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Symbolic equational algebraic thinking in young students

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This paper explores one of the items Kieran (2022) recently highlighted as needing to be addressed in early algebra research, namely detailed intervention studies of the transitional challenges faced by young students in solving equations when they move from manipulative-based to letter-symbolic-based contexts. The paper draws on longitudinal research to investigate the gradual transformation of the concrete context's meanings as the students move with the teacher towards an understanding of the meaning and operativity of symbolic equational algebraic thinking. A Vygotskian inspired embodied semiotic analysis reveals three interrelated transitional themes: transliteration in discursive activity, a denotative-connotative shift of meaning, and an awareness of new cultural forms of sign-production of the alphanumeric semiotic system.

Keywords: Equations, symbols, denotation-connotation, early algebra, algebraic thinking.

Antecedents

In the recent ZDM issue on algebra, Kieran (2022) ends her survey paper with a list of items to consider for further research. The first of them is “detailed intervention studies of the transitional challenges faced by young students as they move from the non-letter-symbolic to the letter-symbolic form of equations, especially equations of the type $ax + b = cx + d$ ” (p. 1145). There is, indeed, a growing interest in introducing young students to equations through concrete materials (see, e.g., Stephens et al., 2022). The challenges of the transition to equations given in alphanumeric form remain, however, to be better understood. This paper is an attempt to explore these transitional challenges. To do so, I draw on previous research where students dealt with equations using a specific semiotic system designed to introduce them to equations: the *Concrete Semiotic System* (CSS)—a system comprised of (a) paper envelopes that contain the same unknown number of cardboard cards, (b) cardboard cards, and (c) the equal sign. In that research, instead of presenting the students with abstract equations, *story-problems* based on narratives of children having hockey cards were used. The students translated and solved the *story-problems* in the CSS. Figure 1, left, shows the translation of a story-problem that led to the equation that can be rephrased as “two unknowns plus one equal to one unknown plus six.” Later, the students used a written version of equations in the CSS, where drawn cards and envelopes replaced the concrete objects (Figure 1, right). Because of the resemblance between object and sign, we call this last system the *Iconic Semiotic System* (ISS). These semiotic systems proved useful to introduce the students to fundamental ideas underpinning the successive simplification of the original equation through two basic algebraic rules: subtracting equal amounts of objects (envelopes and/or cards) from both sides of the equation (see Figure 1) and (when necessary) dividing both sides of an equation of the type $ax = b$ by a same number (see Radford, 2022).

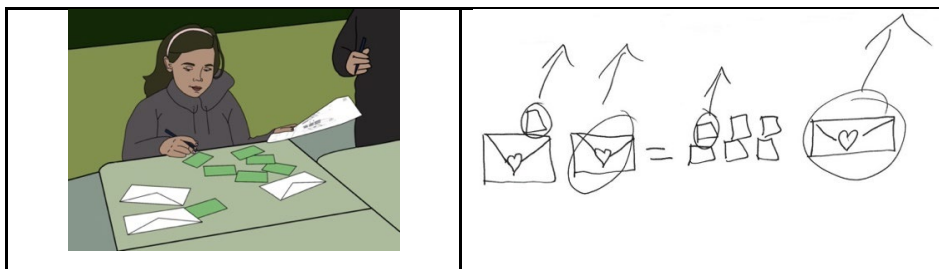


Figure 1: Grade 3 students solving an equation in the CSS and the ISS (From Radford, 2022)

Although previous research has shed some light on the use of letters in sequence generalization activities (e.g., Radford, 2018), the results cannot be directly transposed to the field of equations. The processes of sequence generalization and solving equations are very different. In the former, usually, the goal is *to find* a relational expression (a function $y = f(x)$) where one variable is expressed in terms of another variable (e.g., $y = 3x - 1$). In the latter, the relational expression (e.g., $7x + 2 = 6x + 8$) *is given* (directly or from a word-problem translation) and the goal is *to simplify it* so that x is equal to something. Generalizing and simplifying involve two different thought mathematical processes (Radford, 1996).

Theoretical framework

The theoretical framework followed in this paper is based on the Vygotskian theory of objectification (Radford, 2021). The research question is cast in terms of the investigation of the students' transition to a cultural-historical *mode of thinking* (here an algebraic alphanumeric or symbolic one). Theoretically, ¹this transition is understood as the gradual disclosing of the meaning and operativity of symbolic equational algebraic thinking. The disclosing of meaning at the roots of this transition occurs as the students engage with peers and the teacher in human sensuous and material activity. This activity is, indeed, considered the dynamic locus of the phenomenon of meaning—a phenomenon that manifests itself through its various anchorages in history, material and ideational culture, language, and the body.

