



HAL
open science

On students' ways of dealing with area and perimeter's relation

Jérôme Proulx

► **To cite this version:**

Jérôme Proulx. On students' ways of dealing with area and perimeter's relation. Thirteenth Congress of the European Society for Research in Mathematics Education (CERME13), Alfred Rényi Institute of Mathematics; Eötvös Loránd University of Budapest, Jul 2023, Budapest, Hungary. hal-04419345

HAL Id: hal-04419345

<https://hal.science/hal-04419345>

Submitted on 26 Jan 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

On students' ways of dealing with area and perimeter's relation

Jérôme Proulx

Université du Québec à Montréal, Canada; proulx.jerome@uqam.ca

This paper reports on an experiment conducted in Grade-7 classrooms, where students solved tasks about area and perimeter in a mental mathematics environment. The analysis of students' ways of solving highlights three types of approaches: localized, globalized, and flexible-curve strategies. Close examination of these points to how students' detachment from the figure, as well as their fluency to navigate between 1D and 2D, acted as key elements for solving tasks on area and perimeter.

Keywords: Area, perimeter, area and perimeter relation, student strategies, mental mathematics.

Introduction – area and perimeter entanglements

The study of students' difficulties with concepts of area and perimeter has a long-standing history in mathematics education. Through the years, researchers have raised numerous challenges students face when attempting to give meaning to, or simply distinguish, area and perimeter. Without being exhaustive, these challenges can be grouped along four main categories, as follows:

(1) *The association of variations and equivalences.* Students' frequently associate variations in area with ones in perimeter. For example, it is often believed that any modification (increase or decrease) in the perimeter of a figure has necessarily a similar or equivalent effect on the figure's area, and vice-versa (Douady & Perrin-Glorian, 1989). In the same way, it is often thought that the same perimeter conserves the same area, and vice-versa as well.

(2) *The reference to measure and measurement units.* The choice of units when measuring the surface or boundary of a figure can create difficulties. That perimeter offers a measure in units, and area in square-units, is not always clear for students. On different drawings, the thickness of the boundary of the figure can induce difficulties in determining what is perimeter and what is area, or simply blurs surface and length altogether (Moreira & Comiti, 1994). As well, when a figure is given on a grid, the perimeter can be thought to be the surrounding squares.

(3) *The effect of the shape of the figure.* Piaget's studies alerted us to the effect that an object's shape or its spatial arrangement can have on students. For example, a figure's geometric shape can have an important impact on one's evaluation of its area and perimeter: long or "cumbersome" figures are often considered to have bigger area than compact ones (Douady & Perrin-Glorian, 1989). As such, Marchett et al. (2005) highlight the dominance of the two-dimensional aspects of a figure over its one-dimensional ones: when asked to compare perimeters, students state (previous) strategies used for area, or become confused with the size of area in assessing perimeter.

(4) *The use of formulas and calculations.* Formulas are omnipresent when considering area and perimeter, where for many students these two concepts *are* their formulas or become conceived as a multiplication or an addition of lengths (Douady & Perrin-Glorian, 1989). This often reduces the measurement of area and perimeter to an arithmetic exercise, to the extent that some students have difficulties assessing the area of a figure if they cannot calculate it directly (Marchett et al., 2005). Hence, irregular or composite figures, which do not necessarily have formulas attached to them, have been shown to be challenging for students.

Although more details could be accounted for, these challenges illustrate well the extensive nature of students' experience with area and perimeter. It helps explain how area and perimeter can be conceived as entangled, leading to potential complications. Even if progress has been accomplished in helping students develop more robust understandings relative to area and perimeter (see papers in this CERME WG), varieties of difficulties and misconceptions remain, year after year. This situation points to the need to continue studying how students experience the intricate relationship between area and perimeter, and to draw some insights from it. This paper reports on findings gathered through experiments conducted in Grade-7 classrooms, where students had to solve tasks about both area and perimeter in a mental mathematics environment.

