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Geometrical reasoning and the influence of fluid intelligence over geometrical competence

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Geometrical competence and its relationship with other cognitive variables, like fluid intelligence, has traditionally not been studied to the same extent as other mathematical areas. Aiming to explore this association in the geometry field, this paper examines the geometrical reasoning and fluid intelligence from a sample of high-school students (N=356, ages 13 to 16). A Geometrical Reasoning Questionnaire (GRQ) was design for this purpose, and fluid intelligence was measured using Raven's Advanced Progressive Matrices Test. A comprehensive analysis of the geometrical reasoning was made considering Duval's model and Mason model. Results, on one hand, show both theoretical frameworks enrich the geometrical reasoning analysis, and on the other, confirm the relationship between fluid intelligence and geometrical competence.

Keywords: Geometry, intelligence, adolescents, secondary school.

Introduction

The concept of fluid intelligence (Gf) was introduced by Catell (1971), who states that it “appears to operate whenever the sheer perception of complex relations is involved. It is an expression of the level of complexity of relationships which an individual can perceive and act upon when he does not have recourse to answers to such complex issues already stored in memory” (p. 99). Flanagan et al. (2000) later stated that Gf “refers to mental operations that an individual may use un faced with a relatively novel task that cannot be performed automatically. These mental operations may include forming and recognizing concepts, drawing inferences corresponding implications, problem solving and extrapolating” (p.30).

From all the said definitions, we assume Gf to be the individual's capability to adapt to new situations, the agility in which he establishes new relationships, and therefore its capacity to reason with abstract contents. These are abilities that students need in academic environments, thus various studies proving the influence of intelligence in school learning (Deary et al., 2007). This apply specially to science and mathematics, where not only it has been proved the relationship between Gf and mathematical academic performance, but also the influence over abstract and spatial reasoning (Primi et al., 2010).

With this basis, the contribution of this paper is double. First, this study was designed to explore the relation of Gf and geometrical competence, specifically during adolescence (ages 13 to 16). Competence is understood as the appropriate geometrical understanding, thinking and resolution corresponding to each level of depth in the solving process of the problem. On the other hand, students' reasoning processes upon solving geometrical problems have been a topic of interest (Mason, 2010; Duval, 1995) due to their implications over geometrical learning and teaching methods. Therefore, as a second motivation, this study also examines the reasoning underlying

geometrical tasks based on both Duval's theory the cognitive apprehensions (Duval 1995) and the attention structures developed by Mason (2010).

Theoretical framework

In order to tackle the intricacy of the subject under investigation, a number of theoretical considerations are employed to establish a consistent interpretative framework: Duval's model, Mason's attention structures and fluid intelligence concept. These are two separate theoretical frameworks that open up avenues for a possible future model of integrated analysis.

Duval's model of cognitive apprehensions

Duval's theoretical framework, semiotic theory of representations, faces the cognitive functioning underlying the diversity of mathematical processes. This is considered the basis of the difficulties that many students have with comprehension of mathematics, rooted in the interaction between the special representation of the figure and the abstract conceptualization of its figurative properties (Duval, 1995). The cognitive systems that are required to give access to mathematical objects are explored, establishing a distinction and a relation between visualization and reasoning (Duval, 2006).

For the purpose of this study, from the extensive Duval's theory we will only focus on the four different cognitive reasoning ways related to visualization, also called apprehensions. Following, we define all of them: perceptive, discursive, operative, and sequential apprehension. These categories will be used for the analysis of the answers given and will allow us to evaluate the competence of their geometric reasoning, when, as said, competence is understood as the right geometrical understanding, thinking and resolution corresponding to each level of depth in the solving process of the problem.

First, the visual, intuitive identification of the geometrical object takes the student to a pictured perception. In this first apprehension, called perceptive apprehension, only tasks like recognizing the figure's dimension or its name and classification can be made. However, it is possible that perception of the figure requires more than a given visual figure, and its description shall be complemented, or even substituted, by a verbal statement. The students develop the so-called discursive apprehension when they successfully combine the verbal problem statement with the figure or geometrical object they are meant to work with. This apprehension occurs simultaneously to perceptive apprehension if, in order to successfully solve the geometrical problem, information is required from both the figure and the verbal statement. When the properties and relations of the figure are perceived, it is said that the operative apprehension has started (Duval, 1995). At this point the student is able to establish operations, such as changes in size or orientation. These operations can take the student to the correct solution path. Duval (1995) developed a final apprehension, named sequential apprehension, to describe the point when the student takes all the figure's parts and properties previously perceived, and conceptualize them in an abstract way. The figure's construction process is organized and no longer depends on visual, intuitive remembrance but only on mathematical restrictions and properties.

