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Interaction of spatial and theoretical aspects in spatial geometry problem solving through construction: The case of 3D sketching

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The current study focuses on the action of 3D sketching in spatial geometry problem solving as a means to educate students' mathematical perception. The case study is presented by two vignettes showing different utilization of 3D sketching in an activity where secondary school students answer spatial geometry questions while interacting with given 2D diagrams and 3D models created by them. The study continues previous work on an enactive approach to geometry education. The analytical framework for analyzing the cognitive apprehension of geometric drawings is adapted to the characterization of an enactive approach to learning. 3D sketching as a form of embodied and enactive math activity involving the construction of tangible models evokes coordination between various types of apprehension and thus facilitates learning and teaching spatial geometry.

Keywords: Spatial geometry, 3D sketching, middle school students, embodied learning, cognitive apprehension of geometric drawings.

Introduction and theoretical background

The seminal Battista's (2007) contribution states that teaching and learning geometry depends mainly on two ways of thinking: "invention and use of formal conceptual systems to investigate shape and space" (geometric thinking) and "...the ability to 'see,' inspect, and reflect on spatial objects, images, relationships, and transformations" (spatial thinking, p. 843). Similarly, Laborde (2005) argues that geometry combines spatial-graphical and theoretical aspects, where the former allows humans to interpret real-world phenomena related to shape and space and the latter—to generate and explore problems, questions, and methods within mathematics. This duality can be seen as a rationale for the inclusion of geometry education in school syllabi, a source of difficulty in geometry education, and one of the main foci of research on geometry teaching and learning (e.g., Brunheira et al., 2022).

According to Laborde (2005), students' work with diagrams in geometry problem solving can integrate spatial and theoretical aspects of geometry. This view is concurrent with the recent study of Fujita and colleagues (2020), suggesting that in spatial geometry, coping with tasks demands harmonization of spatial and formal reasoning by domain-specific knowledge and with the work of Mithalal and Balacheff (2019), who explored conditions in which construction tasks in 3D DGE (dynamic graphic environment) stimulate students' transition from working with drawings to perceiving geometric properties of figures. Novel technology, for instance, DGE, allows students to work with diagrams in ways impossible with traditional paper-static medium, leading to significant changes in the relationship between spatial and theoretical aspects of diagrams. This line of research continues to be relevant, as manifested by a recent CERME discussion: "What is the added value of tangible and/or digital/virtual manipulatives in geometry instruction?" (Brunheira et al., 2022, p. 6).

The current study aims to characterize the affordances of the different novel medium—3D pen (a hand-held device enabling spatial sketching with hot plastic) for spatial geometry learning. The study

focuses on the heuristic role of 3D sketching in spatial geometry problem solving as manifested in an activity where secondary school students interact with 2D diagrams (given) and 3D models (created by them). Doing so continues our previous work on an enactive approach to geometry education presented at the last CERME (e.g., Palatnik & Abrahamson, 2022).

3D sketching as a form of enactive/ embodied geometry activity

Ng and Ferrara (2020) studied the potential of the 3D Pens technology from an embodied perspective of “assembling of learners, concepts and tools: moving and learning together with 3D Printing Pens through coordinated hand and eye movements.” (p. 3) when teaching *primary school* students on the number of faces, edges, and vertices of prisms and pyramids. Their work demonstrated that students’ perception of and reasoning about 3D solids was shaped by 3D drawing as diagramming and their ability to use constructed models as manipulatives—by touching, moving, and rotating them. Thus, learning with 3D Pens affords a connection between enactive and iconic representations of geometric forms. However, how this technology can be utilized for *middle school* geometry and not hump but support theoretical-formal-symbolical aspects of geometry reminds an open question.

