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The role of representations in developing a conceptual understanding of trigonometry

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The paper presents first results of the first cycle of a design research project on exploring and developing a learning environment that aims to provide learning opportunities for a conceptual understanding of trigonometry. Following Duval's theory of registers, learning processes are investigated regarding representational transformations to understand the constitution of individual mathematical objects. Right triangle representations from the literature and German textbooks are investigated that reduce complexity and should therefore support learning processes from a theoretical perspective. With the help of a newly developed analysis tool, the importance of these representations in a learning process is revealed.

Keywords: Trigonometry, geometric concepts, representation, design research, learning processes.

Introduction

From a mathematical perspective, trigonometry can be seen as a subject where functions, geometry, and algebra interconnect. For example, Wittmann (1987, p. 356; translated by the first author) described it as the “algebraisation of the congruence theorems”. This thematic complexity must be given particular attention in the classroom so that students develop sound mathematical concepts. Developing a conceptual understanding of trigonometry also involves understanding and developing various representations of trigonometric-related mathematical objects.

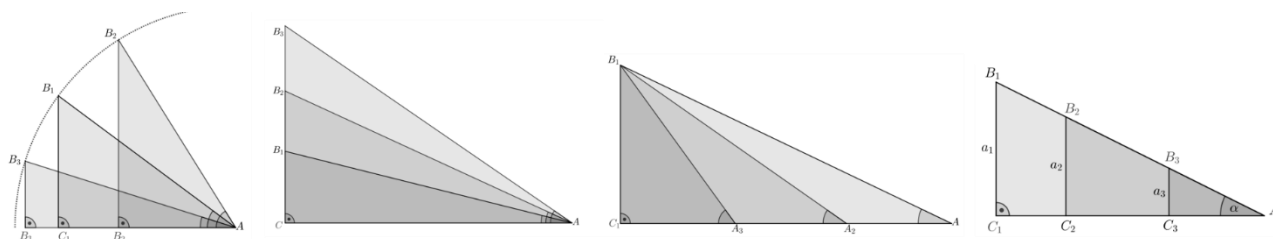


Figure 1: Typical graphical representations which involve a reduction of variability

The graphical representations shown in Figure 1 are a collection of figures suggested by research in mathematics education (e.g. Wittmann, 1987) and German textbooks for developing a conceptual understanding of trigonometry. In the first three configurations, the lengths of the hypotenuse, the adjacent, and the opposite are kept constant, while the fourth configuration shows similar triangles, i.e. the angles are equal. From a theoretical perspective, these graphical representations should support learning processes because they enable the application of the operative principle (Wittmann, 2021). Indeed, they provide a reduction of variability, thus reducing the complexity of the possible variations of a right triangle. Watson and Mason (2006) also suggest that different similar triangles should be analysed so that the invariance of the trigonometric ratio for similar triangles can be understood (4th representation in Figure 1). Following Marton's theory of variation (Marton & Tsui, 2004), it is particularly necessary to also vary the aspect that is to be experienced. In this case, it is

the angle, while other quantities within the triangle should remain constant (1st-3rd representation in Figure 1). In this paper, we investigate to which extent these theoretically deduced graphical representations support students' concept development by answering the following research question: Which role do the representations from Figure 1 play in the learning process during students' first encounter with trigonometry?

Theoretical background: Duval's theory of registers

How can learning processes with regard to the presented thematic complexity and focus on representations be empirically investigated? Duval's theory of registers (Duval, 1999) provides a suitable theoretical basis for this task since it focuses on *semiotic representations* and their production:

Those [semiotic representations] that are intentionally produced by using any semiotic system: sentences, graphs, diagrams, drawings... Their production can be either mental or external. . . . [T]he content of the representations denotes the represented object: it is an explicit selection because each significant unit results from a choice. (Duval, 1999, p. 5)

