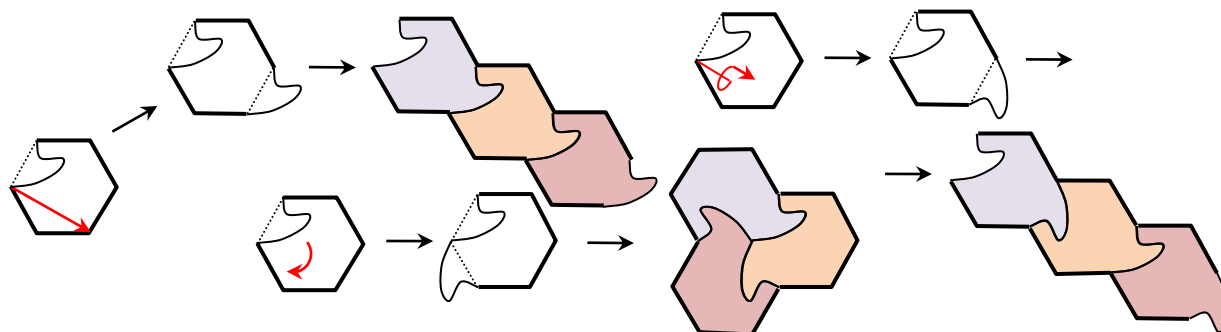
3D printing technology in education supports the idea of multiple representations which help students to construct mathematical concepts through connecting concrete and virtual manipulatives to abstract ones.

### **Polygons, congruence transformations, and tilings**

A polygon is a plane figure bounded by straight line segments as its sides. Usually, there are some other conditions to restrict the definition only to a simple polygon that is a polygon which does not intersect itself. This traditional geometric topic belongs to the study of planar geometry and is commonly included in the curriculum of elementary and secondary schools in the Czech Republic (NPI, 2023a; NPI, 2023b). The topic of polygons includes the study of their types and properties. Other topics from planar geometry are geometric transformations (such as translations, rotations, and axial or central symmetries) which are congruent, i.e. preserve shapes and distances.

A tiling (a tessellation) represents covering of a plane using one or more types of tiles (regular or irregular polygons) with no overlaps and gaps. From the geometric point of view; a tiling is a very interesting topic because it is based on the geometric properties of these polygons from which it is constructed and on congruence geometric transformations. There exist special types of tessellations as regular, semi-regular, or aperiodic. It can be proved that there are only three regular tilings

composed of equilateral triangles, squares, and regular hexagons. Then, we can also create an infinite number of tilings from these three regular ones using congruence transformations. In Figure 1 there are several rules applied on hexagonal tiling.



**Figure 1: The geometric principle for a tiling created from hexagonal tiling**

A tiling is not typically included in the curriculum of elementary and secondary schools but can represent a topic on which we can examine students' conceptual understanding of polygons and their properties and congruence geometric transformations. Some more information regarding planar geometry, specifically polygons, congruence transformations, and tilings can be found in literature (Pottmann et al., 2007).

## **Materials and methods**

### **A construction set of polygons**

For the experiments we newly designed and fabricated a construction set of polygons on a 3D printer. The set consists of different shapes of regular and irregular polygons. In mathematics, we work with an abstraction – a point has no dimension, a straight line is infinitely thin, a polygon has no thickness. However, in 3D printing we have to work with printable so called manifold object that is why we added some height to polygons and get solid figures which represent these polygons. This was explained to students; they admitted this fact without hesitation.

### **Research context and data collections**

The study seeks to understand how collaborative activities help student to develop their conceptual understanding of polygons and what is the role of physical 3D printed manipulatives in solving geometry planar problems. Therefore the study is based on the mixed research method.

In order to explicate the effectiveness of 3D printed concrete manipulatives, research experiments in classrooms with several groups of students were conducted. The experiment was repeated four times with four groups of students of different ages. In total 25 elementary school students and 34 secondary school students took part in the experiment. Elementary school students from one class (age 12-13 years) were one group (within the experiment they were divided into subgroups of 8, 8, and 9 students), three groups were secondary school students always approximately one half from one class – 14 students (age 15 years), 9 students (age 16 years), and 11 students (age 17 years). Since mathematics instructions are taught one a week divided into two. The experiments always took two mathematics instructions, i.e. 90 min, and were conducted in the school year 2021/2022 in the Czech Republic.