Research objectives – studying students' work

As the literature abounds with reports on students' difficulties with area, perimeter, and their relation, one other way to contribute to this growing understanding is to attempt at studying key elements that students put forth when having to manage both area and perimeter. As Lamon argues (e.g. 1993), moreover to focusing on their difficulties, the detailed analysis and identification of students' thinking processes has the potential to enhance understandings of how to work with specific mathematical contents, thus offering "explanatory power for children's performances in the domain" (p. 42). As such, the study of students' mathematical processes when dealing simultaneously with area and perimeter might reveal insightful ways in which to overcome these often-experienced challenges with area and perimeter. This paper aims to report on such mathematical processes.

Our own research work in mental mathematics aligns with this objective. We investigated the potential of doing mental mathematics on topics like equation solving, geometry, statistics, and functions (at CERME: Proulx, 2013, Proulx et al., 2017). This research led to significant outcomes in relation to students' solving processes and the elements they bring forth in these mental mathematics environments: e.g. uncovering specific patterns and regularities to orient the solving of equations (doubling, tripling, cancelling); pointing to and engaging with graphical cues to operate on functions (x-intercept, intersections, parallelism of curves), assessing various characteristics of a distribution before calculating its mean (attending to extrema, clustering and number of data). It thus appears that mental mathematics contexts could afford a space where solvers can engage in specific ways of solving as well as raise elements of mathematical significance. This study, about which aspects are reported here, aims to investigate solving processes that students are engaging in when solving tasks about both area and perimeter concepts in a mental mathematics environment.

Conceptual grounding – mental mathematics and enactivism

Mental mathematics as mathematics without paper-and-pencil

What does mental mathematics on area and perimeter mean? Because most work on mental mathematics is on numbers (often referred to as *mental arithmetic* or *mental calculations*), no definition of mental mathematics that would encompass other mathematical topics appears in the literature. Based on the work on mental calculations, one tentative definition is: Mental mathematics is the solving of mathematical tasks without paper and pencil or other computational/material aids. Thus, this research program is situated in the existing research literature on mental mathematics, where it is the context of study, that is, the fact that there are no paper, pencils or other material aids available, that defines it. For this study, doing mental mathematics on area and perimeter means to

deliberately avoid any use of measurement or construction tools (e.g. rules, tapes, scissors, square tiles), to engage students in what Nunes et al. (1993) call “direct perceptual comparison”. As such, they assert that “different measurement systems will structure both physical and intellectual activities in different ways” (p. 41), leading to infer that alternative measurement contexts (e.g. with or without tools) might trigger students to engage in different or alternative ways of doing. These mental mathematics constraints have in that sense the potential to make emerge key elements for better understanding students’ experiences with area and perimeter. As mentioned, the intention is to scrutinize students’ mathematical solving processes (strategies, key elements) brought forth when solving tasks on area and perimeter in a mental mathematics context.

Mental mathematics strategies and enactivism

Recent work in mental mathematics points to the need to better conceptualize how students develop mental strategies. Faced with significant varieties of students’ creative solutions and dissatisfaction about their “classification” in known categories, researchers have criticized the notion that students “choose” from a toolbox of predetermined strategies in order to solve problems in mental mathematics (the so-called selection-then-execution hypothesis). Threlfall (2002) insists rather on the organic emergence and contingency of strategies in relation to the tasks and the solver (e.g. what he understands, prefers, knows, has experienced with those tasks, is confident with). This view aligns with Lave’s (1988) situated cognition perspective that conceives of mental strategies as flexible emergent responses, adapted and linked to specific contexts and situations. In mathematics education, the enactivist theory of cognition has been concerned with issues of emergence, adaptation and contingency of learners’ mathematical activity (from the work e.g. of Maturana & Varela, 1992; Varela et al., 1991). Therefore, aspects of the theory are used to ground this study in its intention to study students’ strategy processes. Varela’s (Varela et al., 1991) distinction between problem posing and problem solving offers ways to clarify notions related to the emergence of strategies generated for solving tasks in mental mathematics.