Operating from an enactive/embodied learning perspective, Palatnik and Abrahamson (2022) presented theoretical foundations and empirical arguments for a set of embodied spatial-geometry curricular resources for middle school. They conjectured that tasks in which students construct 3D objects are more than “working with manipulatives”—they let students use their natural capacities of multimodal *perception* and collaborative *action*. This argument was supported through qualitative analysis of embodied designs for spatial geometry learning. In particular, when a group of students sketched a model with a 3D pen and explored it, they experimented with their new degrees of modal freedom—looking at objects, pointing at them, touching them, lifting and rotating them. Students’ multimodal perception supported their actions, and students’ actions, in turn, supported and educated their perception (see Chemero, 2020 for the tenets of embodied cognition). Students’ critical insights in spatial problems were characterized as shifts in perceptuomotor attention leading to the refinement of geometric argumentation. Palatnik and Abrahamson implicated students’ realization of available 3D medium affordances catalyzing these shifts. The current study develops this direction and focuses more specifically on the action of 3D sketching as a means to *educate* students’ mathematical *perception* not only by “using perceptually rich objects” but also as “(t)he process of creating and perceptually interpreting the diagrams can substitute for the abstraction” (Goldstone et al., 2010, p.281). For this purpose, the framework of Duval (1995) for analyzing the cognitive apprehension of geometric drawings and the development of mathematical perception was applied to the 3D pen environment and modified per embodied learning perspective.

Analytical framework: four types of cognitive apprehension of a figure

According to Laborde (2005), the type of information that can be extracted from diagrams is theoretical (objects, relations, and operations on these objects) and spatial-graphical (graphical entities on which it is possible to perform physical actions). Duval’s (1995) work on the heuristic role of geometrical figures is closely related to this distinction. Duval distinguished four types of apprehension of geometric figures related to spatial-graphical and theoretical aspects of geometry: perceptual (recognizing a figure and its parts at first glance), sequential (constructing a figure),

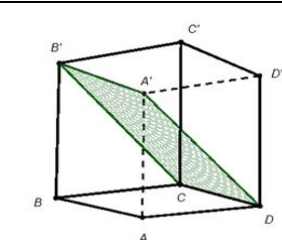
discursive (defining, determining figure properties), and operative (modifying a figure mentally or physically in order to get insight on the proof or solution). Duval claims that mathematical perception development requires distinguishing and coordinating between these different forms of apprehension, and in particular, students need to be taught how to operate on a figure. Considering the fundamental embodied tenet of *perception-action coupling* (Chemero, 2020), this claim receives a new reading: students may develop *different forms of action with figures* and thus perceive/ apprehend different affordances that figure land for their advancement in problem solving.

This study aims to answer the following research questions: What constitutes students' work with 2D diagrams and 3D models when answering spatial geometry questions with a 3D pen? What heuristic role plays 3D sketching in the solution of spatial geometry tasks?

Methodology

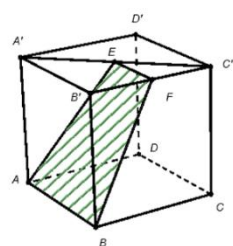
Participants and tasks

Two cases reported in this paper are part of an expanding data set generated in the course of a broader research project (currently more than 50 students from Israel and 20 students from Poland) exploring embodied learning design for spatial geometry education (Palatnik & Abrahamson, 2022; Benally et al., 2022). All the students (9th-10th grade) have not yet studied spatial geometry in an institutionalized way but have already studied planar geometry necessary to solve the given problems. The current paper presents cases of two 10th-grade students: R and Z. R learned mathematics in the 3-unit and Z in a more advanced 4-unit stream (correspondingly 65% and 20% of Israeli students).



$ABCD A'B'C'D'$ is a cube. Answer the true/ false questions and explain your reasoning.

- Quadrilateral $DCB'A'$ has acute angles.
- Quadrilateral $DCB'A'$ is a parallelogram but is not a rectangle.
- Quadrilateral $DCB'A'$ is a rectangle.
- Quadrilateral $DCB'A'$ is a square.



$ABCD A'B'C'D'$ is a cube. E is the midpoint of $A'C'$. F is a midpoint of $B'C'$. Answer the true/ false questions and explain your reasoning.

- B' is a point on AE.
- ABFE is an isosceles trapezoid. The equal sides are _____.
- ABFE is a right-angled trapezoid. The right angles are _____.
- AE is the longest side of quadrilateral ABFE.

Figure 1: Spatial geometry tasks used in the study

Before the main activity, the students were given a 3D pen and were asked to sketch a pyramid to practice 3D sketching. Most students were able to sketch freely with the pen within 10 minutes. The study's main activity includes three spatial geometry tasks (see Figure 1 for examples). Students were given a 2D diagram accompanied by text and a plastic wire model of the cube.