For classifying transformations of semiotic representations, Duval (1999) introduces the notion of *registers*: A register is a producer of specific semiotic representations. For these semiotic representations, the register provides specific cognitive operations that elaborate the meaning of objects by representational transformations. The operations within registers must fulfil two functions: a symbolic and a transformational function (Arzarello, 2006). The transformational function contains two types: transformations within a register (treatment) or between different registers (conversion). "The symbolic function refers to the possibility of interpreting a sign within a register, possibly in different ways, but without any material treatment or conversion on it" (Arzarello, 2006, p. 272). An *object* is thereby defined as the invariant of all its possible representations (Duval, 2017). Since a transformation of representations reveals new properties of an object and these transformations depend on registers, objects are significantly shaped by the registers in which they are transformed (Duval, 2017). As will also be exemplified in the empirical section, the concepts of semiotic representation and transformation are key for understanding learning processes:

The classification of semiotic representations into different kinds and analysis of the semiotic transformations specific to each kind are therefore the crucial issues for the cognitive analysis of mathematical activity, i.e. for the processes of understanding and reasons behind non-understanding in the learning process. (Duval, 2017, p. 45)

The very abstract definition of registers is exemplified in the following for the case of the verbal, graphical, and numerical registers, as these are relevant for later analysis. Since Duval did not refer directly to trigonometry in his work, we present adapted versions. The *verbal register* involves three key operations with a symbolic function: Designation of objects, enunciation of objects, and discursive expansion (Duval, 2017). The designation of objects aims to establish uniqueness in naming. Designation can be done, for example, by deictic, everyday, academic, or technical expressions. The enunciation of objects establishes connections between the designated objects by relating two objects. In addition, the pragmatic value of the statement, such as questioning, confirming, or verifying, may be brought out. The discursive expansion organises the sequence of

sentences into a coherent whole according to the underlying intention. The verbal register plays a particularly important role as “[the three operations] are located at the exact point where knowledge, understanding and awareness – and, therefore, cognitive condition to develop knowledge – are inseparable” (Duval, 2017, p. 54). Operations with a transformational function within the verbal register are, for example, paraphrasing or verbal references to graphical representations. The symbolic function of the *graphical register* is about iconic and non-iconic representations (for a larger discussion see Duval, 2017). An example of a transformational function is dimensional deconstruction (Duval, 2017): In order to refer to certain components of a triangle, such as sides or vertices, the two-dimensional triangle must be decomposed into its one-dimensional (sides) and zero-dimensional (vertices) components. Only then is it possible to work with the representations of smaller dimensions. Another transformational function is the static or dynamic movement of figures or parts of them. An example of a symbolic function in the *numerical register* is the following: When looking at the length of a side, the focus can be on the number or the unit. If both aspects are recognised at the same time, it is called size recognition. Numerical operations with a transformational function are, for example, computational or rounding processes.

The presented empirical research aims to investigate learning processes concerning the constitution of objects and their meaning. Duval’s theory of registers provides a theoretical basis for this purpose, but an analytical tool still needs to be developed which makes representations, transformations of representations, registers, and the individual constituted objects identifiable. For this reason, a second research question is added: How can learning processes during students’ first encounter with trigonometry be investigated based on Duval’s theory of registers?

Research context

The presented analysis is part of a larger project on developing and investigating the teaching and learning of trigonometry that is conducted in the tradition of design research (Gravemeijer & Cobb, 2006). The overarching aims are to develop a learning arrangement for students’ first encounter with trigonometry in German grade nine classrooms with a focus on conceptual understanding and to build local theories on the students’ learning processes.

Annika wonders what the meaning of the % on this sign is.

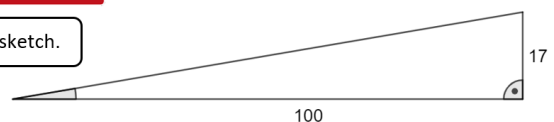


Ben, what does 17% stand for? Is that supposed to be 17°?



The 17% probably has something to do with the slope. Look at my sketch.

1) Can you explain Ben’s sketch to Annika? Does the 17% correspond to 17°?



But I don’t understand why it doesn’t just specify the height of the mountain. The higher the mountain, the steeper the road.

2) How would you answer Ben?

3) Annika and Ben have drawn more triangles on a sheet. Which one shows the largest slope, and which one is the smallest?
Explain to each other how you can recognize this.

