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Does solving arithmetic linear equations offer transferable algebraic competence?

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In this study we have investigated the relation between grade 9 students' responses to a simple linear equation and a visually similar but proportionally-based linear expression task. Analyses, based on data from the results of a test of 359 Swedish year Swedish students from six diverse schools, drew on written responses to the two tasks. Students giving algebra-based solutions to the linear equation are not only more likely to complete the proportional task but also offer better solutions than students giving arithmetic-based solutions. The results highlight the need for curricula, textbooks and teachers to revisit the place of algebraic equation-solving, whilst also encouraging students to verify their solutions.

Keywords: Secondary school mathematics, algebra, linear equations, assessment, proportionality.

Introduction

Evaluating mathematical knowledge is an important enterprise involving large sums of money and the growth of a methodological industry (Engel & Rutkowski, 2020), as seen in, for example, studies employing Rasch analyses to create task hierarchies for evaluating mathematical knowledge (Izsák et al., 2012). However, such studies offer limited support for teachers, typically concerned with the implications of particular tasks for their students' learning. In this respect, recent studies have shown how comparing grade 5 students' solutions to multidigit arithmetic with their solutions to other multidigit arithmetic tasks can expose a range of arithmetical misconceptions (Öhlund et al., 2017). In this paper, we follow in this tradition by examining the relationship between Swedish grade 9 students' successful responses to a simple linear equation and a visually similar, in that it has the appearance of a linear equation, algebraic task requiring elements of proportional reasoning. The latter task, which is rare in Swedish textbooks (Palm Kaplan & Prytz, 2020), may be seen as more challenging for Swedish students than simple linear equations that students typically find unproblematic (Andrews & Palm Kaplan, 2020). The tasks are

Task A: Solve the equation $2x + 3 = 11$ and

Task B: If $4x + 5y = 11$, what is $12x + 15y$?

Relevant literature

The ability to solve linear equations, widely acknowledged as a gatekeeper to higher education and employment, has spawned much research in the United States (Ma & Wilkins, 2007). However, little is known about the impact of such competence elsewhere, particularly Sweden, the site of this study.

The literature on equation solving distinguishes between equations which have the unknown on both sides of the equals sign, and those which have the unknown only on one side of the equals sign. The former, here denoted non-arithmetical equations, require algebraic competence (Fillooy & Rojano, 1989) and are typically solved by means of one of two approaches. The first approach, typically found in studies of students' solutions (Huntley et al., 2007), is a 'swap the side swap the sign' (SSSS) procedure involving a rote-learned transposition whereby the unknown finishes on the left-hand side and a value on the right (Nogueira de Lima & Tall, 2008). The second approach is a 'do the same to both sides' (DSBS) procedure that is conceptually based on the balance scale (Araya et al., 2010; Vlassis, 2002). In contrast, equations with the unknown only on one side of the equals sign, or arithmetical equations, may always be solved by a series of operation reversals. However, this does not exclude that SSSS or DSBS algebraic strategies may be used also for solving arithmetical equations. For example, Humberstone and Reeve (2008) identified a hierarchy of four levels of competence in beginning algebra from arithmetic to algebraic reasoning.

A second topic, also believed to underpin later learning, is proportionality, or the ability to identify multiplicative relationships between two quantities and then extend this relationship to other pairs of quantities (Lamon 2007). In an attempt to connect proportional reasoning to algebra, Burgos and Godino (2022) proposed a hierarchy of understanding from algebraization level 0, involving only arithmetical activity, through to algebraization level 3, involving only algebraic activity. Between these are two 'proto-algebraic' levels; algebraization level 1 involving the calculation of a unit value that is then scaling to obtain a solution, and algebraization level 2 involving the identification of a proportional equation that is resolved through cross-multiplication. In task B, identifying proportionality between the expressions $4x + 5y$ and $12x + 15y$ also requires analysing the expressions as algebraic objects (level 3). Alternatively, B may be solved through more or less general substitution strategies (level 3 or 1). Broadly speaking, students of all ages tend to misapply or fail to recognise proportionality in the tasks presented to them (Lundberg & Kilhamn, 2018; Van Dooren et al., 2009), while preservice primary teachers rarely attain levels of understanding commensurate with Burgos and Godino's (2022) higher levels of algebraization. Overall, however, the relationship between students' solutions to a simple linear equation and a visually similar proportion-based algebraic task is an unexplored field.

