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Deepening preservice teachers' algebraic thinking by noticing children's thinking

Joana Cabral¹, Hélia Oliveira² and Fátima Mendes³

¹Instituto Politécnico de Setúbal, Escola Superior de Educação, Setúbal, Portugal;
joana.cabral@ese.ips.pt

²Instituto de Educação da Universidade de Lisboa, Lisboa, Portugal

³Instituto Politécnico de Setúbal, Escola Superior de Educação, CIEF, Setúbal, Portugal

This study aims to discuss how noticing children's algebraic thinking impacts on two pairs of preservice teachers' (PTs) algebraic thinking in an initial teacher education course. The study follows a qualitative methodology, with two pairs of PTs as participants. The research methods were participant observation of classes, complemented with audio and video recording, and document collection of PTs' written work. The results show that by seeking, simultaneously, the development of PTs' algebraic thinking and their ability to notice children's algebraic thinking, the teaching experiment has allowed PTs to deepen their knowledge of algebraic thinking dimensions that are compatible with an early algebra perspective that they should mobilise in their future practice.

Keywords: Algebraic thinking, initial teacher education, noticing, elementary preservice teachers.

Introduction

In the context of initial and in-service teacher education, the need for articulating content and pedagogy has been highlighted (Ponte & Chapman, 2008) and thus the traditional division between mathematical content courses and methods courses in mathematics education should be reconsidered (Li & Superfine, 2018). Specially in courses that aim to develop teachers or preservice teachers' algebraic thinking, the foreseen articulation becomes even more relevant because quite often PTs are not familiarised with aspects related to relational or functional thinking in a compatible perspective with the work that they are expected to carry out when teaching for the early years (Hohensee, 2017). The literature has also widely discussed the importance of the noticing skill and the need for efforts to help (preservice) teachers learn to notice in new or more sophisticated ways (Jacobs et al., 2018). Acknowledging the importance of developing both PTs' algebraic thinking and their ability to notice children's algebraic thinking, this study is guided by the following research question: how does the PTs' noticing of children's algebraic thinking impact on their algebraic thinking?

Theoretical framework

Algebraic thinking

The characterisation of algebraic thinking has been an important aspect in mathematical education research and, in particular, Blanton and Kaput (2004) define it “as a habit of mind that permeates all of mathematics and that involves students' capacity to build, justify, and express conjectures about mathematical structure and relationship” (p. 142). Many authors identified the fundamental areas of algebra in the early years that promote the development of algebraic thinking. Blanton et al. (2018) consider three areas: “(1) generalized arithmetic; (2) equivalence, expressions, equations, and

inequalities; and (3) functional thinking” (p. 32). Articulating areas (1) and (2), relational thinking is centred on a vision of generalised arithmetic, focusing on the generalisation of arithmetic operations and their properties, and looking at the structure of arithmetic relationships (Pitta-Pantazi et al., 2020). Functional thinking focuses on key aspects such as generalisation and the use of different representations and Blanton et al. (2018) consider that thinking functionally includes “generalizing relationships between co-varying quantities and representing, justifying, and reasoning with these generalizations through natural language, variable notation, drawings, tables, and graphs” (p. 33). As a context to functional thinking development in the early years, the literature has highlighted the importance of exploring pictorial patterns (Hunter & Miller, 2022), namely because of the relevance of the visual component in the development of the generalisation ability (Rivera & Becker, 2007).

Research shows that many teachers or PTs had little or no contact with the development of algebraic thinking in an early algebra perspective (Hohensee, 2017). Thus, teacher education courses should promote opportunities for PTs to develop their own algebraic thinking, and also to learn appropriate ways to promote meaningful learning to their students regarding aspects and dimensions such as generalised arithmetic, interpretation of the equal sign and functional relationships.

(Preservice) teachers’ mathematical knowledge and the noticing skill

In the scope of content courses there are advantages in articulating aspects of mathematical knowledge and aspects related to the analysis of students’ work because this can positively contribute to PTs’ learning, in various domains, namely in the development of their own mathematical knowledge (Appova & Taylor, 2019; Depaepe et al., 2018). According to Schoenfeld (2020) beyond a deep knowledge of the content they will teach, teachers need to understand how students approach different mathematical concepts and be prepared to respond to their needs. By analysing students’ productions or interventions (preservice) teachers are developing the ability to notice their students’ thinking (Jacobs et al., 2018). Following Jacobs and colleagues (2010), in this study we consider the noticing of students’ thinking as the ability of describing and interpreting students’ thinking. Although fundamental to teachers’ practice, research has shown that the noticing skill is not automatically possessed by teachers, namely, mathematics teachers (Jacobs et al., 2018), being crucial that teacher education programs allow them to develop this ability (Buform et al., 2022).

Methods

This study follows a qualitative methodology and was developed within the scope of a teaching experiment that occurred in a mathematical content course named Patterns and Algebra (Cabral et al., 2021), where the first author assumed the role of teacher educator. This content course is set on the third and last year of an undergraduate program in elementary education. The teaching experiment consisted of 12 sessions with the main goal of promoting the PTs’ algebraic thinking and their ability to notice children’s algebraic thinking, simultaneously. During the teaching experiment there were two types of interrelated training tasks: i) those with an incidence on mathematical knowledge (algebraic thinking) and (ii) those related to the noticing of algebraic thinking. When these PTs attended preschool or primary school, the mathematics syllabus did not consider the early algebra domain, so the mathematical knowledge tasks are intended to deepen the PTs’ relational and functional thinking. The noticing tasks concern the analysis of 4-10 years old children’s algebraic

thinking from their written work or lesson videos where they solve mathematical tasks similar to those solved by the PTs.

