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# Do students' ways of solving linear equations influence their later mathematical learning? Evidence from a study of Swedish upper secondary students

Tyge Sjöstrand<sup>1</sup>, Jörgen Niklasson<sup>1</sup>, Christos Kyprianou<sup>1</sup>, Attila Szabo<sup>2</sup> and Paul Andrews<sup>2,3</sup>

<sup>1</sup>Kungsholmens Gymnasium, Stockholm, Sweden; [tyge.sjostrand@edu.stockholm.se](mailto:tyge.sjostrand@edu.stockholm.se)

<sup>2</sup>Stockholm University, Stockholm, Sweden

<sup>3</sup>VIA University College, Aarhus, Denmark

*In this paper, by means of a mixed methods approach to data collection and analysis, we examine the impact of linear-equations-related competence on Swedish upper secondary students' later mathematics achievement. Quantitative data were 307 student solutions to an equations-related word problem posed at the start of their programme and the same students' terminal grades to the first compulsory general mathematics course. Qualitative data were the written solutions to the word problem and interviews undertaken with purposively selected students. The results showed that students who solved the problem algebraically were not only confident in their solutions but significantly more successful on the terminal assessment than those who solved it by other means. Moreover, students who approached the word problem algebraically but solved it incorrectly, were no less successful than those who solved it successfully by other means.*

*Keywords: Algebra, linear equations, Sweden, upper secondary school, mathematics achievement.*

## Introduction

For the last quarter of a century researchers have been evaluating the widespread belief that algebra in general (Dougherty et al., 2017) and linear equations in particular (Capraro & Joffrion, 2006) stand as *gatekeepers* between school mathematics and higher education and employment. In particular, the topic of linear equations offers early opportunities for students to connect arithmetic to the symbolism of formal mathematics (Wasserman, 2014). Broadly speaking, the literature on equation solving makes an important distinction between two forms of equations. On the one hand, arithmetical equations, which have an unknown on one side of the equation only and can always be solved by operation reversal, require little algebraic insight other than recognising the unknown. On the other hand, algebraic equations have unknowns on both sides, cannot be solved by such approaches, and require algebraic insights (Andrews, 2020). Historically, algebraic equations have been solved by one of two methods. The first, a 'swap the side swap the sign' (SSSS) procedure, is based on the transposition of terms from one side of the equation to the other. The second, conceptually underpinning a 'do the same to both sides' (DSBS) procedure, relies on operations undertaken on both sides of the equation simultaneously (Caglayan & Olive, 2010; Vlassis, 2002; Wasserman, 2014). The former requires students to have no more than an operational understanding of the equals sign, while the latter requires a relational understanding (Alibali et al., 2007). Didactically, the former has been criticised for masking mathematical understanding (Star & Seifert, 2006), while the latter appears the preferred approach of many European teachers (Andrews & Sayers, 2012; Marschall & Andrews, 2015). In sum, the literature seems to suggest that linear equations stand as a gatekeeper to

later learning, and that being able to solve algebraic equations draws on a ‘do the same to both sides’ approach, then the aim of this paper is to examine the veracity of this implication in the Swedish context.

### Swedish students’ linear equations-related competence

Recent studies have shown that the linear equations-related competence of Swedish students, at all levels of the system, is variable. For example, the ministry of education periodically releases details of individual national test items undertaken by grade 9 students shortly before the end of compulsory school. For example, in 2010, 67% of all students solved  $13 - 3x = 7$  successfully. In 2013, 86% of all students solved  $\frac{1}{2}x + 1 = 5$  but only 30% solved  $2(x + 1) = 5 - 2x$  correctly. Also, several thousand Stockholm students transfer annually from compulsory to upper secondary school (hereafter USS) and, at the start of the new school year, undertake a diagnostic test before receiving any teaching. Over the ten-year period 2012–2021, the success rates for solving  $2x + 7 = 25$  varied little, from a minimum of 84% in 2014 to a maximum of 89% in 2019 and 2021. Over the same period, indicating a general improvement, the success rates for solving  $5x - 8 = x + 3x - 8$  rose from a low of 30% in 2012 to a high of 48% in 2021 (Sweco, 2021). Overall, these studies show that while students can typically solve arithmetical equations, their competence with algebraic equations is limited.