Methods

Coherently with the theoretical framework, the methodology considers teaching-learning activity as the *explanatory conceptual category*. This means that we seek to shed light on the students' transition to symbolic equational algebraic thinking through the prism of teaching-learning activity. In terms of procedures, the teaching and learning activity is investigated through a *multimodal analysis* of action and *inter-action* of the teacher and the students. We use four to five video cameras to record the whole class and small group discussions. We start from full transcriptions of the video recorded lessons. The transcriptions are then subjected to fine-grained video analyses revolving around language, prosody, gestures, perception, body position, action, and artefact use (details in Radford, 2015).

The data presented below come from a longitudinal research program carried out in a French public school in Sudbury, Canada. This school has been part of a long collaboration between the school board and our university. These data come from when 9–10-year-old Grade 4 students started using

letters. In Grade 3, the students used the CSS and the ISS to study equations of the type $ax = b$ and $ax + b = cx + d$ (see Figure 1). The central goal was the understanding of the operation-based rules involved in the simplification of equations (see Radford, 2022).

The Grade 4 teaching-learning activity that is the focus of this paper started with a general discussion where the students engaged with the teacher about representing and solving with letters equations of the type $ax + b = cx + d$ given in the Iconic Semiotic System. Then, the students were invited to solve, in small groups, equations given directly with letters. The teaching-learning activity moved then to a general discussion of ideas. Four small groups of two to three students were videotaped. Full transcripts were made; salient episodes featuring the problem under study were identified and subsequently enriched with a frame-to-frame video multimodal analysis of meaning formation. In this paper I focus mainly on one of the small groups—a kind of case study supplemented with information from general discussions. Case studies are limited in terms of the generality of their results. However, they have the virtue of allowing in-depth explorations of the research problem. The focus in this paper is on the work of a group comprised of Cora, Céleste, and Kent.

Solving the equation $7n + 2 = 6n + 8$

The equation appeared on the activity sheet, leaving a space underneath for the students to make their calculations.

- 1 Céleste: (*Writing the equation again while reading it out loud*) Seven numbers plus two equals six numbers plus eight. Now we'll do our calculations down there, OK? (*Pointing to a space under the equation*) . . . If we have seven numbers, seven minus six . . .
- 2 Cora: Seven minus six?
- 3 Céleste: Yes, because it's seven envelopes.
- 4 Kent: But how do we know that it [n] is envelopes or cards?

In their previous experience with the CSS and the ISS, to solve equations, our students were led to *eliminate* signs (envelopes and cards) on both sides of the equation. During this process, the successive simplified equations appeared *immediately* (see Figure 1; more on this below). In the ANSS the cultural solution process requires, by contrast, to *write* the simplified equation underneath the previous one. In other words, in the ANSS, the forms of sign-production change. This is why, during the general discussion that preceded the previous dialogue, the teacher insisted that, in solving equations with letters, both of the calculations required to simplify an equation and the resulting simplified equation needed to be written. The space of its location is what Céleste refers to in Line 1: “Now we'll do our calculations down *there*.”

In addition to an emergent awareness of the new form of sign-production that the ANSS entails, the previous dialogue shows that the alphanumeric equation becomes an object of discourse through a transliteration in which “n” is rendered as “number.” The transliteration is not a mere naming device (a mere nominalization process). It is an essential part of meaning formation. The letter “n” is thought of as a *generic number*. But Line 1 contains more than that. Indeed, in Line 1 Céleste suggests a first step towards the simplification of the equation “seven minus six,” and explains, to counter Cora's skepticism, that $7n$ means *seven envelopes*. The letter n (and associated signs, like $7n$) acquires hence a *denotative* meaning. That is, the letter n refers to a specific referent—a definite description of

something in the cultural world (an envelope). Kent pushes further the denotative intended meaning: does n mean cards or envelopes? In her response to Kent, Céleste explains that n means envelopes: “because number means unknown. And the envelope is unknown.”

This short excerpt points towards three interrelated themes that will be present in the students’ transition to symbolic equational algebraic thinking: transliteration in discursive activity, meaning, and an awareness of the ANSS cultural forms of sign-production. Let us see how these themes play out as the solution of the equation proceeded.

In the following dialogue, the teacher has joined the small group.

