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Are geometric concepts of vectors bookable? An analysis of the didactic transpositions in mathematics textbooks

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In Germany, the use of vectors is traditionally introduced in lower secondary education in Physics and in upper secondary education in Mathematics. In this curriculum, a geometric point of view plays an important, but somewhat unclear role whereas at the beginning of the Bachelor's degree programmes structural-symbolic approaches dominate in courses like "analytic geometric and linear algebra". Based on an analysis of Mathematics textbooks in upper secondary education from a point of view of geometry, differential geometry, and dynamical systems, the underlying geometric concepts are worked out as praxeologies. Some typical problems in working with vector geometry relate to the change between different conceptual models in the organisation of the textbooks.

Keywords: Geometric concepts, secondary school mathematics, mathematics instruction.

Introduction

The term "analytic geometry" describes geometry with vectorial methods. In upper secondary education or in the first year of tertiary the area "analytic geometry and linear algebra" has a fixed place. At universities, in courses like "Analytical Geometry and Linear Algebra", traditionally the share of linear algebra dominates. As an area of teaching, linear algebra developed as a counter-concept to Euclidean geometry since Dieudonné's "A bas Euclide!" and distinguished from the view and geometric thinking in school and university, see e.g., Herbst et al. (2017), Chapter 1. Mathematics education research results are limited in the field of analytic geometry.

Problems with the concept of vectors in analytic geometry have been intensively studied from the point of view of students' difficulties at school (Wittmann 2003). In tertiary education there is research on the teaching and learning of linear algebra in higher education in Germany (Fischer, 2006) and in the United States in using geometry to teach linear algebra (Gueudet-Chartier, 2004) and especially from the working group in San Diego, see Wawro et al. (2013).

Vector geometry is also often not the starting point in mathematics textbooks in upper secondary education before linear equations and structural properties of vectors appear. We follow (Kang & Kilpatrick, 1992) on the didactic transposition by varying Kilpatrick's question "Is problem solving bookable?" of 1980 for geometric concepts of vectors: Are geometric properties of vectors bookable? In order to transmit knowledge about a given subject - in this case, the use of vectors - one follows a path that begins in society, continues in school, continues through an area within a discipline in which the subject is studied, through a certain sector within the area and through a subject of the said sector. To simplify the approach, we cannot take systematically into account the different levels of education, starting in lower secondary education, where vectorial concepts appear first in Physics, because most

mathematics textbooks do not focus on this link, but assume a start from scratch in the course for upper secondary education where “analytic geometry” is the core content of mathematics for half a year in Germany.

The aim of this paper is to understand more systematically which geometric approaches have to be developed in a course on analytic geometry in upper secondary education. Since the main interest is devoted to the variety of aspects of geometry, several textbooks on the same curriculum – here from Germany – are considered.

For this, two research questions are in the focus:

- (1) Which geometric concepts for vectors appear in the textbooks and how can these be described in mathematical terms?
- (2) What is the technical know-how that the books provide for the learners in dealing with these concepts?

Theoretical framework

The theoretical framework of this project is based on the anthropological theory of the didactic (ATD). A mathematical work, also called mathematical organisation, is born as a response to a type of question or problematic task and is made up of technical, technological and theoretical elements (Chevallard et al., 1997). The ATD establishes that the tasks require different procedures, called techniques, which make it possible to provide answers to the proposed activities. Techniques allow problems to be grouped into what is called types of tasks.

In order for a technique to exist in an institution, it must not only provide answers to certain characteristic questions, but it must also be justified. The arguments that are developed to describe and justify the use of each technique are called technological elements. Just as each technique is associated with a technology, each technology is associated with a theory. Thus, a mathematical organisation consists of four categories of elements in a “praxis” block and in a “logos” block: tasks (T_i) and techniques (τ_i) describe the praxis block, whereas the logos block consists of technologies (θ_i) and theories (Θ_i). Here i is an enumeration parameter.

According to Chevallard et al. (1997), textbooks belong to didactic devices insofar as each of them affects the structuring and development of the process of studying mathematics. In this study, textbooks for school mathematics constitute the empirical material of the mathematical organisation of the knowledge to be taught (“Noosphere”).