However, both these studies report only answers and pay no attention to students’ solution strategies, an issue, developed in the following paragraphs, that Andrews and his colleagues have sought to address. For example, in a group interview study of 39 USS students, undertaken some months after the completion of the first compulsory mathematics course, Andrews and Öhman (2019) presented students with the algebraic equation shown below alongside a solution written with no annotations to indicate the hypothetical solver’s thinking. Students were asked to consider, as a group, how they would help a friend who had been absent when such equations had been introduced. The equation, written here over one line and not four, was

$$x + 5 = 4x - 1; \quad 5 = 3x - 1; \quad 6 = 3x; \quad 2 = x.$$

The study found that all students recognised how the equation had been transformed and typically discussed the nature of the missing value or unknown. Further, nine of the twelve groups volunteered a DSBS narrative to support their friend’s learning, while the remaining three abandoned an initial SSSS perspective in favour of a DSBS due to concerns that a reliance on the former was likely to impede their friend’s understanding. Also, albeit interviewer-prompted, students construed the balance scale as a powerful metaphor for underpinning a conceptual understanding of the equation solving process. Later, Andrews (2020) analysed first year primary teacher education students’ written responses to the same task. Undertaken prior to any mathematics teaching, students were asked to write an explanation for their absent friend. Unlike the USS students discussed earlier, many of these drew on experiences from many years earlier. Indeed, the median of the 156 participants was 23 and the oldest was 48. In other words, exposure to linear equations was unlikely to have been recent for the majority of participants. Andrews (2020) found most students expressing clear objectives concerning identifying the value of the unknown. Moreover, while around a third wrote a narrative referring to SSSS, twice that proportion wrote something attributable to DSBS. Further, two fifths of those who wrote about SSSS, referred initially to DSBS. In sum, few students, including

those who had no exposure to equations for several years, did not have a clear understanding of the purpose of equation solving and a procedure underpinned conceptually by DSBS.

Collectively, the above studies show that at around the transition between compulsory and USS, students have a secure understanding of arithmetical equations and an emergent, but not yet secure understanding of algebraic equations. By the time they leave USS and for some years afterwards, the majority has not only acquired but recalled a conceptually robust understanding of algebraic equations sufficient to recognise and provide a DSBS narrative for its solution. However, despite such evidence, little is known about the impact on their later achievement of such students' linear equations-related competence, something this paper aims to address. In undertaking the study, we were guided by the question:

How do differences in students' approaches to an algebraic equation-related word problem impact later mathematical achievement?

## Methods

The study presented in this paper is a result of a long-standing collaboration between Stockholm's various municipalities and Stockholm university. Its aim is to enable teachers to undertake classroom-based research within a supportive environment. The study reported here was based on a sequential mixed methods approach to data collection and analysis, whereby quantitative data yielded by a large sample of USS students' responses to an algebraic equations-related word problem were supplemented and illustrated by interview data derived from a purposive sample of around ten percent of examined students. The quantitative element of the study involved the entire first-year cohort (ages 16 or 17) of a well-regarded Stockholm USS. As indicated earlier, on transfer from compulsory school and within the first two weeks of their USS programmes, all students in upper secondary schools administrated by the city of Stockholm sit a compulsory, 90-minute, diagnostic test, based on the mathematics they would have experienced during their final years of compulsory school. The USS under scrutiny typically admits high achieving students. For example, over the period 2014 to 2021, its results on the diagnostic test ranked it consistently in the top five of Stockholm's 46 USSs. The diagnostic test, unmodified since 2012, includes the following task, translated by the authors:

Klara went to the grocery store. She spent a third of her money on fruits and she bought bread for 20 kr (SEK). After that, she had half of her money left. How much money (SEK) did Klara start out with? Show your calculations in the box.