In this paper, our aim is to investigate how students' correct responses to a simple equation predict their capability of solving a proportion based algebraic task. In particular, the inclusion of task B, with its superficial similarity to a linear equation, would allow us to see whether students recognised proportionality or unnecessarily applied equations-related reasoning. In so doing, we present a first pass at examining such a relationship, framed by the question:

What insights can be gleaned from a comparison of students' solutions to an arithmetical equation and a visually similar but proportionality-based linear expression task?

Methods

This paper draws on 359 grade 9 (15–16 years) students' written responses to a general mathematics achievement test developed as part of the first author's doctoral study. All tasks were adapted from those found in earlier Swedish national tests and, to ensure as equitable participation as possible for

students with Swedish as a second language, written with as few words as possible. Two of these tasks, for which calculators were prohibited, were the tasks displayed above. Participants, all of whom had given appropriate consent, were drawn from six demographically different Swedish schools. To minimise student anxiety, the test was administered during class time by the class teacher and students instructed to document their solutions.

A multi-step analysis of students' responses

All scripts were scrutinised to identify those students who had completed task A correctly. This process eliminated 103 students (29% of the sample, 95 of whom also failed to complete Task B correctly), from the analysis. The remaining 256 scripts were subjected to the following procedure. Due to our interest in determining the relationship between solutions to Task A and solutions to Task B, and acknowledging the power of low-inference codes for supporting complex analyses of students' linear equation solving behaviours (Andrews & Larson, 2020), Task A solutions were coded against the inductively-derived strategies A1–A3, according to whether students had written (A1) a correct answer only ($x = 4$), (A2) a verified correct answer ($2 \cdot 4 + 3 = 11$), or (A3) an SSSS or a DSBS solution like that shown in Figure 1.

$$\begin{array}{l}
 2x + 3 = 11 \\
 2x + 3 - 3 = 11 - 3 \\
 2x = 8 \\
 \frac{2x}{2} = \frac{8}{2} \\
 x = 4
 \end{array}$$

Figure 1: An exemplar DSBS solution to task A, strategy A3

Table 1: Response categories for task B

Task B	B1	Correct answer only (= 33)
	B2	Explicit use of a factor of three, such as $12x + 15y = 3(4x + 5y) = 3 \cdot 11 = 33$. This strategy does not require the assignment of any values to the unknowns.
	B3	Explicit assigning of useful values to the unknowns, for example, let $y = 1$ and solve the equation for x , which gives $x = 1.5$. Substituting these values into the second expression gives $12 \cdot 1.5 + 15 \cdot 1 = 33$. This code also included minor arithmetic errors such as erroneous multiplication of $12 \cdot 1.5$ or erroneous addition of $18 + 15$.
	B4	Solution of the 1 st expression with respect to one variable, for example $y = (11 - 4x)/5$. Substitution of this solution into the 2 nd expression. This gives $12x + 15(11 - 4x)/5 = 33$.
	B5	Parsing error, whereby a multiplicative term is typically interpreted additive ($4x$ as $4 + x$). Otherwise, correct argumentation. Errors typically led to answers 29, $27xy$, $27y$ or 27.
	B6	Other errors or no solution.

Next, acknowledging Burgos and Godino's (2022) four levels of proportion-related algebraization, all responses to task B, also shown in Table 1, were coded for the strategies B1–B6, according to whether students had (B1) written a correct answer only; (B2) recognised and correctly used the factor three relation between the two expressions; (B3) explicitly assigned a value to one of the variables, created and correctly solved an arithmetical equation, and then substituted both values in the second expression; (B4) solved the first equation with respect to one of the variables, and then substituted the outcome into the second expression; (B5) made a parsing error, whereby, for example, the multiplication implicit in $4x$ was interpreted as $4 + x$ with, otherwise, correct reasoning; or (B6) offered either other incorrect, typically uninterpretable, errors or no response. Significantly, the three codes B3, B2 and B4 ranged from the middle to the highest of Burgos and Godino's (2022) levels of algebraization. Initiated by the first author, all four authors subsequently took part in the process of analysis until the coding was agreed upon.