Data collection and analysis

After completing the questions, the students participated in a task-based interview about coping with questions and their overall experience. In two cases presented in this article, during the interview, the students were prompted to use the video recordings to recall their thoughts during the task (stimulated recall interview SRI, e.g., Stough, 2001). Both problem-solving and SRI sessions were audio and video recorded (face-off frame) and juxtaposed with the researcher's field notes to capture and analyze the students' actions, arguments, and emotions. A multi-layered protocol was prepared to comprise (a) the transcription of the students' verbalizations, (b) actions, including a detailed description of model rotations and adjustments, (c) screenshots, and (d) comments from task-based interviews. It was segmented into the following units for each task: (I) sketching the 3D model, (II-V) answering questions 1-4. In each of the units, following Laborde (2005), we identified student moves as referring to spatial-graphical or theoretical aspects or establishing a link between them. We also differed between actions and utterances referring to 2D diagrams and 3D models and establishing a link between them. Finally, we coded students' actions related to the work with diagrams and models according to four types of figure apprehension (Duval, 1995): perceptual, sequential, discursive, and operative. This multi-level coding served to identify a connection between students' actions and mathematical perception related to work with diagrams using 3D pen sketching.

Findings

To answer the research questions, I present two vignettes. The first demonstrates a heuristic role of 3D sketching in R's interaction with a diagram and the model when coping with the Rectangle Task (Figure 1 top). The second vignette develops this theme by presenting Z coping with the Trapezoid task (Figure 1, bottom). Both vignettes establish how an action of 3D sketching educates students' perception by facilitating their learning to coordinate different types of figure apprehension.

The case of R: 3D sketching educates perceptual and sequential apprehension of a figure

R started the activity by looking closely at the diagram on the computer screen and reading the question, "are there acute angles in the quadrilateral $DCA'B'$?" He decided to "draw this" with a 3D pen. R pointed with a 3D pen at the vertices of a model while gazing at the screen, orienting himself, and started drawing a diagonal of the cube ($B'D$, Figure 2 left).

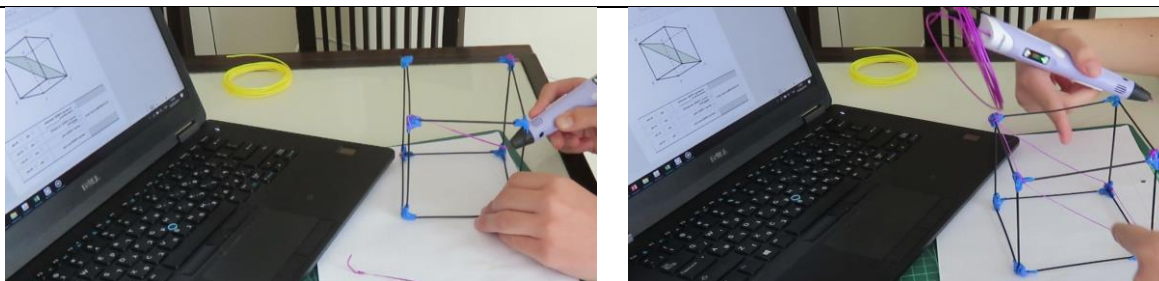


Figure 2: The Rectangle task

Note that $B'D$ is not a part of the diagram, and drawing it to solve the problem is unnecessary. The following exchange happened when the interviewer asked R what he was drawing.

R: [Gestures with a 3D pen. Points at a screen] $B'D$.

- R: [Looks closely at the drawing. Hesitates] Well, (pause), wait a second.
 R: [Smiles, shakes both arms] OK! No, no, it is not!
 R: [Disconnects a diagonal and pulls it out of the model]. Nice, nice! I liked this. That's not it. It goes through, but not that. That's not it. Not quadrilateral sides. Nice.

During the SRI, R explained: (I realized) that what I draw is not what I thought it is. I understood there is no $B'D$ there [points on a screen]. There are $B'C$, CD , and DA' . And this (a diagonal of the cube) is not it. It is a diagonal that is inside. I like the problem. It forced me to use this (3D sketching).

Analysis: In this episode, R's actions are related exclusively to the spatial-graphical aspect of geometry. When R started to draw the model, it triggered a need to establish a correspondence between the diagram and the model. This episode is characterized by constant alternation from a diagram to the model and back, which was necessary for simultaneous perceptual apprehension of the diagram and the model. R's initial perceptual apprehension of the diagram was misleading. He mistook a cube's diagonal for its face diagonal—"what I draw is not what I thought it is". In his words, the problem and the action of sketching "forced" him to surface a perceptual misapprehension.