Our quantitative analyses are based on data from the 307 students who gave informed consent for their solutions to be included in the study. As is typical with such tasks, it was expected that students would adopt algebraic approaches and, say, let  $x$  represent Klara's total and derive an algebraic equation like  $\frac{1}{3}x + 20 = \frac{1}{2}x$ . However, as we show, the problem proved amenable to a range of approaches, both algebraic and numeric. Following analyses of their solutions, around 10% of students participated in semi-structured interviews designed to elicit their understanding of the task and the thinking underlying their solutions. To this end, audio-recorded interviews, undertaken during the weeks immediately following the diagnostic test by the school-based authors of this paper, were structured around broad questions like: What were your thoughts when solving the task? Why did

you choose your method? Have you seen this kind of task before? Would you do it differently if asked to do it again? Finally, to establish any connection between students' approaches to the word problem and their later achievement, the results from the word problem, completed at the beginning of the year, were compared with data from the terminal assessment of the first mathematics course undertaken by participating students, typically at the end of the schoolyear. This first upper secondary course (hereafter FUSC), which is compulsory for all Swedish USS students, is intended to revisit material covered during compulsory school, provides a broad base on which subsequent courses are built and, in particular, emphasise algebra as a problem-solving tool.

## Data analysis

All student responses were anonymised, and word problem solutions coded as correct, incorrect or no solution. Next, each solution, whether correct or incorrect, was examined for the solution approach and coded as either algebraic or numeric. In practice, as can be seen in the particular cases presented below, this distinction was simple to operationalise, with algebraic solutions explicitly drawing on some form of non-numerical symbolic manipulation. For example, Adam's solution, which can be seen in Figure 2, was coded as numeric because his initial, and deleted, algebraic attempt was superseded by a correctly completed numerical solution. These data, alongside students' subsequent FUSC results, were incorporated into a standard spreadsheet for analysis. To facilitate the calculation of simple descriptive statistics, the letter grades of the FUSC assessment, which range from a high of A to a failure of F, were recoded numerically with A as five down to F as zero. In so doing, we justify converting nominal into ordinal data based on the achievement hierarchy implicit within them.

## Results

In the following, we first present the results of an initial quantitative analysis before offering illustrative examples from the ongoing qualitative.

### Quantitative results

**Table 1: The distribution of different solutions against final FUSC grades**

	A (5)	B (4)	C (3)	D (2)	E (1)	F (0)	Total	Percent	Mean grade
Algebraic correct	61	19	5	2	0	0	87	28	4.6
Numeric correct	35	41	37	8	4	0	125	41	3.8
Algebraic incorrect	17	13	8	5	2	0	45	15	3.8
Numeric incorrect	4	5	10	5	4	0	28	9	3.0
No answer	0	3	6	4	2	7	22	7	1.8
Total	117	81	66	24	12	7	307		
Percent	38	26	21	8	4	2			

Table 1 shows the frequencies of the various solution strategies, both correct and incorrect, set against students' FUSC grades. The first two rows show that 212 (87+125) students (69% of the cohort) solved the word problem correctly. The figures also show that students who gave correct algebraic solutions to the word problem achieve higher FUSC grades (4.6) than all other students. In addition, students who presented incorrect algebraic solutions achieved comparable FUSC scores (3.8) to those who offered correct numeric solutions (3.8).