Results

The figures of Table 2 show that of those students who had completed task A correctly, 49% had offered a SSSS or DSBS solution (A3), implying that they not only recognised the task for what it was but also recalled the strategies they had been taught to solve it. The figures also show that 32% of all the task A-successful students, despite instructions to document their solutions, only wrote an answer (A1), implying either that they had undertaken a mental operations reversal without representing the process symbolically, or that they had employed a mental guess-and-check strategy. This second possibility may also have applied to the 18% of task A-successful students who offered a verified correct answer (A2), although our interpretation is that this, too, was likely to have been a consequence of a mental operations reversal. Overall, our interpretation of these results is that the majority of the task A-successful students had either undertaken a mental operations reversal, albeit represented in different ways, or applied a familiar algebraic routine.

Table 2: Crosstabulation showing frequencies and percentages of the two sets of outcomes

		A1		A2		A3			
		Answer only		Verified answer		SSSS or DSBS		Totals	
B1	Answer only	18	7%	5	2%	16	6%	39	15%
B2	Factor three	6	2%	4	2%	37	14%	47	18%
B3	Assigned values	11	4%	5	2%	20	8%	36	14%
B4	Substitution	0	0%	0	0%	0	0%	0	0%
B5	Parsing error	7	3%	10	4%	20	8%	37	14%
B6	Other	41	16%	23	9%	33	13%	97	38%
Totals		83	32%	47	18%	126	49%	256	100%

Moving to the implications of a successful completion of task A for the completion of task B, the results of Table 2 offer some interesting insights. The first is that not one student who had successfully completed task solved a general equation and then substituted the outcome into the second expression

(B4). The second is that almost two-fifths (38%) of the task A-successful students offered either uninterpretable or no attempts on Task B (B6). Third, one in seven (14%) task A-successful students made parsing errors (B5) on task B, indicating that any algebraic competence or, indeed, confidence demonstrated during the completion of task A was relatively insecure. Fourth, the strategy adopted for task A clearly influenced the likelihood of success on task B, with correct answer only (A1), verified correct answer (A2), and SSSS or DSBS solution (A3) yielding success rates on task B of 13%, 6%, and 28% respectively. That is, a student who approached task A with SSSS or DSBS was twice as likely to succeed on task B than a student who offered a correct answer only, and more than four times as likely as one who offered a verified correct answer. Fifth, and this is not unrelated to the fourth, a student who completed task A with SSSS or DSBS (A3) was seven times more likely to have recognised and used the factor three (B2) strategy than students adopting either of the other successful approaches.

Table 3: Crosstabulation showing frequencies and percentages of the combined outcomes

		Task A strategies			
		A1 & A2		A3	
Task B strategies	B1 (answer only)	23	9%	16	6%
	B2 (factor three)	10	4%	37	14%
	B3 (assigned values)	16	6%	20	8%
	B5 (parsing error)	17	7%	20	8%
	B6 (other)	64	25%	33	13%
Sum		130	51%	126	49%

In order to understand more fully the interactions represented by the figures of Table 2, and acknowledging the fact that B4, the substitution strategy for task B, was missing in students' solutions, the data in the columns represented by the codes A1 and A2 were collapsed into a single column to create Table 3. The rationale for this decision was that A3 explicitly draws on algebraic arguments while neither A1 nor A2 comprise any algebraic argument and, as indicated above, are likely to be arithmetic-based. Two chi-square tests were run on the figures of Table 3 to explore the likelihood that the distributions were due to something other than chance. The first ($\chi^2 = 22.5$, $p = 0.0002$) included both correct and incorrect solutions to Task B, while the second ($\chi^2 = 13.0$, $p = 0.0015$) included only the correct solutions represented by B1, B2 and B3. Both tests indicate that arithmetic-based and algebra-based strategies for task A are likely to impact in significantly different ways on the outcomes of task B, whether successful or otherwise. In this respect, the figures of Table 3 offer several important insights. First, students who solved task A with SSSS or DSBS (A3) were more than three times more likely to recognise and use successfully the insight-based factor three solution (B2). Second, these same students were less likely to offer a correct answer only (B1) to task B. Third, they were half as likely to offer either an uninterpretable solution or no solution to task B (B6). Fourth, the use of the assigned value strategy (B3), which is a general strategy in mathematical problem solving, occurred in similar proportions across the two groups. Fifth, parsing errors (B5) seem to occur in similar proportions, irrespective of the original strategy choice.