- 5 Céleste: Now I have to take out six, because that’s the smallest [number] . . . So, seven minus six.
- 6 Teacher: OK, so you’re going to take out six n (*pronouncing with emphasis “n”*). Don’t forget to add n because we want to make sure you understand that it’s envelopes.
- 7 Céleste: (*Murmuring, eyes closed, doing calculations in her head*) (Figure 2, Picture 1) Ok, so, it’s going to be (*opens her eyes and looks at the symbolic equation*) . . . minus six n on two sides (*she writes $-6n$*) (Figure 2, Picture 2) . . . I visualized it (*she closes her eyes with the pen in the air*). You have two cards on one side; then you have the other eight cards on the other side. (*Still looking up with eyes closed*) you take out two (Figure 2, Picture 3) (*pointing in the air to the imagined objects*) . . . And then (*opening her eyes and writing on the left side of the symbolic equation*) we’re going to go . . . two minus two, and eight minus two (*after speaking, she writes -2 on number 8*) (Figure 2, Picture 4).

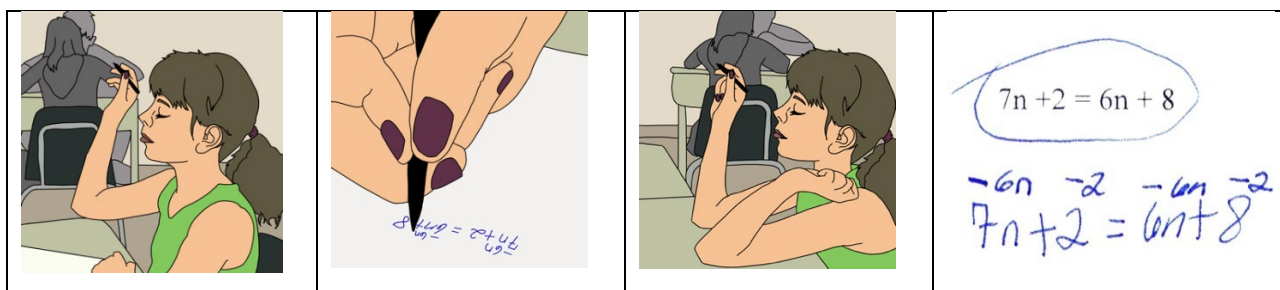


Figure 2: Céleste’s embodied semiotic actions performed on an imagined cards-envelopes equation

The episode shows that the equation in the ANSS serves as a *reference* to build an imagined equation along the lines of equations in the CSS. Transliteration in discursive activity and pointing gestures are used to visualize the objects and guide the algebraic operations of removing envelopes (Line 5) and cards (Line 7). In Line 6 the teacher wants to reinforce the link between the meanings in the CSS and the emergent meanings in the ANSS: “We want to make sure you understand that it’s envelopes.” In Line 7, Céleste includes the teacher’s feedback and says, “It’s going to be . . . minus six n on the two sides.” Then, finding support in the CSS, she proceeds to imagine that she is removing an equal number of cards on both sides of the equation. A successful application of the removing rule in the ANSS is to determine first the suitable number of objects to remove *at once*. This is what Céleste does in Line 5, where she says: “I have to take out six, because that’s the smallest [number].” She is finding the smallest amount between the n -monomials. And this is what she does later, when she says, “you take out two,” referring this time to the constant terms of the equation.

In terms of *meaning*, the four monomials of the alphanumeric equation have been endowed with contextual meanings thanks to their interpretation in terms of cards and envelopes. As to the theme

of *transliteration*, we see that it switches back and forth referring to letters and constants of the ANSS and to envelopes and cards. Concerning the students' *production of signs in the ANSS*, the CSS interpretation of the ANSS equation has opened a window to write and express, in the ANSS, the *algebraic actions* of simplification through small signs above the monomials. This way of writing/expressing the algebraic actions was introduced by the teacher in the general discussion and is now pursued by the students. The size of signs allows the students to distinguish between the reference object and the action performed on that object. However, the *operative* character of the actions remains in the background. That is, while the actions are written (e.g., $-6n$ and -2), their *results* are not shown. These results are vital as they will serve to write the deduced equation. The students' understanding of the ANSS forms of sign production have yet to be pushed forward. Therefore, after Céleste (Line 7) found the solution, pointing left and right, eyes closed, eyes open, to the imagined cards-and-envelopes equation in the air, the teacher encouraged Céleste to write the "next step."