For Varela, problem-solving implies that problems are already in the world, “out there”, waiting to be solved. In contrast, he explains that we specify, we pose, the problems that we encounter through the meanings we make of the world in which we live: we do not “choose” or “take” problems as if they were lying “out there,” objective and independent of our actions, we bring them forth. The problems that we encounter and the questions we ask are thus as much a part of us as they are a part of our environment: they emerge from our interaction with it, as we interpret events as issues to address, as problems to solve. In that perspective, we are not acting on preexisting situations; our interaction with the environment creates the possible situations for us to act upon, and the ways we engage with them. Hence, it is claimed that reactions to a task do not reside inside either the solver or the task: they emerge from the solver’s interaction with the task, through posing the task. If one adheres to this perspective, one cannot assume that “instructional properties” are present in the (mental mathematics) tasks offered and that these will determine solvers’ reactions. Strategies are emerging in the interaction of solver and task, influenced by the task but determined by the solver’s experiences and understandings: in his solving habits for similar or different tasks, in his successes in mathematics with specific approaches, in his understanding of the task, etc. With this perspective, the solver is not seen as choosing from a group of predetermined strategies to solve the task, but as engaging with the problem and as generating a strategy tailored to the task posed:

As a result of this interaction between noticing and knowledge each solution ‘method’ is in a sense unique to that case, and is invented in the context of the particular calculation – although clearly influenced by experience. It is not learned as a general approach and then applied to particular cases. [...] The ‘strategy’ [...] is not decided, it emerges. (Threlfall, 2002, p. 42)

Students are then seen to generate strategies in order to solve the(ir) tasks, seen as adapted responses, locally tailored and emerging in interaction with these tasks. This enactivist perspective orients this study, aiming to investigate students’ solving processes when engaging with area and perimeter tasks in a mental mathematics environment, for drawing out key elements they bring forth in solving.

Methodological considerations

Over three days, six 75-minutes sessions were held in two Grade-7 classrooms (each with 26 to 27 students of 12-13 years). Although they had obviously discussed issues of perimeter and area in their elementary school years, it was the first time these two concepts were discussed in their Grade-7 class. The research sessions were conducted by the PI (J.Proulx). These sessions included presenting various tasks for students to solve mentally; tasks that engaged on both area and perimeter simultaneously. In line with previously conducted studies on mental mathematics, the activities had the following structure: (1) The PI offers orally, or on the white board when an image came into play, the task to solve; (2) The students have about 15 seconds to solve the task, without access to paper-and-pencil or any other material; (3) The students, when signaled, are invited to describe orally their answer in detail; (4) The PI notes the solution on the white board and confirms with the student that this adequately represents their strategy (in some cases, students come at the front board to better explain their solution); (5) The PI invites other students who may have solved differently to offer their solutions (and step (3) continues).

The tasks given in all six research sessions were chosen and designed in collaboration with the teacher. The criteria in selecting these tasks was that they needed to simultaneously call forth both area and perimeter, thus directly immersing students in the relation between both concepts to study their ways of managing it. The mental mathematics context forced selecting tasks along Nunes et al. (1993) mentioned “direct perceptual comparisons”, as no material was allowed for concretely measuring or physically manipulating the perimeter or area of figures. One typical example of a task given to students, and about which the following analysis focuses on, is the hexagonal task (Figure 1): students had to transform the figure in order to reduce its area while enlarging its perimeter.

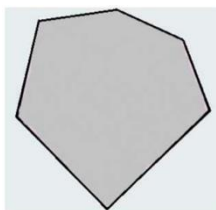


Figure 1. The hexagonal task: how can you reduce its area while enlarging its perimeter?

The sessions were all videotaped. Classrooms used electronic boards, which enabled to get electronic copies of students’ solutions noted on them. Students’ responses (in the form of verbal explanations, and the different notes made on the board) generated the data for the study. Strategies engaged in for both Grade-7 classrooms were similar, and no difference between classrooms is made in the analysis. The data is analysed in relation to students’ solving processes (strategies and key elements) brought

forth when solving tasks. The intention, following Lamon (1993), is to gain a better understanding of ways of handling both area and perimeter concepts. As well, because this analysis is dependent on the type of mathematical objects worked with, available theoretical constructs found in the literature were used to guide and enhance the analysis: examples of such are Douady and Perrin-Glorian's (1989) figure's "imposing presence" and Konya's (2015) 1D-2D view of perimeter.