Mason's attention structures

Mason (2010) defines mathematical learning as attention shifts, both in the form and focus of attention. To characterize attention, Mason considers not only the object by itself, but also the way

that object is attended (Mason, 2010). In order to clarify the process (the "how" question) Mason (2008) distinct five different structures of attention. First, he proposes a holding holes structure, in which the student attends and perceives the problem, but without details. This is characteristic to the second structure, discerning details. Here, the student is able to distinct to the point of decompose the figure but lacking a background algorithm or sequential logic (Mason, 2008). The third attention structure is sequentially developed, and from the details previously perceived, relations are recognized, henceforth the name: recognizing relationships. When, besides relations, mathematical properties are recognized, the fourth attention structure is considered, called perceiving properties. Finally, the student is able to reason based on those properties and to use that reasoning, arriving at the last attention structure: reasoning on the basis of agreed properties (Mason, 2008).

This movements or swifts between the different attention structures, not only to the object attended, explain the geometrical thinking of the students. The reasoning model Mason developed has been previously used to analyse students' reasoning facing mathematical problems (Palatnik and Koichu, 2014). We will use it later in the study to evaluate the students' answers in GRQ, classifying them in different categories according to the most complex attention structure they prove to be competent.

Fluid intelligence

Higher-order thinking abilities has significant influence in school learning, a fact sustained by studies focused on the influence of intelligence (g-factor) and academic performance (Deary et al. 2007). Fluid intelligence (Gf) is understood as the capacity for abstract reasoning and establishing new relationships between multiple mental representations. This variable is of interest for our study in Geometry due the influence on abstract and spatial reasoning. Results of studies by Primi et al. (2010) associate high intelligence in 11-14 year olds with higher initial scores in numerical reasoning, abstract reasoning, verbal reasoning and spatial reasoning. Exploring this association in relation to geometric competence is an essential contribution. For this Raven's Advanced Progressive Matrices Test (Raven et al., 1995) is used, the standard index for Gf and visual-spatial reasoning.

Method

Objective

The purpose of this study is to analyse the geometric reasoning processes of high school students. In this paper we have two objectives. First, to identify the development of geometric reasoning of students in grades 9 to 12 (13-16 years old) by using two theoretical frameworks (Duval and Mason) for the analysis of cognitive reasoning ways related to visualization and representation on geometry figure. We expect that the use of both theories will allow a better understanding of the reasoning patterns. Second, to explore the relationship between students' geometric competence and their measures of fluid intelligence over the years. This will be a new contribution in relation to previous studies where this relationship has been referred to general mathematical competence, or to mathematical fields other than geometry (arithmetic, algebra).

Participants

A total of N=356 students (13-16 years old) were recruited from a private school in average socioeconomic level urban area of Madrid (Spain). On this initial sample, 82 subjects were removed

from the initial data set due to their failure to complete the tasks. This resulted in 274 students (103 males, 171 females) from four consecutive secondary school academic years: year 9 (n=103), year 10 (n=66), year 11 (n= 69) and year 12 (n=36).

Instruments and analysis

The task designed to measure the geometrical reasoning and competence of all the students was called Geometric Reasoning Questionnaire. It consisted of 11 problems, most of the 2-dimensional (surfaces, perimeters, geometrical place, metric transformations, etc) and occasionally 3-dimensional (surface of a polyhedral). As an example, we present the *Goat Problem*”, whose verbal statement says: “A goat is tied with an 8m long rope to the wall of a squared parcel of 4m as side. The rope is tied 1m way from a corner. Outside the parcel, there is grass the goat can eat. Draw the space outside the parcel the goat can reach to eat grass”. This is followed by the image in Figure 1.

The development of geometrical reasoning analysis, via an item *Goat Problem*, will be studied from the perspective of two theoretical frameworks (Duval and Mason) through the analysis of their representations. First, an analysis will be made using the Duval’s model of cognitive apprehensions. Later, a parallel analysis of the same problem based on Mason’s attention structures will be explained.