R continued to solve the problem and drew in succession four sides of the rectangle in contrasting violet plastic (Figure 2 right) while constantly conferring with a diagram and naming vertices and sides " $B'C$, CD , DA' ". During the SRI, he explained: "Now, when I understand where the sides are, I'm drawing them. I'm checking here to see what is happening. To be sure that what I do is what I think. That it is really like this and not again something that looks like, but in fact, it is not." He finished the task by answering all the questions correctly.

Analysis: R drew all four sides, not just the diagonals of the faces, indicating his need to preserve consistency and perceive two of the cube's edges as sides of the quadrilateral $DCB'A'$. While the diagram is given, 3D sketching of a model, when one element appears after another, forces sequential apprehension of a figure. A continuous 3D sketching was accompanied by simultaneous naming of the sides, establishing a correspondence between the vertices (marked on the diagram and not on the model). Thus, sequential and discursive apprehension of a figure combined. When R drew the model, he moved it, educating his perception through orientational and perspectival variations of the figure. These are characteristics of operative apprehension, illustrated even more vividly by the case of Z.

The case of Z: 3D sketching educates operational apprehension of a figure

Z was very proficient with a 3D pen. She drew a face diagonal $A'C'$, then sketched the midpoints (first E and then F), by marking them with small plastic lumps and connecting B and F on a cube face. However, she had to plan her action to sketch a side AE that is not on the cube's surface.

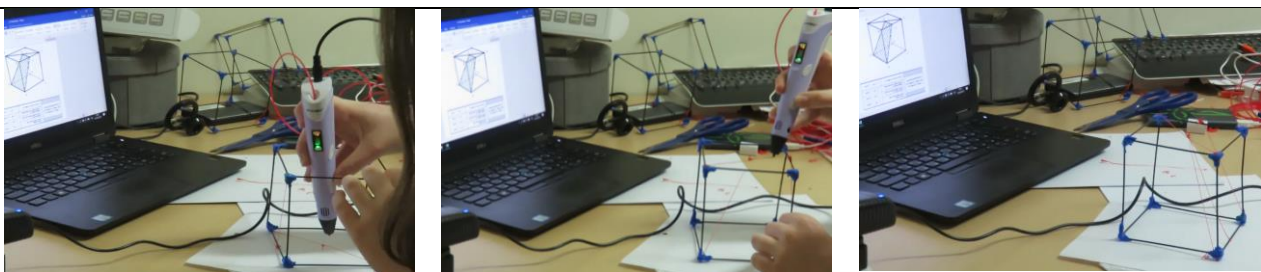


Figure 3: The Trapezoid task

- Z: [Holds a model by an edge.] OK. From point E to point A.
 Z: [Looks closely at the model. Hesitates] Well, (pause), wait a second.
 Z: [Slightly rotates the model to align with the diagram] [Plans to connect points A and E. Inserts a pen inside the model from an upper face; hesitates to draw and pulls a pen out]. [Flips the model, Moves the model slightly to align with the diagram]. [Flips the model]. Flips the model again, with the upper face down]. [Moves a pen quickly to point E and draws EA (Figure 3 left). Rotates the model and flips it].

Analysis: When sketching with a 3D pen, Z has to learn how to keep correspondence between the diagram, the text, and the model. Z coordinates between perceptual apprehension of the diagram and the model by adjusting the model's position to resemble the diagram's orientation (see Figure 3 center). Her sketching keeps coherence with and emulates the given text: she sketches first E and then F. Thus, simultaneous perceptual apprehension of the diagram and the model is combined with sequential and discursive apprehension. A sequence of actions (including complex rotations, flips adjustments, preliminary attempts) antecedent to sketching EA modifies a perception of model in space and is interpreted as commencing an operational apprehension of the model.

Due to the page limit, I do not analyze Z's successful responses to the initial three questions and focus on how she answered the fourth one: Is it true that AE is the longest side of quadrilateral ABFE?