Methodology

Study design. Analytic geometry, which goes beyond linear algebra, is found in many mathematics courses at most universities. In a first step, concepts of vector geometry are considered in the following courses in tertiary education:

Linear algebra and analytic geometry

Geometry: Euclidean, plane, projective, spherical, axiomatic geometries, curves and surfaces, differential geometry, linear group actions

Differential and integral calculus: Linear differential equations, fluxes, curves in space, multidimensional derivation with Jacobi matrices and chain rule, submanifolds, volume and area integrals with the transformation formula

Applied mathematics: differential equations, linear optimisation

Stochastics: vector-valued random variables, correlation matrices.

The empirical basis for the study on the practical-technical block are mathematics textbooks for the teaching and learning of analytic geometry in upper secondary education in Germany. Three textbooks were chosen because they took different point of views on vector geometry. In order to have a broader point of view than the latest developments en vogue, a fourth textbook was added which appeared in the same textbook series, but is actually quite different because they have undergone several curriculum changes over two decades. It should be noted that these books are designed for intensive courses (“Leistungskurse”) in which usually more contents can be covered because stronger students enroll. A choice like this was also necessary to ensure a common curricular basis.

Analysis procedures. The analysis proceeds in several steps: The logos block is described as an epistemological analysis of a curricular study of where contents of analytic geometry are actually used in mathematics. The second step capitalizes on the logos description to establish inductive categories to describe the praxeologies in mathematics textbooks for tasks on vectors. To establish those the inductive category development (Mayring 2015) was used to extract typical tasks (T_a) in the textbooks and techniques (τ_b) to solve them. This is described with the help of suitable technologies (θ_c) and theories (Θ_d), where a, b, c , and d are enumeration parameters. Starting out with the research question on how praxeologies of mathematics textbooks for vectors can be described by mathematical theories, mathematics textbooks are examined on tasks for vectors. To limit ourselves, the use of scalar products was involved at the beginning, but only as far as they contribute to the understand of vector geometry. The aim of the analysis is to understand the underlying concepts for vector geometry with the theoretical block (technologies (θ_c) and theories (Θ_d)) and then to describe the practical block (tasks (T_a) and techniques (τ_b)) on the treatment of vector tasks. Taking Mayring (2015) as an example, the current state of the project can be explained as follows: the inductive categories have been revised after screening about a third of the material. A formative check of reliability lead to several cycles to change the formulated “logos” as well as the praxeologies. To complete Mayring’s approach, the ongoing work will involve more textbooks and preferably other resources to document the praxeologies in the future.

Analyses

The technological-theoretical block. In the cyclic qualitative analysis of tasks described and posed in textbooks, the first attempts failed to have in most of the cases unique theories to be associated with the material. However, it was possible to work out categories which appear frequently. The theories are described according to the typical contents of the above-mentioned courses:

Theories	Technologies
Θ_1 : Synthetic geometry as a model of an affine space	$\theta_{1.1}$: Necessary and sufficient conditions to determine vectors as transformations $\theta_{1.2}$: Necessary and sufficient conditions to determine lines uniquely in the plane $\theta_{1.3}$: Necessary and sufficient conditions to determine lines and planes in the 3-dim. space
Θ_{2a} : The standard affine n-dimensional space A_n Θ_{2b} : The transitive group action of $T A_n = \mathbb{R}^n$ on A_n	$\theta_{2.1}$: Role of \mathbb{R}^n as set of points $\theta_{2.2}$: Role of \mathbb{R}^n as space of translations $\theta_{2.3}$: Arithmetic of points and translations
Θ_3 : The arrow class model	$\theta_{3.1}$: Equivalent class of ordered pairs $\mathbb{R}^n \times \mathbb{R}^n$ $\theta_{3.2}$: Parallelism, length, orientation
Θ_{4a} : The affine tangent bundle Θ_{4b} : Embeddings of affine spaces	$\theta_{4.1}$: The tangent bundle $T\mathbb{R}^n$ and $\mathbb{R}^n \times \mathbb{R}^n$ $\theta_{4.2}$: The bijection of point set and tangent space $\theta_{4.3}$: The parallel transport by translations
Θ_{5a} : The cotangent bundle Θ_{5b} : The normal bundle Θ_{5c} : Submersions	$\theta_{5.1}$: The cotangent bundle $T^*\mathbb{R}^n$ and $\mathbb{R}^n \times \mathbb{R}^n$ $\theta_{5.2}$: The normal vector for a codimension 1 space $\theta_{5.3}$: The bijection of point set and cotangent space
Θ_6 : Ordinary differential equations	$\theta_{6.1}$: Constant vector fields $\theta_{6.2}$: Integral curves and velocity