Figure 2: First tasks of the learning arrangement (translated by the first author)

Figure 2 shows the first three of the ten tasks of the learning arrangement from the first cycle which are also the tasks that the students are working on in the presented empirical insights. The different

ways of describing the slope by angle or tangent are intended to provide first access to trigonometry. The everyday context of interpreting a common street sign in Germany was chosen because of its authenticity, accessibility for the learners, and the possibility of the variety of right triangles that can be produced. Representations from Figure 1 were not directly given in any of the tasks because we wanted to investigate if the students create these representations on their own during their exploration.

Methods of data collection

A total of 16 upper secondary school students from four different eighth grade classes in a city in central Germany participated in the first cycle of design experiments. At the time of the study, the topic of trigonometry had not yet been covered in students' regular mathematics classes. Students worked in pairs for around 90 minutes on the provided learning materials. The first author was present during all sessions but intervened as little as possible in the learning processes. All sessions were videotaped and students' written products were collected.

Methods of data analysis

A qualitative analysis pattern (Prediger & Wessel, 2013) was applied to answer the research question. The first step contains a basic interpretation in the form of a sequence analysis that aims to describe the individual and common processes of constructing mathematical meanings and developing a holistic understanding of the data. The second step contains an analysis with a special focus. Here, the focus was on the individual constitution of representations based on Duval's theory of registers. The analysis aims at identifying the registers in which the different objects were transformed. Therefore, a multitude of individual representations of a mathematical object is obtained and the type of representational transformations is specified. To identify the representations and registers, transcripts and written products were examined in three steps.

First, the transcript is examined at the expression (but not the whole sentence) level. Thereby, technical, academic, and everyday language expressions, as well as gestures, deictic expressions and references to the tasks that refer to certain mathematical objects are noted. In combination with the sequence analysis, initial conjectures are made about which objects and registers are addressed. The inclusion of the sequence analysis is essential in this step so that all object references can be found at the expression level. Utterances that do not show potential object references are deleted to reduce complexity. In the second step, the individual transcript lines are read as a whole on the turn level and their pragmatic value is attributed to them. In this way, uncertainties or reinforcements of the assignments of the previously posed objects and registers become transparent. Furthermore, it is analysed whether the identified objects are causally or logically related. The goal of this step is to reach certainties about the previously postulated objects and registers and to make transformations of registers transparent. Finally, the discourse level is analysed. This means that the communication is examined concerning the question "What function does the turn have regarding the whole discussion?". In this way, conclusions can be made about the intentions of the transformation of representations. For a better understanding, the steps of analysis are illustrated by the fictional transcript excerpt: "I think that these [*points at the graphical representation of the angle*] are not 17. What do you think?". At the expression level, the deictic expression "that" and "17" are relevant. Based on the pointing gesture, "that" addresses the geometric representation of the angle. This is a

recognition in the graphical register, although it is not clear which exact operation (cf. theoretical background) is used. Thus, pointing supports the conversion between the graphical and the verbal register. “17” indicates the recognition as a number in the numerical register, but a size recognition could also have taken place in conjunction with the pointing gesture. Both ambiguities could be clarified by going back to the sequence analysis. Furthermore, both representations probably refer to the object of the angle, which was thus processed in a verbal, geometric, and numerical register. At the turn level, a hypothesis (“I think”) and a question (“What do you think?”) are formulated. In the hypothesis, the incompatibility of the numerical register of the angle as an object in conjunction with the “17” is expressed (“not”). However, the subsequent question expresses some uncertainty about this rejection, so this register assignment could still change. Since the example only covers one turn, the discourse level cannot be further exemplified here.

Empirical insights

The following transcript excerpts illustrate the construction and transformation of representations of a learning group during a session. The interpretations are the final result of the analysis steps presented above. In the transcripts, # marks the starting point of overlapping speech; - an abrupt termination; ././... a 1/2/3sec. pause; [*xsec.*] a xsec. pause; [*comment*] marks other comments.

1st transcript excerpt: Constitution of the slope as an object

The students Lisa and Anja read the first task from Figure 2 aloud, then correctly measured the 10° angle marked in grey in the presented triangle using a set square. Lisa posed the central question: What does the sketch have to do with 17 degrees?