The problem we are observing is the following: in the CSS, the actions are performed kinesthetically. This is what Céleste does in her imagination when she moves the pen in the air (Figure 2). In the CSS, actions are not symbolized. They disappear as soon as they are carried out. By contrast, in the ANSS, the forms of sign production require that actions be symbolized. The first focus of attention is, hence, on the symbolization of the actions (e.g., $-6n$). Through the symbolization of actions, actions do not disappear. Symbolizing transforms the evanescence of kinesthetic actions into presence. As a result, the terms of the equation *and* the actions that are carried out on them, have a same presence (a *co-presence*) that they did not have before. The proper use of the ANSS forms of sign production requires, however, a *second* focus of attention: results not only need to be calculated—they also need to be *written*. In the CSS, the result showed itself to the view. In Figure 1, left, one of the students (then in Grade 3) removes one card from each side kinesthetically, then she removes one envelope. The result is *there*—it does not need to be calculated. This is why, in her discussion with Céleste's group, the teacher points successively to the expression $7n^{-6n}$ and a space on the activity sheet below this expression where the written result could go, while asking, "What's left in your calculation?" (Figure 3, Picture 1). Céleste answers with a pronounced "Ooooh!" A key meaning of the ANSS is in the process of being disclosed. In the ANSS, writing the steps occurs in a temporal dimension that does not coincide with the one of the kinesthetic actions.

So, followed attentively by her teammates, Céleste embarks on writing the deduced equation. She writes "1n." Then, she considers the expression 2^{-2} and is immediately confronted by a new obstacle: must all results be written? Céleste whispers "zero." The teacher asks: "Do you need to write zero?" The forms of sign production in the ANSS are such that only some results must be written. As one student from another group reasoned when skipping the writing of zero from 2^{-2} , "this has no more lives." To the teacher's question, Céleste answers: "No, it's one n equal ..." (Figure 3, Picture 2). The group moves to the right side of the equation:

- 8 Céleste: (*looking at the first term on the right side of the equality*) Six n...
- 9 Teacher: Six n minus six n is what?
- 10 Kent and Céleste: Zero.
- 11 Céleste: Zero, so nothing [to write].
- 12 Teacher: (*Considering the last expression of the equation*) Then, eight minus two? (*The teacher leaves at this point to check on another group*).

- 13 Kent and Céleste: Six.
 14 Cora: What do we write? Six n?
 15 Céleste: So, one n equal to six n? (Figure 3, Picture 3) . . . There are six cards in an envelope.

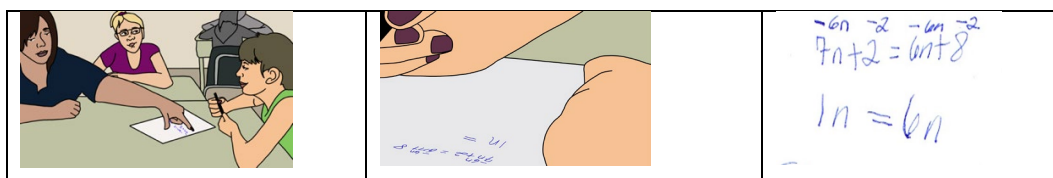


Figure 3: Towards the disclosing of the operativity of symbolic equational algebraic thinking

After her intervention in Line 12, the teacher left the group and walked to the whiteboard to organize a general discussion. Her dialogue with Céleste’s small group helped the students to focus on the alphanumeric equation and actions to solve it. Indeed, the previous excerpt shows how *reasoning* occurs now *within* the ANSS. It is only at the end (Line 15) that Céleste returns to the envelopes-cards context—even though writing $1n = 6n$, she interprets the solution correctly. *Reasoning within the ANSS* is a great accomplishment and one, I would like to suggest, that is required to transition to symbolic equational algebraic thinking.

Reasoning within a symbolic system

To better grasp the tremendous step that the students are taking, it is worth turning to the history of algebraic symbolism to distinguish between notations and genuine algebraic symbols. Notations (such as *co*, *m*, *p*) appeared in manuscripts as abbreviations of words: *cosa* (meaning a thing, i.e., the unknown), *minus*, and *plus*, respectively. They appeared in a time where solving equations was done through natural language, eventually helped by diagrams, as Høyrup (1995) suggests. The Renaissance manuscripts of the abacist teachers were first verbatim transcriptions of the problem-solving procedures. Bit by bit, they incorporated abbreviations (like those mentioned above), which led, later on, to a search for a full-fledged symbolism that would run in a largely autonomous manner vis-à-vis discursive activity (Radford, 2001). A key difference between Lines 1 to 7 and Lines 8 to 15 is that the meaning of letters is slowly moving from notations to algebraic symbols.