Table 1: Theories and corresponding technologies for analytic geometry

The technologies mainly describe how the theoretical concepts are related to the analytic geometry of vectors. In the course of the textbook analysis, it was decided to involve scalar products for right angles and normal vectors because these are closely related to concepts of lines in the plane and planes in the 3-dimensional affine space.

Example analysis for the "logos" in school books. the introduction of vectors. Vectorial approaches in plane and spatial geometry have found their way into German-language textbooks since the days of New Math in the 1970s. Of course, they change continuously in their modes of representation and the models used. While in the beginning mainly arrow class models were used, models are increasingly appearing that place the translations in the center and those that use the arithmetic vector spaces \mathbb{R}^2 and \mathbb{R}^3 and only want to interpret them geometrically.

Books	Employed Technologies
Lambacher-Schweizer (Appel et al. 2018) Neue Wege (Körner et al. 2011)	$\theta_{2,1}$: Role of \mathbb{R}^3 as set of points $\theta_{2,2}$: Role of \mathbb{R}^3 as space of translations $\theta_{2,3}$: Role of \mathbb{R}^3 as set or space of position vectors $\theta_{2,4}$: Arithmetic of position vectors and translations
Lambacher-Schweizer (Baum 2001)	$\theta_{3,1}$: Equivalent class of ordered pairs $\theta_{3,2}$: Parallelism, length, orientation
Mathematik Oberstufe (Bürger et al. 1991)	$\theta_{2,1}$: Role of \mathbb{R}^3 as triple of number interpreted as points or arrows

Table 2: Starting point in the logos for vectors in textbooks

With the exception of Bürger et al. (1991) all approaches we examined use the concept of "position vectors", $\theta_{2,3}$. In connection with the description of straight lines and planes vectors describe the "position" and the "direction". If parameters have to be determined, they are sometimes called "bound" (to a certain object) or "free" even though these constructs do not appear in the technical mathematical description.

For reasons of space, we take an exemplary approach and look at the introductory theory of vectors and the introduction of straight lines in the widely used textbook Lambacher Schweizer (Appel et al. 2018). Prior to the section presented here there is a subchapter in which points in space are given coordinates using a Cartesian coordinate system. With respect to a given coordinate system, the set of points is identified with the \mathbb{R}^3 . A point P with its coordinates is given in the usual school form e.g., as $P(1|2|3)$ for example. The point belonging to the triple of coordinates $(0|0|0)$ is denoted with O as usual. We consider the introduction of vectors as shown in Fig. 3. Vectors are introduced as follows: A vector is first of all a number triplet that is to be understood differently than a number triplet that describes a point: The vector number triple operates on a point through displacements in the directions of the coordinate axes. Technically speaking, the vector \vec{v} is introduced as an element of the \mathbb{R}^3 , which operates on elements of the \mathbb{R}^3 , which are interpreted as point coordinates. This procedure can be easily represented in the standard affine model Θ_2 , see Table 1: the point space $\mathbb{A}_3 = \mathbb{R}^3$ and the vector space $T(\mathbb{A}_3) = \mathbb{R}^3$ has been introduced, $\theta_{2,1}$ and $\theta_{2,2}$ in Table 1, (the latter only as a set, still without vector space operations). The transitive group action of \mathbb{R}^3 on \mathbb{A}_3 , see $\theta_{2,b}$, has been introduced by an example (the vector \vec{v} operates on a point A giving a point D). For two points A and D the vector \overrightarrow{AD} is introduced as the displacement that maps A to D and is expressed in coordinates. Remarkably, the assignment of a pair A, B to the vector \overrightarrow{AB} receives its own notation. This corresponds to the notation in the standard model Θ_2 , and it is operationalized with the help of the entries in the point triples. In contrast, the operation of vector \vec{v} on point A giving point D – although introduced and central to affine spaces – does not receive its own notation here or throughout the book. Thus, supposedly obvious notations such as $\vec{v}(A) = D$ or also $A + \vec{v} = D$ are dispensed with.