- | | | |
|-----|------|---|
| 021 | Anja | And I don't understand what the 17 . oh so they mean the height |
| 022 | Lisa | That is the [<i>points at the traffic sign</i>] 17 percent |
| 023 | Anja | But then that shouldn't be there [<i>points at the opposite or the number 17 in Ben's sketch</i>] |
| 026 | Anja | Maybe it's in metres . so 17 but he means well ... whether it's enough if you specify the height . maybe that's [<i>points at opposite</i>] the height and that [<i>points either at the presented angle or the hypotenuse</i>] somehow the measured- [<i>performs a kind of folding movement with forearm</i>] |
| 027 | Lisa | angle |
| 028 | Anja | [<i>nods</i>] path |

The excerpt shows the first attempt of the students to constitute the object of the slope. This is initiated by Anja's attempt to answer the question by attributing “17” as a “height” (line 21). Lisa replies that it is “17 percent” (line 22), which opens up a new problem for the students: To which graphical representation can a percentage be attributed? Anja tries to solve this problem in line 26 by processing the 17 in Ben's sketch as the size “17 metres” and by attributing it to the opposite. This means a back-and-forth transformation between numerical and graphical registers. Processing the 17 percent turns out to be more difficult: Anja addresses a new object by pointing at the angle or hypotenuse and additionally gesturing with her forearm. An indication that she intends to assign the 17 percent to the graphical representations she pointed to second (line 26) is given by the use of the word “measured” (line 26), which addresses a numerical register. At first glance, it seems strange for Anja to affirm Lisa's answer “angle” (line 27) with a nod and at the same time uses the word “path” (line 28) for it. Going back to the sequence analysis, only one reading seems to make sense: Anja constitutes a new

object by identifying it with 17 percent in the numerical register, representing it by the angle or the hypotenuse in the graphical register, and addressing it with the word “path” in the verbal register. The use of the word “path” (line 28) can be explained by the fact that the ascent is phenomenologically known to her through different steep paths, which is also congruent to the gesture used in line 26. The folding movement of her forearm is meant to symbolise different steep slopes. The word “path” belongs to the graphical representation of the hypotenuse and indicates an iconic interpretation of the slope. But by also referring to the graphical representation of the angle (“[nods]”, line 26), the slope is also interpreted in the graphical register in a non-iconic way. The transcript excerpt shows that the (pre)object of the slope was constituted by a variety of transformations between different registers, as postulated in the theory.

2nd transcript excerpt: Using graphical representations for further constitution of the slope

Before entering the following excerpt, both students were observed several times quantifying the slope using a percentage within the numerical register and representing it in the graphical register using an angle. Both students stated that the slope of the mountain has nothing to do with its height after working on the second task.

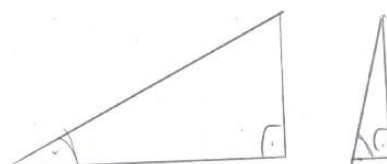


Figure 3: Anja’s drawings for task 3, on the left DA1 and the right DA2,

- 077 Anja *[directly after task 3 was read out]* I would say the wider the mountain is ... the less the slope is because if you now have two times three centimetres *[draws opposite of DA1 and DA2]* then you have
- 079 Anja yes it’s crooked now doesn’t matter ... then this here is this long *[draws adjacent of DA1]* but here *[draws adjacent of DA2]* that long #
- 081 Anja both are three centimetres *[points at DA2]* but here the slope is # *[draws hypotenuse of DA1 and DA2]*
- 082 Lisa #then it actually depends on the height *[points at opposite of DA1]* where they had 17 *[points at the triangle of the task sheet]*
- 083 Anja yes so the height has again a little something to do with it
- 084 Lisa uhm yes because then also the angle changes *[points at the angle of the triangle from the task]*
- 085 Anja yes the alpha here *[points at the angles of DA1 and DA2]*
- 086 Lisa or it depends on the angle then the height changes if *[points at angle and opposite cathetus of the triangle from the problem]* ... no no no that makes no sense
- 087 Anja I think it depends on the angle *[points at the angles of DA1 and DA2]* ...
- 088 Lisa actually more on the height of the side *[points at the opposite of the triangle from the problem]*
- 090 Lisa *[7sec.]* on both
- 091 Anja yes well yes well I think it depends on how wide it is
- 092 Lisa *[5sec.]* which is the biggest slope and which is the smallest?