- Z: [Reads the question] AE is a longest side of a quadrilateral ABFE. It is true. Because, in general, diagonals in a cube are always longer than the edges. So (pause). That (AE) is not a diagonal. Of course, it is not a diagonal. But still, it is long enough [Diagonal gesture bottom-up], (pause) it continues beyond [Vertical gesture following AA']. I also see this with my eyes.
 I: Do you have some justification? Let's say I would ask you, can you build some auxiliary construction that can help? To prove this? ...
 Z: [Tilts a model to face her and draws an auxiliary line from E to AB] It is supposed to be an altitude. [Points at the model decisively and gestures along the drawn altitude] So, I know that I have an altitude here. So, I know that the triangle that I created is a right triangle. That means a side EA is a hypotenuse and, thus, the longest side in the triangle because it is against the angle of 90 (degrees).
 I: [Points on a new rectangle] What kind of quadrilateral is his?
 Z: What? It should be a rectangle. It does not look like it, but it is a rectangle. So, I know that this side [traces EA with her finger] is longer than this side [traces the altitude], which is equal to this side [traces BF], and thus this side [traces EA] is longer than this side [traces BF].

Analysis: In this episode, Z works solely with a model that no longer aligns with the computer screen diagram (see Figure 3 right). Z answers quickly and correctly, which reflects her adequate apprehension(s) of the problem. However, her first attempt to justify her answer in theoretical terms, "diagonals in a cube are always longer than edges," uses a valid but irrelevant claim. She then returns to spatial-graphical arguments: "it continues beyond", "I also see this with my eyes," and use of gestures. The interviewer prompts for other justification and hints that an introduction of an auxiliary line can help. When Z sketches the altitude from E to AB as an auxiliary line, she modifies the model to give her an insight into the proof—an operative apprehension. This action also enables her to combine spatial-graphical and theoretical aspects of problem solving. Z perceives and justifies that the trapezoid is split into a right-angle triangle and a rectangle, demonstrating the operative and discursive apprehension of the figure. Finally, Z constructs a theoretically valid argument (based on

the properties of right-angle triangle and rectangle) supported by a transitivity rule expressed by spatial-graphical means and embodied in a sequence of rhythmical gestures.

Discussion

This study sought to characterize students' work with 2D diagrams and 3D models when answering spatial geometry questions with a 3D pen. Several tacit processes, such as mental rotations and transformations, surfaced by observing students' work with a 3D pen. Five types of physical action can be discerned: adjusting the model with the diagram, pragmatical preparing the model for sketching, sketching the model, sketching auxiliary constructions, and removing parts of the sketch.

Two vignettes demonstrated that a substantial part of this work constituted actions for coordination between figures in two different mediums: the tangible-dynamic-3D *model was adjusted* to match the screen- static-2D diagram. These actions were directed at improving perceptual apprehension. Even perceptual apprehension (Duval, 1995), this immediate recognition of a shape and its parts was challenging but mendable, as demonstrated by R's initial mistake and eventual success.

During the initial stages of the task, to draw a particular segment, the students, in R's words, were "forced" to *prepare the model for sketching*, vividly illustrated in Z's case of sketching EA. A pragmatic need to draw inside the model led to complex rotations of a model while keeping in mind the invariant relative position of the segment(s). These actions, in turn, resulted in updated operational apprehension of the model. This finding strengthens the heuristic role of the model's position in space as crucial for students' problem solving (see Palatnik, 2022, Palatnik & Abrahamson, 2022).

When R and Z *sketched the model*, this action mobilized a sequential and discursive apprehension. In turn, naming sides and vertices, reading the given text, and recreating the diagram in 3D helped them prepare for further problem-solving action (c.f. Chemero, 2020 perception-action coupling).

Z's first attempt to justify her claim about AE referring to spatial-graphical aspects of concurs with Duval's (1995, p. 155) claim that "Difficulties in moving away from perceived features of a figure can mislead the students as to the mathematical properties and objects represented by a drawing, and can obstruct appreciation of the need for the discovery of proofs". However, Z's subsequent attempt was based on *auxiliary construction* (on the heuristic role, see Palatnik & Sigler, 2019) mobilizing both spatial-graphical and theoretical aspects (c.f. Fujita et al., 2020, Mithalal & Balachev, 2019).

Goldstone et al. (2010) claimed: "A credible and worthy hope for education is to teach students to take the natural affordances of our long-tuned perceptual systems, which are at their core spatial and dynamic, and retask them for new purposes" (p. 280). This study demonstrates a subtle and laborious way to educate students' mathematical perception by honing their natural ability to interact and create tangible objects for exploring spatial geometry models. 3D sketching as a form of embodied/enactive math activity involving the construction of tangible models facilitates coordination between various types of figural apprehension supporting spatial geometry learning and teaching.

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