Number triples can not only be understood as points, but also as a displacement, i.e. as a "route description" from one point to another. One speaks of a vector and writes the coordinates of the vector in column form, e.g. $\vec{v} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$.

The point $A(1|1|0)$ was moved with the vector

$\vec{v} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ by one unit of length (UL) in x_1 -

direction, by three UL in x_2 -direction and by 2 UL in x_3 -direction to the point $D(1+1|1+3|0+2)$, i.e. to $D(2|4|2)$. One also says that the point A is mapped by \vec{v} onto its image point D (also called A').

The vector \vec{v} also maps C to F and B to E . For the two given points A and D , the vector that moves A onto D is called \overrightarrow{AD} . One can determine the coordinates of the vector \overrightarrow{AD} from the coordinates of the points A and D like this: $\overrightarrow{AD} = \begin{pmatrix} 2-1 \\ 4-1 \\ 2-0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$. In the coordinate system, vectors are illustrated as arrows (directed lines). All arrows of the same size and parallel illustrate the same vector.

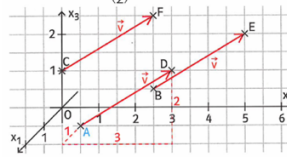


Figure 1: Introduction of vectors in Appel et al. (2018) (translation by the authors)

In the following chapter, the vector space structure for the set of displacements is established and in this context the notion of position vector is introduced, see Fig. 4.

Position vector. The vector \overrightarrow{OP} , which translates the origin O to the point P , is called position vector of the point P . It has the same coordinates as the point P . The position vector \overrightarrow{OP} is often referred to as \vec{p} .

Figure 2: Introduction of position vectors in Appel et al. (2018). (translation by the authors)

If we relate this to the theory of affine space, first of all a point O needs to be distinguished because in the theory of affine spaces all points are equivalent, there are no special points. This leads to the bijection $\iota: \mathbb{A}_n \rightarrow T(\mathbb{A}_n)$, $P \mapsto \overrightarrow{OP}$ which of course refers to the distinguished point O . Similarly, for a given translation vector \vec{X} one passes to the associated point $\iota^{-1}(\vec{X})$. For this, \vec{X} is applied to O , for which, a general notation is dispensed with, a special designation is introduced which depends on the use of the distinguished point O : The vector \vec{X} has as its number triplet the same coordinate entries as the vector \vec{X} and is therefore simply denoted by X . The equation between two points $B = A + \overrightarrow{AB}$ can only be described by the vector equation $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$. The graphical representation of the simple situation $B = A + \overrightarrow{AB}$ in where the point O has no role, towards the representation of the vector addition becomes more elaborate, see Fig. 5. Further comparisons of more complex situations can be found for example in Malle (2005).

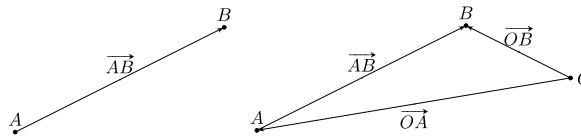


Figure 3: Representations of the displacements from A to B with vector-on-point operation and with location vectors

In summary, the concept of position vectors replaces the concept of vectors operating on points. This is possible from a technical mathematical point of view, but it is in no way necessary and does not seem "natural" from the point of view of the underlying theories. The construct of the position vector is widely criticized, in particular in Malle (2005). There it is associated with the arrow class model,

Θ_3 , and the proposed solution is that no geometric vectors and vector spaces are considered, but the operations in the standard vector space \mathbb{R}^n are interpreted geometrically in a variable way. The position vector is not necessary for the arrow class model or the translation model and the operation of vectors on points. This belongs to the core of affine geometry and, on the other hand, there is the danger that "ungeometrical point additions" are considered: How to justify that additions of points such as $\frac{1}{3}(A + B + C)$ makes geometric sense, but the expression $\frac{2}{3}(A + B + C)$ not?

