An introduction to TWG4: Geometry teaching and learning

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The TWG4 "Geometry teaching and learning" focuses on studies related to geometry instruction, ranging from early childhood education to higher education, including the professional development of teachers. In CERME13, we sought to advance themes raised in previous ERME conferences, including, but not limited to, specific aspects of mathematical activity in geometry (visualization, interrelation between visual-spatial and formal aspects of geometry, problem-solving, proving, the role of diagrams and models); geometry curriculum in different countries (textbooks, didactic situations, tasks, activities, and competencies); teacher education in geometry (contexts, practices, challenges, and perspectives). Along the traditional themes, the call for papers for the current conference reflected growing research interest in dynamic geometry environments (DGE, emphasizing geometric components). It also encouraged contributions focusing on teaching non-Euclidian geometry and adjacent branches of mathematics, such as trigonometry and vectors.

Our group received 23 contributions (21 papers and two posters), showing a growth of interest compared to the last (virtual) CERME12 (14 contributions). Nineteen of the submitted papers and both posters were accepted for presentation. Of those, eighteen papers will be presented in the Proceedings. Approximately 30 scholars from various regions – Western, Eastern and Central Europe, the Middle East, Asia, and South and North America – engaged in TWG4's sessions. This diversity was also reflected in the constitution of the co-leaders team: Israel, Portugal, the United Kingdom/ Japan, Germany, and the Czech Republic. The group's activities were organized in seven sessions following the main themes of the contributions. The themes of *Geometry in teacher education*, *Geometric reasoning and problem-solving strategies*, and *Curriculum and textbooks* were addressed in separate sessions. Three sessions were dedicated to different forms of *geometry teaching and learning with tools* (physical and digital), with emphasis on the interrelation between different forms of tools. Most of the sessions were of the same format: a short presentation of (three to four) contributions, participants' reactions, and a joint discussion of the raised themes.

During the resuming session, the participants decided to concentrate and summarize the work of the group along the four following themes:

- The role of tools and geometric activities with a combination of different tools
- Networking theories and negotiation of meaning for basic theoretical notions
- Research-inspired teaching and learning geometry in the 21st century
- International vs. local curricula for Geometry

The role of tools and geometric activities with a combination of different tools

The group discussed this topic based on nine papers. The range of tools that were explored spanned from traditional Geoboards (Gambini & Viola), clay and skewers models (Bimova et al.) to novel and gaining popularity 3D sketching and printing (Surynková) and dynamic geometry environments (Papadaki). Note that all these contributions investigated non-primary school students. Instead, the participants were middle- and high-school students and pre-service teachers (PST).

The findings indicate that different tools and their combinations can help students facilitate problem-solving, otherwise less approachable (Sua et al.; İmamoğlu et al.) This concurs with the work of Bimova et al., who explored whether students need support tools in tangible or virtual 3D geometry models to solve spatial geometry problems. Their findings highlight the importance of finding appropriateness and correspondence between the model medium and problem types. Indeed, various tools supporting geometry teaching and learning are beneficial in affording students to use more senses, create different points of view, and enable different ways of figural apprehension (Palatnik; Papadaki). There is an interrelation between the geometric tasks and tools that are used for teaching. For instance, DGE can be utilized for fostering deductive reasoning and not only for the exploration of properties, e.g. for making constructions (İmamoğlu). Sua et al. explored problem-solving activities in DGE featuring robust and soft (i.e., not preserved under dragging) constructions. Creating a tangible model and introducing an auxiliary element in real time may influence how students apprehend 3D geometric objects (Palatnik). In this respect, the findings of Vízek and Samková stand out, since in their study students reflected more on geometric characteristics of the activities they engaged into, rather than on their dynamic environment design.

Most findings in the papers were related to tools' influence on the cognitive domain of geometry instruction. Furthermore, authors (e.g., Surynková on students' work with 3D printed tiling) indicate that using tools (manipulatives) support students' motivation and engagement in learning. It is suggested that emotion, body movement, and discourse should receive more focus in research related to TWG4. Our group discussions raised several points related to the affordances and constraints associated with particular manipulatives. The participants suggested that using physical tools can be beneficial and should not be limited to primary school. However, we expect that in different grades spatial and formal aspects of geometry will interrelate differently through using tools. Another issue that was discussed was the role of the teacher as designer/adaptor of tools, as initiator, or facilitator of inquiry, and the role of the students as designers vs only users of the tools.