Students were given tasks on types of polygons, their properties and applications in tilings. All students learned about polygons in regular mathematics instructions in their previous studies. The experiment was divided into four phases and the aim was to explore and to compare students' success rate in solving geometric tasks with and without concrete manipulatives and to observe their discussions and collaborations. First, (1) students were inquired about the types of polygons (regular and irregular) and their properties. Second, (2) students were asked to search for the tilings without any visual aids. Third, (3) they were allowed to draw images. Finally, (4) students were encouraged to use concrete manipulatives from 3D printer. First and third tasks were solved individually of each student on a sheet of paper. Second and fourth tasks were discussions among students with the guidance by the author.



**Figure 2: Students' activities with the construction sets of polygons**

(1) The first phase was based on open-ended tasks on types of polygons and their properties that aligned with national curriculum standards for polygons and the concrete group of students. Students individually wrote their answers on a sheet of paper. Students' written materials were collected and evaluated quantitatively. The open-ended tasks were scored and analyzed to identify students' potential lack of conceptual knowledge.

(2) The second phase was realized as a discussion among students and a teacher (the author of the article). The teacher inquired verbally students which tilings they can imagine and what properties of polygons have to be taken into consideration. Discussions were video recorded; students' ideas were then coded and processed qualitatively.

(3) The third phase was again based on open-ended questions on types of tilings. Students individually drew their ideas on a sheet of paper. They were given the tasks to find as many tilings as possible, to justify their correct constructions, and to decide that they have found all tilings of a given property. The open-ended tasks were scored and evaluated.

(4) The fourth phase was the main phase of our experiment. Students were allowed to work with the construction set of polygons. The teacher encouraged them to comment their steps. This collective activity was recorded; students' answers and ideas were coded and processed qualitatively. Students' activities and their collaboration when they were trying to find various types of tilings and were using the construction sets of polygons can be seen in Figures 2.

We used Strauss's (1987) framework of grounded theory to classify students' responses in the qualitative part of the experiment. Several coding categories appeared and were relevant for our study: (a) method of trial and error, (b) systematic searching for a tiling based on the sum of interior angles, (c) systematic effort to find all solutions.

## **Results**

The results from the first phase of the experiment (the quantitative part of the experiment) showed that students are well equipped with the factual knowledge on polygons and their properties. Almost all students in all groups were able to name every polygon and determine basic properties of polygons (number of sides, sizes of interior angles, regular and irregular, convex and concave, area of polygons ...). In the factual knowledge, the oldest students achieved the best results. In the second phase of the experiment (the qualitative part of the experiment), students immediately knew that tilings can be made from equilateral triangles, squares, and rectangles. Then students hesitated with other regular polygons. Several students insisted on the possibility to make tilings from regular pentagons. Some students remembered a tiling made from regular hexagons from the nature. When the teacher asked students about irregular polygons, nobody was able to come to the idea whether such tilings exist or not; and what is the condition if so. In the third phase of the experiment (the quantitative part of the experiment), students individually drew their own tilings and wrote the ideas what has to be satisfied that the tiling can exist. Most students came up again with elementary tilings from equilateral triangles, squares, and also regular hexagons. Only individuals formulated the rule for the sum of interior angles of polygons in vertices where polygons meet. Finally, the fourth phase of the experiment (the qualitative part of the experiment) was the most interesting. Secondary school students were surprised at the beginning that they were asked to work with the visual aid (some students noted that this is just for children) but immediately they started use it and finally appreciated the models a lot. After several trials and errors students were able to find even semi-regular tilings which they had not come across in the previous phases. The main result was that students started formulated the general rules for polygons and mathematical proofs when they were searching for tilings without any teacher's intervention. Groups of secondary school students (and significantly the group of the oldest students) formulated the statement of the sum of interior angles in irregular quadrilaterals (based on the division of quadrilaterals into triangles) and were able to draw the geometric proof of the existence of a tiling made from an arbitrary quadrilateral.

The majority of students were eager to find general rules and it could be seen that they used the concrete manipulatives just as the aid for the confirmation of their ideas.

## **Discussion**

The experiment revealed that students remember facts well but lack conceptual understanding of polygons. We see the benefits of using manipulatives in concretizing abstract mathematical concepts which is in compliance with literature (Kontaş, 2016). Using multiple representations (figures and manipulatives) students were able to make great progress in revealing mathematical concepts by themselves. Not to mention students' reactions on concrete manipulatives; based on the author's observations students were much more motivated to solve geometric tasks if they were allowed to use some visual aid.

## **Conclusion and future plans**

In this study we introduced an experiment with a newly designed construction set of polygons as concrete manipulatives for mathematics instructions. This experimental research showed that the use of concrete manipulatives has a great potential in the process of mathematics education and



students' learning, namely in the topic of polygons and their properties. I plan to continue using concrete manipulatives and also virtual manipulatives as visual aids in my mathematics instructions. I consider this experiment as a pilot version, so I plan to repeat it with bigger samples and with modified versions of open-ended tasks.

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