Example analysis for the practice block: the "octahedron of horror". The following final school exam assignment was posed 2008 in the federal state of North Rhine-Westphalia in Germany. The press referred to it as the "Octahedron of Horror".

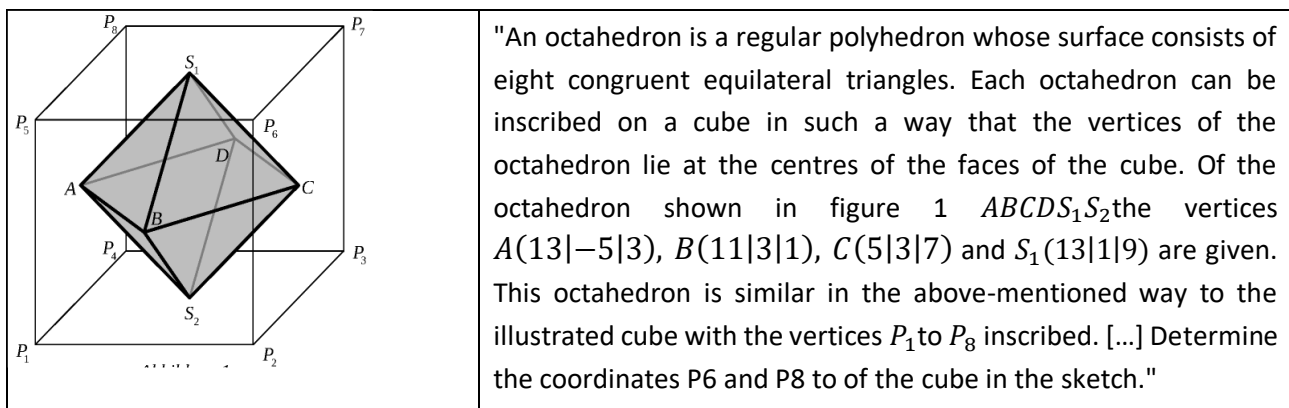


Figure 4: The "Octahedron of Horror" (translation by the authors)

For this an a priori analysis for a solution of the task "Determine the coordinates of the corner points and of the cube in the sketch" using the models for vector geometry is to be created. For this analysis we use the tangential bundle model Θ_4 and distinguish whether parallel transport $\theta_{4,3}$ is used or not. In the model using parallel transport \overrightarrow{AB} can be shifted to $\overrightarrow{S_1 P_6}$ and $P_6 = S_1 + \overrightarrow{S_1 P_6}$ can be determined since A, B, S_1 are given. In a similar way, P_8 can be computed.

The much-practiced approach via the intersection of planes leads to the bounding planes of the side faces of the cube through the points P_2, P_5, P_6 resp. P_2, P_3, P_6 or P_5, P_6, P_7 . This leads in the tangential model to three parameter shapes, for which with a similar effort as above – only sixfold – spanning vectors have to be determined, or cotangentially – with threefold effort – normal vectors for the normal forms. The intersection of two of the planes yields an intersection line, which in turn is intersected with the third plane to obtain the coordinates of P_6 .

Similar considerations could be made in the translation model and distinguish whether the vector is applied to the point S_1 , which leads to a fast solution, or not.

Outlook

The analyses illustrate that vectors appear in various contexts. There are several, well-tried concepts for the introduction of vectors, but all have side effects if other approaches are neglected. Even though the mathematical theories might seem at first glance exaggerated, they make underlying concepts visual because they describe the way in which vectors are actually used. In retrospect, we regard the

analyses across the institutions as fruitful to obtain insights into the underlying concepts. It remains a challenge to differentiate more between the institutions of schools and universities in the analyses.

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