Characterization of strategies related to perimeter – The hexagon task

The analyses of solving processes engaged in for the hexagonal task are illustrative of students' ways of dealing with area and perimeter concepts in the study. Students made salient particular ways of handling the figures, which acted as key elements for solving the tasks given to them. The following subsections presents students' strategies along three approaches, namely the localized, globalized and flexible-curve ones. (Expressions in "quotation marks" are taken from students' actual wordings.)

Localized strategies

One of students' frequent approach is to focus on or isolate a specific part of the figure (e.g. side, corner, measures, symmetry, height, etc.) – in contrast to considering the entire figure – and working on it to give meaning to the task through this isolated part. Most of the time successful, these strategies lead to a local and specific focus, somehow enabling a reformulation of the task's challenge according to these local aspects. For example, students made use of localized approaches as they focus on one side of the hexagon to create a dent in it. On the far left of Figure 2, one student explained that the dent on one side creates added length to the perimeter while taking away some area. For other students, the same dent could be repeated numerous times to have even less area and more perimeter, as shown in the middle of Figure 2. Or, for every "angle you do a triangle" as one student explained for the hexagon on the right-side of Figure 2.

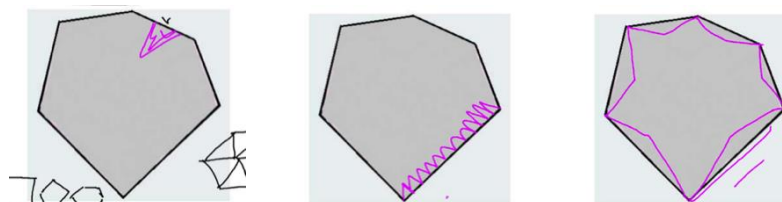


Figure 2: Localized solutions

In so doing, these localized strategies fixate on a specific aspect (e.g. side, corner) of the figure to anchor the solving of the task; straying away from considering the figure as a whole. Without setting it aside, the figure appears less important than what happens with its sides or corners, for example.

Globalized strategies

In contrast, other students engaged with the figure as a whole when handling the task, taking the total geometric shape into account. Also frequent in students' ways of solving, these strategies attempt to manage simultaneously all or most elements in the figure (e.g. looking at all the figures' sides at once). Students' use of globalized strategies led to difficulties, caused by the simultaneous handling of too many elements about the figure and its shape. For example, some students attempted to move around each side of the hexagon so as to produce another hexagonal figure that would have less area and more perimeter. This was done, e.g., by aligning some sides in a straight line (first-left of Figure 3, in red; blue arrows added to show the movement of sides) or by "pushing the low triangle part inside" (second-left of Figure 3, in purple; blue arrows added). Other students attempted to use the

sides of the hexagon to create another shape. In the right-side of Figure 3, the student aimed “to transform it in a rhombus” while joining sides together and moving others. The same for the far right-side of Figure 3, where the student aimed to “flatten all corners” to create a triangle.

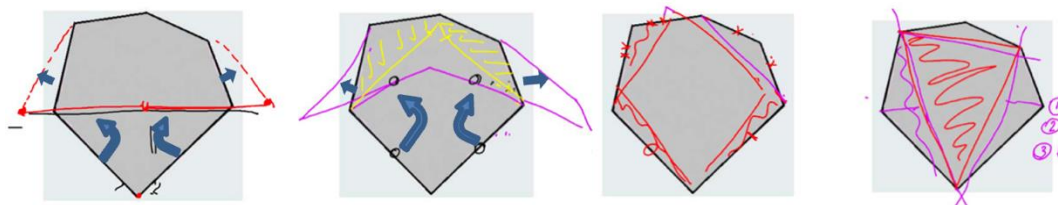


Figure 3: Globalized solutions

In these strategies, the corners are used as some sort of joints linking each side, as students attempted to move these sides around along the joints while preserving their lengths. Somehow, in these strategies, the figure is “dictating” the possibilities, where the joints are maintained as well as the sides of the figures (e.g. length, number of). Difficulties were experienced in confirming the loss in area and gain in perimeter in the resulting figures. At most, these students could assert that the perimeter was kept constant, since it was the same sides that were moved around or combined, but they lost sight of area. These difficulties led students to propose additional adjustments for adding perimeter (e.g. moving the sides even more). This made them realize that doing so might however increase area instead, provoking some puzzlement concerning their success or not in solving the task.