To examine the significance of the differences shown in Table 1 and acknowledging the high frequency of cells with fewer than five elements, the data of Table 1 were collapsed into the categories shown in Table 2 and three chi square tests undertaken. The first, undertaken on all three rows of Table 2, confirmed that the distribution represented in the table was highly unlikely to have been due to chance ( $\chi^2=36.2$ ,  $p<0.0001$ ). The second and third, incorporating the Bonferroni corrected threshold probabilities necessary when undertaking a series of pair-wise tests, compared the first two rows and the second two rows respectively. The test undertaken on the first two rows, effectively comparing the impact on later achievement of correct algebraic solutions with all other correct solutions, confirms ( $\chi^2=25.89$ ,  $p<0.0001$ ) that the distribution was exceptionally unlikely to have occurred by chance, implying that students who successfully completed the word problem algebraically were more likely to achieve the higher FUSC grades than those who successfully completed the task in other ways. The second, undertaken on rows two and three and effectively comparing the impact on later achievement of correct numeric solutions with all incorrect solutions, yielded no significant distributive difference ( $\chi^2=2.8$ ,  $p=0.247$ ), implying that solving the word problem correctly by means of any numeric approach seemed to have no greater impact on students' later FUSC grades than solving the task incorrectly.

**Table 2: A simplified Table 1 for calculating chi square**

	A/B	C/D	E/F
Algebraic correct	80	7	0
Numeric correct	76	45	4
Incorrect attempts	39	28	6

### Qualitative results

The image shows two handwritten algebraic solutions. The left solution, by Simon, starts with the note 'x = alla klaras pengas' and shows the equation  $\frac{x}{3} + 20 + \frac{x}{2} = x$ . It then multiplies through by 6 to get  $2x + 120 + 3x = 6x$ , which simplifies to  $120 = x$ . The right solution, by Alison, starts with  $x - \frac{x}{3} - 20 = \frac{x}{2}$  and shows steps to isolate x, including  $x^2 = \frac{2}{3}x + 20$  and  $x^2 - \frac{2}{3}x - 20 = 0$ , leading to  $x = 120$ .

**Figure 1: The algebraic solutions of Simon and Alison on the left and right respectively**

In the following, we present a selection of solutions, alongside selected interview utterances, to illustrate how students interpreted the task and warranted their responses. Most of the students who submitted correct algebraic solutions, not only achieved the two highest grades on the FUSC test, as shown in Table 1, but offered remarkably similar solution strategies and interview narratives. Figure 1 shows two such solutions. Simon (FUSC grade A), solution on the left, asserted that  $x$  would represent all of Klara's money and derived a correct equation. His multiplication throughout by six and accurately executed subtraction of the terms involving the unknown, alluded to a solution tacitly located in a DSBS conceptualisation, alongside a clear competence with generalised fractions. When asked about the task, he commented that "one could formulate it in other ways, (but) I chose this one because it was the easiest for me". Alison's (FUSC grade A) solution, on the right of Figure 1, also

presented a correctly derived equation. Her solution, however, appeared to incorporate elements of both SSSS and DSBS. The second line, in which terms are moved correctly from one side of the equals sign to the other, could be interpreted as an SSSS operation, accompanied by accurately executed generalised fraction arithmetic. Next, having placed her terms within brackets, she multiplied throughout by six in a DSBS manner, before finishing with another SSSS-like operation. During her interview, she commented that she could have approached the solution graphically, which she believed “would be pretty hard and would take time”, before adding that she had “learned... that algebra can solve pretty much anything. So yes, I used algebra, it was the easiest, quickest way”.

$\frac{1}{3} + 20kr \Leftrightarrow \frac{1}{2}$  (Mgn: 6)  
 $\frac{2}{6} + 20kr \Leftrightarrow \frac{3}{6}$   $20kr \Leftrightarrow \frac{1}{6}$

$\frac{1}{3} = \frac{2}{6}$   $\frac{2}{6} + \frac{1}{6} = \frac{3}{6}$   $\frac{3}{6} = \frac{1}{2}$   
 $\frac{1}{6} = 20kr$   $20 \cdot 6 = 120kr$