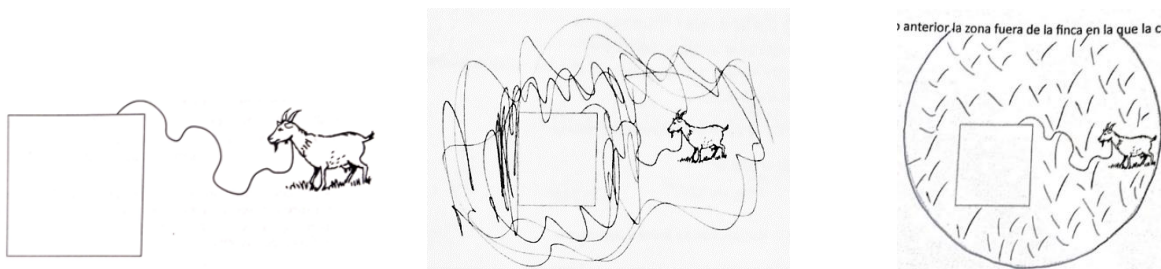


Figure 1: The Goat Problem

Figure 2: No shape solution

Figure 3: Circle solution

Therefore, considering Duval’s theoretical model, we understand that the first and superficial solution the student can form is an open region, without a specific shape or form (a real example of a year 9 student can be seen in Figure 2). In this case, the student has been able to understand the verbal statement and the purpose of the problem (even when the solution is wrong) and its reasoning has been sustained in the first visual perception. The participants with this type of solution are considered to reach both perceptive and discursive solution. However, the students who persist in the solving process are considered to make a conscious effort to comprehend the problem, perceiving the relationships between the different geometrical objects and consequently their properties. These students reach the operative apprehension, and their solutions show closed forms in the shape of geometrical figures. A real example of this type of solution by another year 9 student can be found in Figure 3. Finally, the cardioid-shape solution could be reached. In this case the student has arrived at the maximum level of abstraction of this problem, therefore organizing the information based on their mathematical properties and reaching the final and sequential apprehension.

Considering now Mason’s theoretical point of view, a parallel analysis is developed. From the attention structures framework, it is assumed that the first attention structure, holding holes, is shown when any type of solution is draw by the student (even if it has an inconclusive shape). An example of this situation can be seen in Figure 2. As soon as the answer is a closed shape, it is understood that

the student perceives the limitations of the geometrical composition, therefore advancing in its reasoning and perceiving details (second attention structure). Recognizing the relations between the different objects that are involved in the problem is also manifested in a closed-shaped solution, generally with a relatively clear geometrical shape (like a square). This suggests the graphical proof of the attention to details structure and recognizing relationships structure is the same (second and third attention structures). As the student deepens into the problem and perceives the geometrical limitations of the rope that holds the goat, their drawing refines into a more circular form, as seen in Figure 3. It is understood that, at this point, the properties of the geometrical objects are clear to the students, covering the fourth attention structure (perceiving properties). Moving from a plain circle to the final cardioid suggest that those properties had been reasoned upon, so the student has reached the fifth and final attention structure (reasoning on the basis of agreed properties).

Finally, Raven’s Advanced Progressive Matrices (RPMT; Raven et al., 1995) was used as a test measuring non-verbal abstract reasoning. The RPMT consists of sixty visual analogy problems. To solve each problem of the test, the participant is required to identify the relevant features of an array of visual abstract figures and shapes, plus an empty box, and choose the correct missing element from amongst several alternative responses arranged below the matrix. The dependent variable was the number of correctly solved items.

Results

Geometrical reasoning

Results in Table 1 offer the evolution of the different Duval’s apprehensions shown by all the participants in the *Goat Problem*. Mean values show an increase with age in overall reasoning capability, although the increase differences are lower in perceptive and discursive apprehensions (.89 to .97) than in operative (.37 to .61) and sequential (.04 to .24) apprehensions. Deviation values, however, gives especially relevant information concerning perceptive, discursive and sequential apprehensions. In the first two, they suggest polarizing levels and the early years, and a homogenic tendency as time passes (DT=.31 to DT=.17). In the sequential apprehension, however, the movement is shifted: early years are relatively homogenous but later ones tend to polarize (DT=.19 to DT=.35).

Table 1: Duval’s apprehensions measures of the goat problem by academic year

Year	Perceptive		Discursive		Operative		Sequential	
	Mean	DT	Mean	DT	Mean	DT	Mean	DT
9	.89	.31	.89	.31	.37	.49	.04	.19
10	.94	.24	.94	.24	.5	.5	.06	.24
11	.97	.17	.97	.17	.43	.5	.07	.26
12	.97	.17	.97	.17	.61	.49	.14	.35

This suggests a reasoning pattern where students are usually competent in initial reasoning processes, and as they mature most become proficient in them. This means their difficulties appear in further and more abstract phases of the problem, that are initially out of reach for most of students. However, as their mathematics education makes progress, a division appear between those who become overall competent, and those who stop their reasoning at lower stages.