Discussion

In this paper, we have investigated how Swedish grade 9 students' solutions to a particular arithmetical linear equation, namely, $2x + 3 = 11$, influence their approaches and ability to solve accurately a visually similar but proportional task, namely, If $4x + 5y = 11$, what is $12x + 15y$? Of the 359 students who undertook the test, 256 (71%) completed the first task successfully. This figure resonates with other studies of Swedish students, including national tests (Andrews & Palm Kaplan, 2020), and provides a warrant for the analysis presented in this paper. From the perspective of the simple linear equation, the analyses yielded both expected and unexpected results. With respect to the expected and acknowledging the instruction for students to document their solutions, we regard it as encouraging that half of all students who completed task A correctly, used an algebraic strategy (A3). Given the instructions to document their solutions, the high proportions of students who offered answer only (A1) or verified answer (A2) were unexpected. Overall, they demonstrated a similar range of linear equations-related competence to that found by Humberstone and Reeve (2008).

From the perspective of the interactions of the two tasks, the key focus of the study, some important insights emerged. First, no student who successfully completed task A approached task B by solving a general equation and then substituting the general result into the second expression. In other words, no student showed evidence of attaining Burgos and Godino's (2022) third level of algebraization, which was unsurprising, since it is a general strategy for solving systems of equations, a topic not only known to be difficult (Ogbuehi & Fraser, 2007) but typically outside the experience of Swedish grade 9 students. Second, task B seemed to confuse almost two-fifths of the task A-successful students (B6), who either documented an uninterpretable solution or simply ignored the task. For these students, any algebraic insights they may have brought to task A seemed not to transfer to an unfamiliar context. That being said, students who explicitly solved task A algebraically (A3) were half as likely to seem confused by task B, as those who did not. Third, students who solved task A algebraically were almost four times as likely to recognise and use successfully the insight-based factor three solution (B2). This is interesting because there is no obvious connection between algebraic approaches to linear equations and the recognition of the proportional structure of task B. In other words, many of the students who solve linear equations algebraically may have acquired some form of transferable competence lacking in their colleagues. Despite B2 explicitly involves only multiplication by 3, it does this in ways that underpin the broader development of mathematical competence (Hickendorff, 2018). Fourth, students offering algebraic solutions to task A were much less likely to have offered a correct answer only to task B, than other task A-successful students. This is not surprising as they had originally demonstrated a tendency to document their solutions. Fifth, students' use of the assigned value strategy for task B (B3), which we argue falls into the 'proto-algebraic' levels of Burgos and Godino's (2022) framework and could be construed as a generic problem solving strategy, seems independent of students' approaches to Task A. Sixth, the tendency to make parsing errors on task B (B5) also appeared unrelated to students' approaches to task A, indicating that their solutions to task A may have been based on an incomplete understanding of algebraic notation (MacGregor & Stacey, 1997).

Concluding thoughts

The evidence of this study indicates that the ability to solve arithmetical linear equations does not, of itself, guarantee success on algebraic tasks. However, being able to explicitly solve them algebraically raises the likelihood of such success. That being said, with hindsight and now representing the next phase of our work, we believe this study was limited by the generic ‘algebraic’ solution code for Task A. Preferably, A3 should have been subdivided according to whether students offered SSSS or DSBS approaches, as they are embedded in different cognitive and behavioural approaches to equation solving (Nogueira de Lima & Tall, 2008) and, we hypothesise, likely to have differential outcomes in respect of Task B, particularly in the case of equation-solving strategies based on understanding rather than a rote-learned response. However, this was not an option since it was difficult to distinguish between SSSS and DSBS strategies in the student solutions. In undertaking our analyses, we also exploited elements of Burgos and Godino’s (2022) framework connecting algebra and proportionality. This was not unproblematic, since task B could be solved not only as a proportionality but also as an algebra task. Finally, there are didactical implications. This study has highlighted for teachers the importance of encouraging students to document their solution in for example tests, in ways that offer the possibility for further learning as an element of formative assessment (Foster, 2016).

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