The analysis of this excerpt focuses on Anja’s success in making connections between the new object of slope and previously known objects using the semiotic representation in Figure 3. Anja uses a qualitative the...the...statement, in which the opposite acting as a mediating variable, i.e. the opposite, is kept constant while the adjacent and the slope are varying. In particular, the operational

principle (Wittmann, 2021) is used by Anja. In technical language, this could be described as “The longer the adjacent line, the larger the slope, as long as the opposite line remains constant in length”. Anja could not verbalise a full technical description; however, she manages to paraphrase the statement’s content through her drawing and using associated pointing gestures (lines 77–81). She addresses the object of the adjacent through a reference to reality (“wider the mountain”, line 77) and the opposite is processed in the graphical register within the drawing process. On the one hand, the use of different registers and their transformation into other registers (e.g. the adjacent was first described verbally and then processed (drawn) in the graphical register) could indicate a successful constitution of meaning. On the other hand, it could also indicate an overwhelming cognitive effort that would occur when trying to formulate such a complex statement. The drawing and the use of pointing gestures in connection with the produced semiotic representation could have had a cognitively relieving effect. While Anja switches very fluidly between the verbal and the graphical register (line 83, line 87, line 91), she can correctly describe the effects of the different variables on the slope.

Discussion

The analysis of the first transcript shows that it is possible to describe learners’ processes of constituting meaning with the presented analysis tool. The strength of the approach lies in the separation into different registers, which makes it possible to capture the different characteristics of the individual processes of meaning-making. Both students showed viable mathematical approaches, and problems in the learning process were also revealed. Anja’s phenomenological approach to the slope of paths led her to think about a mathematical object of the slope. The intuitive association of the slope with the graphical representation of an angle, which is also articulated later by Lisa, builds a strong foundation for grasping the meaning of trigonometric relations. Thereby the variety of different representations and their transformations within and between different registers were crucial for the process of constitution. The variety of semiotic representations from different registers should be considered in further developing the learning arrangement in future design cycles.

What role did the graphical representations from Figure 1 play in the learning process? Anja’s drawing (cf. Figure 3) corresponds to the third representation shown in Figure 1. However, Anja only drew two triangles and displayed them side by side. This kind of representation could be even more conducive for the initial access to the subject since recognising and comparing the two separate triangles could be easier. Here, Anja’s representation served two functions: On the one hand, the visuality made pointing gestures possible and thus relieved the verbal discussion. This was shown by the very complex the...the...statement, which could not be completely verbalised and had to be supported by pointing gestures. On the other hand, by keeping the opposite constant, the number of variables to be considered was reduced, which was suspected in the theory: The change of the adjacent, hypotenuse, or angle of the slope only affected the two other quantities. This made it possible for Anja to compare the slopes of both triangles (line 81). Lisa might also have a mental representation in the sense of the second representation from Figure 1 available because she correctly recognised the effect by the variation of the opposite on the angle of the triangle (cf. line 84). But the graphical representations are not enough for a successful learning process: In the discussion, Anja’s graphical representation formed a symbiosis with the complex the...the...statement. This statement

developed based on the graphical representation, but also led to the consideration of variations of certain quantities, as the statement was to be verified in the discussion (lines 84–91). The simultaneous consideration of graphical (and numerical) representations and verbal expressions also shows that Duval’s theory of representation works well in bringing these domains together but allows one to describe them separately. However, the impact of gestures (e.g. line 26 and pointing gestures) and phenomenological considerations about the learning process shouldn’t be underestimated. Here Arzarello’s holistic and multimodal approach in the form of semiotic bundles (Arzarello, 2006) could contribute to further insights.

In the other groups, the impact of the graphical representations from Figure 1 was very low and in some cases misleading: In one group, a drawing corresponding to the second graphical representation was produced, which only slightly supported the learning process and led to incorrect findings concerning the second task of the learning material. Thus, future learning arrangements should focus on enforcing and supporting engagement with such graphical representations. But learners need more guidance in order for them to develop sound concepts.

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