Flexible-curve strategies

Of a different nature, but less frequent, were the flexible-curve strategies. The key difference in these strategies is that they deal with the perimeter as a flexible object, often spoken of in terms of a string, a tape, a rope, a bendable ruler, etc. This leads to the shape of the figure, or the figure itself, to be temporarily left aside in order to assess the perimeter, and to mold it into other shapes if need be. For example, the line creating a square might be changed into a circle, preserving the perimeter (i.e. isoperimetric figures). The figure’s sides then become a highly malleable soft-object, foldable and adaptable as one pleases. One example of this strategy happened when a student suggested producing an elongated shape with the hexagon to obtain a thin rectangle that had the same perimeter but much less area. Referring to a task from the previous day, where they had to produce numerous rectangles with a 60 cm perimeter, the student said: “we could lengthen it...the rectangle yesterday the thinner it was the less area it had” (Figure 4, on the right is the thinner rectangle aimed for).

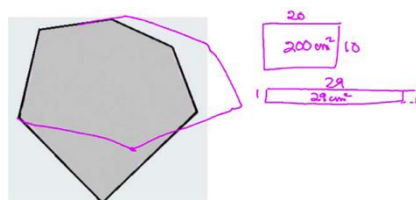


Figure 4: A flexible-curve solution

Because the perimeter of the thin rectangle would be the same as the initial hexagon, he explained that “you simply add some little more” to the longer sides since lots of area would have been lost when slimmed down into the rectangle (see the dotted part on the right side of the 29x1 rectangle). In this flexible-curve strategy, the figure’s corners disappeared in the production of the rectangle. In short, the initial hexagon imposed no constraints in the production of the resulting thin rectangle.

Discussion: figure's detachment and fluent 1D-2D interplay

The way students have dealt with the figures appear of utmost importance in whether or not they succeeded in solving this and other tasks. Two key elements are of particular significance in students' engagements with these tasks. The first element relates to the role students' give to the figure, and the needed detachment from it when considering perimeter. This first element leads to consider a second one, related to the fluent 1D-2D navigation when dealing with tasks about area and perimeter.

In localized strategies, students, at least momentarily, do not stray away from the shape of the figure and focus on isolated parts of it (i.e. not considering the entire figure altogether). By pinpointing at some sides, by looking at and focusing on a specific part of the figure, these students succeeded almost every time in solving the task. For localized strategies, as the sides or corners were the focus of attention, the task became about dealing with these local attributes (e.g. making a dent on a side, modifying or comparing one side with another). In the globalized strategies, the omnipresence of the figure, taken as a whole, constrained students' solving and usually blurred the meaning given to perimeter (and subsequently for area). For example, the consideration of the hexagon as a whole, as a more or less rigid 6-sided and 6-cornered polygon, disrupted the capacity of some students to satisfy the task's requirement for diminishing area while augmenting perimeter. The shape made difficult the final assessment to be drawn. In contrast, in the flexible-curve strategies, the initial figures did not seem to play a role any longer. Some students were in a complete detachment from the hexagon to create another shape. Their detachment from the figure appears key to overcome the "imposing presence" of the figure, as Douady and Perrin-Glorian (1989) calls it. As such, the salient presence of the figure and its attributes constrained most globalized strategies: the conceived rigidity of the figure's corners and sides refrained the globalized strategies from solving the task. And, this may also be how localized strategies differ: through paying attention to local aspects, such as one side, the whole figure's attributes are set aside for a while. This *detachment of the figure* acted as a key element in students' ways of overcoming possible entanglements between concepts of area and perimeter.