In his historical studies on algebraic symbolism, Heffer (2014) notes that “the first steps towards symbolization of algebra were [...] taken by the use of non-discursive representations of algebraic operations” (p. 108). Something like this is what we see in the new expression $7n^{-6n}$. “Minus six n” (Céleste, Line 6) is not a common expression in natural language. $7n^{-6n}$ can still be transliterated, but it does not make it a common expression that you will hear at the gas station or the supermarket. In the case of our students, the problem of the transition to a symbolic algebraic language, as seen in the previous dialogue, is deeply related to the possibility of thinking and reasoning *within* this system. Heffer has studied some 15th century manuscripts where a clear distinction between solving the problem discursively (*per scriptura*, i.e., by normal writing) and symbolically (*in forma figurata*) is made. In one of them, a 15th century mathematician says, “I showed [how to solve the problem] symbolically [...] not to make things harder but rather for you to understand it better” (cited in Heffer, 2014, p. 108). As Heffer notes, in the eyes of the mathematician, “the ‘symbolic’ solution adds something to the discursive one” (p. 107).

The general classroom discussion

The general classroom discussion was an occasion to further the gradual disclosing of the meaning and operativity of the symbolic mode of algebraic thinking. Guillermo was invited to solve the equation $3n + 3 = 1n + 9$. He went to the whiteboard.

- 16 Guillermo: I looked at what I had. I saw that I had three numbers, three unknown numbers plus three equals one number plus nine. So, I looked at how many unknown numbers I had here. It [the lesser] was one. So, I did minus one, so I took this one out (Figure 4, Picture 1) . . . and then here, I made minus one (above 3n) (Picture 2).
- 17 Teacher: Why do you remove minus one n on each side?
- 18 Guillermo: Because here [pointing to 1n], I can't take off three, because I just have one number, one n. So... um, here, I look, there's three and here there's nine, so I take the smallest number, and I do minus three and then, here at nine, I do minus three (Picture 3).
- 19 Teacher: What's left for you to do?
- 20 Guillermo: Because we have, we have two numbers remaining, I make two n (Picture 4) . . . so that equals . . . six. So, I make equals six (Picture 5).
- 21 Teacher: What do you have left to do?
- 22 Guillermo: I divide that. And one number equals three (Picture 6). Because the double of three is six.

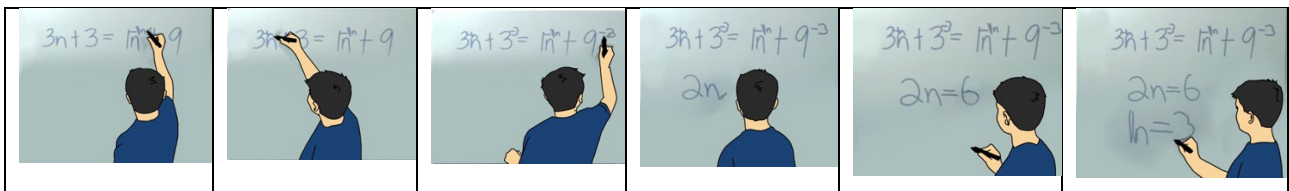


Figure 4: Guillermo reasoning through and through within the alphanumeric symbolic system

An important feature of Guillermo's reasoning is that it is carried out within the alphanumeric symbolic system. Transliteration refers to the decontextualized numbers (unknown numbers, unspecified 3, 9, etc.) as opposed to the previous examples where transliteration was still referring — although with decreasing intensity—to cards and envelopes. In Guillermo's solution, the letter n and the various numerals do not need to refer to something else. In semiotic terms, signs no longer denote; now, they connote (Eco, 1976). That is, instead of referring to something else, and meaning in terms of something else (denoting) (as when n means envelopes), signs (letters, numerals, the equal sign, and derived signs) mean in terms of the value they acquire within a system of differences and cultural rules of sign-use (connoting). In the ANSS signs and operations acquire an internal syntax-based coherence and meaning.

Concluding remarks

The results presented here point towards the presence of three interrelated themes in the transition to a cultural-historical mode of equational alphanumeric algebraic thinking: transliteration, meaning, and an awareness of new modes of sign production. The results put into evidence a gradual shift from denotative to connotative meanings and some of the pedagogical actions that foster such a shift. In this shift, denotative meanings do not necessarily disappear. They become embedded (or sublated, as Vygotsky would say) in the new connotative meanings. Thus, at a certain point of the discussion, explaining to his colleagues the meaning of $3n$, Guillermo says "because we don't know what the

number is; because it's three envelopes and we don't know what the number is." He moves easily from one meaning to the other. The transition investigated here rests, I suggest, in the increasing possibilities of *reasoning within the ANSS*; that is, to reason connotatively, freeing imagination from the continuous attention to denotative meanings.

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