Networking theories and negotiation of meaning for main theoretical notions

The theoretical landscape of the TWG4 remains diverse and pluralistic. Many participants find various aspects of Duval's work, particularly figural apprehension and transition between different types of visualization (iconic and non-iconic) helpful in understanding and characterizing cognitive processes related to geometry learning (e.g., Aydinyer & Ubuz; Büttner & Erath; Gambini & Viola; Palatnik; Papadaki). Van Hiele (1986) levels remain an important but less frequently used reference point. For instance, Arnal-Bailera and Manero devised a questionnaire to assess the acquisition of the fifth Van Hiele level concerning the definition process.

In most of the contributions, some forms of networking theories (Bikner-Ahsbahs & Prediger, 2014) can be found. In particular, some elements of local geometry theories are combined or synthesized with more general ideas from mathematics education and cognitive sciences. For instance, to assess students' geometric reasoning, Rubio-Sánchez et al. combined Duval´s (1995) model of cognitive apprehensions and Mason´s (2008) attention structures model. In work on geometric problem solving, Proulx used theoretical ideas of enactivism, Palatnik drew on embodied learning, and Papadopoulos and Ventistas focused on the range and quality of control strategies.

In the group discussions, the participants indicated a need for a more coherent understanding of theories from the community (e.g., various aspects of Duval's theory). The participants suggested undertaking collective work, creating a "dictionary/guide" of the main background and field theories and their concepts. Such a tool could help researchers to get a coherent overview of the local theories and field of research on geometry teaching and learning.

International versus local curricula for geometry

Our group's contributions provide an exciting opportunity to be introduced to aspects of diversity of geometry curricula in different countries. There are various approaches to curricula: practice-based, theoretical, proof-based, etc. The overall feeling is that in some cases instead of dealing with challenges presented in geometry instruction, the geometry part is being reduced in the curriculum.

In her research, Kuzle scrutinized the German national curriculum and a combined curriculum from two German federal states for primary mathematics education. She evaluated the diversity and coherence of geometric topics by applying the fundamental ideas of the geometry framework (Schweiger, 2006). While Kuzle's findings indicated an acceptable level of topic diversity, they also highlighted issues with curricular coherence. This approach can be extended for comparative analysis of curricular frameworks in the geometry domain. It is noteworthy that the inventory of fundamental geometric ideas and their definitions is subject to scrutiny and modification. Educational resources remain an important area of curriculum studies. Büttner and Erath's work took right triangle representations from German textbooks as a starting point. It was the first cycle of a design research project to explore and develop a learning environment aiming at a conceptual understanding of trigonometry following Duval's (2017) theory of registers.

The group discussion highlighted the following points: There is a need for an international study surveying and characterizing geometry curricula in different countries. Several approaches can be used (e.g., fundamental ideas—Kuzle; anthropological theory of the didactic, as in the work of Bauer and Halverscheid). What do different stakeholders perceive as the teaching of geometry for 21st-century curriculum? What tools can be used in 21st-century curriculum?

Research-inspired teaching and learning geometry in the 21st century

The questions of applicability, impact, and relevance of research conducted by the TWG4 participants to teaching and learning geometry in the 21st century were central to almost all discussions. These themes were prominent in studies of pre-service teachers (PST). Many studies took an active and design-based approach. For instance, Vargas et al. raised a question of establishing connections between geometry and real-life contexts for PST and, in general, of understanding geometry as a way

of understanding the world. Imamoğlu et al. tested the PST course designed following the Geometric Working Spaces model (Kuzniak, 2018). Bretscher and Geraniou explored teachers' mathematical digital competency by analyzing responses to a survey item in which they had to manage rounding errors using a dynamic geometry app. One of the most exciting discussions was related to a question about the explicit inclusion of digital tools in the curriculum, linking to the question of what geometry is for the 21st century. The introduction of free software supported by thousands of enthusiasts producing and modifying instructional resources (e.g., GeoGebra) is unique to the realm of geometry compared to other mathematical domains (e.g. arithmetic, probability).

We conclude this introduction with several questions indicating possible venues for future research: Does introducing and developing novel tools (such as extended reality, DGE, and tangible models in new mediums) provide new ways to enact and perceive geometry? How do new findings in cognitive and neurosciences influence the ways of geometry instruction? In particular, how can teaching and learning geometry be connected to embodied cognition and enactivist approaches – especially in 3D geometry (e.g. 3D haptic tools)? How do spatial-visual and formal aspects of geometry evolve across the grades? Should school geometry be grounded in the Euclidean world? How can the results of our research reach our curricula and our classrooms and influence the teaching and learning of geometry? How can teachers be supported in choosing appropriate tools and in designing and/or implementing tasks for their learners? How can the findings from various countries be synthesized to develop a more efficient approach to geometry teaching and learning?

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