Through working with the figure and its curve, both localized and flexible-curve strategies point also to Konya's (2015) presentation of perimeter as a 1D object while being an attribute of a 2D figure. These strategies "happen" in 2D, that is, in the figure, while being as well about 1D. Flexible-curve strategies are concerned with the line and its flexible arrangement to make the figure fit. Something similar about this 1D-2D fluency can be seen in localized strategies, where students directly focus on parts of the figure, e.g. its sides, and thus are zooming in on 1D attributes in a 2D figure. In doing so, these localized strategies work on 1D segments that can first be bent, cut, dent, etc., which then afterwards are brought back at the level of the 2D figure. The localized strategies make fluent use of the 1D and 2D interplay, focusing on 1D aspects of a 2D object, distancing for a while from the 2D figure to come back to it afterwards. This illustrates some students' constant oscillation from 1D to 2D, making salient the needed fluency to go from one to the next when dealing with area and perimeter in the same figure. On the opposite, this can be seen as what most, if not all, globalized strategies fail in doing. These strategies reside constantly in the 2D world, undetached from the 2D figure (this also relates to what Duval, 2005, calls the needed dimensional deconstruction to overcome the priority of 2D over 1D for transitioning from one to the other). Hence, like the ease of detachment from the 2D figure, the *fluent back-and-forth between 1D and 2D* appears to act as another key component brought forth by students when simultaneously managing concepts of area and perimeter.

Concluding remarks

The mental mathematics context, through what Nunes et al. (1993) call “direct perceptual comparison” tasks, offered a way to investigate students’ strategies related to area and perimeter, while doing away with geometrical constructions and manipulations. Students’ strategies in this study made salient the significance of one’s detachment from the figure, as well as the fluent 1D-2D interplay, when considering issues of area and perimeter. These two elements, far from being ultimate keys that can unlock all students’ enduring challenges and difficulties with area and perimeter, offer however additional lenses to consider in further studies on students’ work with area and perimeter.

References

- Douady, R., & Perrin-Glorian, M.J. (1989). Un apprentissage du concept d’aire de surfaces planes [About the learning of the concept of area of planar surfaces]. *Educational Studies in Mathematics*, 20, 337–424.
- Duval, R. (2005). Les conditions cognitives de l’apprentissage de la géométrie [Geometry learning cognitive conditions]. *Annales de didactique et sciences cognitives*, 10, 5–53.
- Konya, E.H. (2015). The level of understanding geometric measurement. In K. Krainer & N. Vondrová, (Eds.), *Proceedings of the Ninth Congress of the European Society for Research in Mathematics Education* (pp. 536-542). Charles University in Prague, Faculty of Education and ERME.
- Lamon, S. (1993). Ratio and proportion: Connecting content and children’s thinking. *Journal for Research in Mathematics Education*, 24(1), 41–61.
- Lave, J. (1988). *Cognition in practice*. Cambridge University Press.
- Marchett, P., Medici, D., Vighi, P., & Zaccomer, E. (2005). Comparing perimeters and areas. In M. Bosch (Ed.), *Proceedings of the Fourth Congress of the European Society for Research in Mathematics Education* (pp. 766–776). FUNDEMI IQS – Universitat Ramon Llull and ERME.
- Maturana, H. R., & Varela, F. J. (1992). *The tree of knowledge* (Rev. ed.). Shambhala.
- Moreira, P., & Comiti, C. (1994). Difficultés rencontrées par des élèves en ce qui concerne la dissociation aire/périmètre pour des rectangles [Students’ difficulties about the area/perimeter dissociation for rectangles]. *Petit x*, 34, 5–29.
- Nunes, T., Light, P., & Mason, J. (1993). Tools for thought. *Learning and Instruction*, 3, 39–54.
- Proulx, J. (2013). Mental mathematics and algebra equation solving. In B. Ubuz, C. Haser, & M. A. Mariotti (Eds.), *Proceedings of the Eighth Congress of the European Society for Research in Mathematics Education* (pp. 530–539). Middle East Technical University and ERME.
- Proulx, J., L-Lamarche, M.-L., & Tremblay, K.-P. (2017). Students’ and teachers’ mental solving of equations. In T. Dooley & G. Gueudet (Eds.), *Proceedings of the Tenth Congress of the European Society for Research in Mathematics Education* (pp. 488–495). DCU Institute of Education and ERME.
- Threlfall, J. (2002). Flexible mental calculations. *Educational Studies in Mathematics*, 50, 29–47.
- Varela, F. J., Thompson, E., & Rosch, E. (1991). *The embodied mind*. MIT Press.