**Figure 2: The numeric solutions of Adam and Amalia on the left and right respectively**

The figures of Table 1 also show that the largest group comprised those students who solved the task numerically. A not atypical solution from this group, shown on the left of Figure 2 and highlighting the student’s recognition of an algebraic expectation, was that of Adam (FUSC grade A). Having abandoned an algebraic solution, Adam offered a concise and accurate numerical solution indicative of a robust understanding of fraction arithmetic and, in his marginal comment, a DSBS conceptualisation. During interview, acknowledging his idiosyncratic use of the ‘if and only if’ symbol, Adam said that “I tried to use  $x$ , to use a variable... I realized that it was complicated ... That is why I used this symbol... which shows equality but with a different meaning”. In similar vein, Amalia’s (FUSC grade A) solution on the right of Figure 2 also shows a clear understanding of fraction arithmetic. However, when asked during her interview whether she had considered using algebra, she replied, implying a lack of confidence in her algebraic competence, that “I didn’t think of that... I think we’ve done similar stuff, but I wasn’t used to it in the way that we are now”.

$50\% + 20 + 33.3\% = 100\%$   
 $100\% - 50\% - 33.3\% = 16.7\%$   
 $16.7\% \approx 17$   
 $20\text{ kr} = 17\%$   
 $\frac{20}{17} \approx 1.1$   
 $1\% \approx 1.1\text{ kr}$   
 $100\% \approx 110\text{ kr}$

**Figure 3: Samuel’s numerical solution**

Other students’ numeric attempts showed how a correctly initiated solution could become confused when students encounter the inconvenient numbers yielded by their approaches. For example, Samuel’s (FUSC grade D) solution, shown in Figure 3, began with a correct summation of the respective percentages involved in the task. However, due to the calculative inconvenience of the

decimal representation of one-third, Samuel's reasoning became confused when he rounded 16.7 to 17 and then incorrectly divided 20 by the result. In other words, while his procedure was conceptually correct, it was undermined by a failure of execution. Finally, there were solutions, like Linda's (FUSC grade D), which employed a form of trial and improvement. During her interview, she confirmed that she had seen similar tasks before, but that "I probably didn't conclude right away that it was 120... I probably tried with 140... Maybe I saw some connection between the numbers and tested with 120".

## **Discussion**

In this preliminary study, we have undertaken a mixed methods approach to uncover the relationships between students' approaches to an algebraic equation-related word problem, completed at the beginning of the first year of their upper secondary experiences, and their later mathematical achievement as measured by a national test of broad competence at the end of their first year. The quantitative analysis shows that students who successfully adopt an algebraic approach at the beginning of the year are more likely than all other student groups to achieve the highest national test grades at the end of the year. Moreover, students who attempted an algebraic solution unsuccessfully were as likely to achieve the higher grades as those who solved the task successfully numerically. Such findings resonate with earlier studies showing that linear equations-related competence (Capraro & Joffrion, 2006) acts as a gatekeeper to students' later learning of mathematics. In sum, the evidence indicates that the ability to formulate and solve an algebraic equation is a predictor, albeit one in need of further research, of later mathematics success as measured by national tests of broad competence. However, further research will be needed to determine whether, as in the United States (Ma & Wilkins, 2007), Swedish students who are algebraically successful are more likely to achieve academic success beyond upper secondary school.

The interview component yielded additional insights, each of which requires further investigation. First, students who offered successful algebraic solutions seemed confident in their approaches, asserting that algebra offers the "easiest" ways to solve tasks. Second, students like Adam and Amalia, despite the quality of their numerical solutions, rarely expressed the same degree of satisfaction with their chosen approaches, typically lamenting their lack of algebraic competence. In other words, there seemed to be a tacit awareness on their part that they should have attacked the task algebraically but lacked the competence to do so. Third, resonating with earlier studies of Swedish students (Andrews, 2020; Andrews & Öhman, 2019), most students who attempted algebraic solutions, and many who did not, alluded to a DSBS underpinning of their approaches (Caglayan & Olive, 2010; Vlassis, 2002). Indeed, even Alison's apparent use of SSSS seemed underpinned by DSBS. In conclusion, the decision to employ a mixed methods approach seems vindicated. Admittedly, by way of caution, the study confirms that the school involved is highly successful in facilitating students' achievement of the higher FUSC grades. However, the wider results prompt a need for further research into the impact of algebraic equation-related competence, particularly the impact of students' later achievement.



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