The data collected from all the participants regarding the attention structures used in the *Goat Problem*, sorted by year, is shown in Table 2. Mean values show a general increase in the attention capability of the participants as they develop, but the gap is not uniform. The increase differences are especially wide in the middle structures like attention to detail and relationships (.37 to .61) and properties (.17 to .5) while in initial and final structures like holding holes (.89 to .97) and reasoning (.04 to .14) the gap is much lower. Deviation values show, at the beginning of the reasoning process, a separation among the results of early years students that tends to close as they age (.31 to .17). This tendency to homogenize the results does not apply to higher stages of reasoning, where the tendency is the opposite as deviation widens along the academic years (.19 to .35)

Table 2: Mason’s attention structures of the Goat Problem by academic year

Year	Holding holes		Attention to detail		Relationships		Properties		Reasoning	
	Mean	DT	Mean	DT	Mean	DT	Mean	DT	Mean	DT
9	.89	.31	.37	.49	.37	.49	.17	.38	.04	.19
10	.94	.24	.5	.5	.5	.5	.38	.47	.06	.24
11	.97	.17	.43	.5	.43	.5	.33	.47	.07	.26
12	.97	.17	.61	.49	.61	.49	.5	.51	.14	.35

From those descriptive measures, a pattern of the geometrical reasoning can be deduced. The competence of lower reasoning stages is initially good and becomes excellent with time. However, as the complexity of the process increase, so it does the difficulties the students face. The higher reasoning processes are practically unobtainable to younger students. Older ones, however, hit those levels more frequently, but socially separating the group in those who are able and those whose reasoning process cease beforehand.

Descriptive and correlational analyses

Table 3: Descriptive statistics

	Year 9		Year 10		Year 11		Year 12	
	Mean	DT	Mean	DT	Mean	DT	Mean	DT
GRQ	1.55	1.2	2.23	1.33	2.86	1.52	2.89	1.72
Gf	47.75	5.26	47.02	6.91	48.6	7.79	52.54	3.13

Descriptive statistics, mean (M) and standard deviation (SD) for the study measures are presented in Table 3. Under the column GRQ are the direct results of the 11-item Geometric Reasoning Questionnaire of all participants, sorted by year. This will help us understand the progress of the geometrical competence shown by the students along the years. Under the column “Gf” it appears the direct punctuations of Raven’s Test, the measure of fluid intelligence. Correlation coefficient calculus shows strength of the association between Gf and the student’s geometrical performance, with a moderate and positive Spearman correlation value of $p=.32$ ($p<.01$).

Discussion and conclusion

Geometrical reasoning processes relative to visualization of high schoolers were investigated to great extent, thus covering the first goal of this study. We found complementary information about the developing geometrical reasoning in adolescent both from Duval’s and Mason’s theoretical frameworks. This advances our understanding of geometrical thinking in the direction of a new theoretical perspective that can grow, unify, and enrich from both frameworks here proposed. The common points found on them allow us to converge the stated analysis into four main dimensions of the geometrical reasoning: visual dimension, verbal dimension, relational dimension, and organizational dimension. In this study these emerging classifications might be useful as future framework for geometrical reasoning research of the GRQ, considering the theoretical basis underneath: Duval’s model of cognitive apprehensions and Mason’s attention structures.

First, we propose the visual dimension, which is developed over the first encounter the student has with the geometrical figure. The solution process starts from this point, in which features such as subparts, names, classifications and dimension can be recognized. However, it may be necessary to complement the visual incentives with a verbal statement. Given this, the verbal dimension refers to the correct link made by the student from the verbal statement to the geometrical figure. This dimension might be simultaneous to the visual dimension if the correct interpretation of the problem requires both the information from the geometrical object and the verbal indications from the problem statement. Once the figure, as an object to study, is understood, the student proceeds to recognize its mathematical relations and properties. At this stage, operations such as metric transformations or modifications are possible. The student makes a deliberate effort to explore the properties of the geometrical object, setting the point is where relational dimension takes place. Finally, the final dimension can be developed, the organizational dimension, where the parts and subparts of the figure, along its mathematical properties, are conceptualized by the student and the decision paths are mathematically shaped. This means the abstract process and mental organization are in such high levels that superficial restrictions are relegated in favour of the logic of mathematical restrictions.

An example of the similarities between Duval, Mason, and the emerging dimensions can be made using the previously explained *Goat Problem*. In Table 4 it is shown the relation between every possible solution given by the student, their associate apprehension, their related structure of attention and the emerging dimension that identifies with. Although here the relations between every stage are clear, further research should be made, exploring this new theoretical proposal to other situations and problems. On the other hand, the aim to understand the relationship between intelligence and geometrical thinking was also investigated. Not only evolves along the adolescent period here

studied, but the reliable positive inter-correlations obtained sustained that relationship. This offers a stepping point for future investigations related to the connections of cognitive factors with geometrical competence, and the search for other variables that also influence geometrical learning.

Table 4: Theoretical contrast between Duval and Mason frames and the emerging dimensions.

Theory	Duval				Mason					Emerging dimensions			
	Perc.	Disc.	Oper.	Sec.	Hold.	Det.	Rela.	Prop.	Reas.	Visual	Verbal	Relat.	Organ.
Cardioid	X	X	X	X	X	X	X	X	X	X	X	X	X
Circle	X	X	X		X	X	X	X		X	X	X	
Rectangle	X	X	X		X	X	X			X	X	X	
Open	X	X			X					X	X		

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