A class of 20 PTs has been involved in the teaching experiment. Two pairs of PTs were selected as participants of this study: Anabela and Bianca, and Beatriz and Júlia (20-21 years old). The selection of these pairs was based on: their different performance in diagnostic tasks at the beginning of the teaching experiment, where Anabela and Bianca had a similar performance in opposition to Beatriz and Júlia, and the diversity of their paths in secondary school education, as both Anabela and Bianca attended mathematics as a subject in secondary education, while Beatriz and Júlia did not.

The data collection methods were participant observation of classes, complemented with audio and video recording, and document collection of PTs' written work. In this paper, we analyse PTs' written productions related to the exploration of relationships in numerical equalities and equations, in the scope of two mathematical tasks about relational thinking and their resolutions of a mathematical task about a pictorial growing pattern. The data also contains a few written productions related to the noticing skill where PTs analyse students' resolutions and interventions of similar mathematical tasks, as well as the dialogues between the members of each pair.

Based on the theory, a data analyses framework that comes out from crossing the two domains considered in the teacher experiment was designed: mathematical knowledge (algebraic thinking) and the ability to notice children's algebraic thinking, according to the dimensions of algebraic thinking in each case (relational and functional thinking). In this paper, in the scope of relational thinking, we focus our attention in *exploring numerical relationships* which refers to the perception and generalisation of numerical relations, the identification of the variation between elements of a numeric expression, such as compensation or numerical decomposition (Kieran, 2007). Regarding the functional thinking, where the main focus is on pictorial growing patterns, the analysis categories are *exploring relationships* which concerns the identification of the pattern structure, and of the variables and how they relate to each other (Walkoe, 2015) and *generalising* which concerns to the extension of reasoning from the initial domain, through the identification of a commonality between cases and to the formulation of a general rule about a certain set of data (Radford, 2008).

Results

Relational thinking – Anabela and Bianca

The first task asks for the identification of the logical value of various equalities and its justification. Previously, the teacher educator suggested PTs to try to establish numerical relationships between the two sides of the equalities without calculating the value represented by each expression. Anabela and Bianca respond that the equality $28+36+79=28+35+80$ is true and present the justification: “the sum of the terms on the left side and those on the right side gives the same value” (Anabela and Bianca's written production). Despite the previous teacher educator's suggestion, these PTs focus on computing the value represented by the expression in each side of the equality and thus showing an interpretation of equalities related to calculation.

After this task, the PTs solved two tasks that comprehend the analysis of 1st grade students' relational thinking as they explore different equalities and explain how they thought. The requested analysis

from the PTs focused on the perception of the development of the students' relational thinking and Anabela and Bianca contacted with different approaches by the children. When analysing a resolution where the student (Sara) does not need to proceed to calculation to assess the logical value of an equality, they refer "Sara begins explaining that the equality is true . . . concluding that it is not necessary to carry out the two additions to know that it is an equality" (Anabela and Bianca's written production), which shows that PTs were progressively recognising different approaches that aimed a relational perspective.

The next mathematical knowledge task asked for the values that complete numerical equations, and Anabela and Bianca, individually, in the case of the equation $5 + 8 + 17 = 5 + 10 + \underline{\quad}$, indicate that the missing value is 15 by establishing compensation relationships. Anabela explains that:

The number that completes the equality is 15. When comparing the left side to the right side, we see that 5 is the 1st term in both. If we look at the 2nd term of each side, we see that from the left side to the right one two units have been added, so we must subtract two units to the value of the 3rd term [10] on the left side to maintain the equality. (Anabela's written production)

Bianca justifies her answer saying that:

The missing number on the left side is 15 because: the first term is the same on the two sides and the other two terms are different on both sides, but if we notice, on the left side, we have 8 and on the right side we have 10 (which corresponds to 8 plus 2). Thus, the last term on the right side must be 15 because, from 17 on the left side, we will have to subtract 2 that we added to 8 on the right side. (Bianca's written production)

Bianca also expresses this idea directly in the equation (Figure 1):

$$5 + 8 + 17 = 5 + 10 + 15$$

Figure 1: Bianca's answer

At this point, both Anabela, who refers exclusively to terms without mentioning the numbers in question, and Bianca, who specifically points them out, understand that the second term on the right side of the equation is two units bigger than the corresponding term on the left side of the equation and use these relationships to determine the missing value. Thus, they do not simply made calculations but search for numeric relationships that allow them to find the missing number.

Functional thinking – Beatriz and Júlia

The task "The necklaces" presents a pictorial growing pattern (Figure 2) and implies the determination of close and distant terms and the order of a distant term, as well as the general rules that express the functional relationship between each order and its term and the inverse relationship.

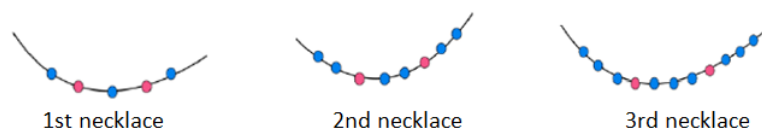


Figure 2: Growing pattern in the task "The necklaces" (Mestre, 2014)

Regarding the category *explore relationships*, we observe that, from the pictorial terms, Beatriz and Júlia identify the structure of the pattern:

- Beatriz: She always adds a blue one [bead] between the red ones. . .
Júlia: Here is one blue, one red [1st necklace], here is two blues, one red [2nd], and here is three blues and one red [3rd].

From the perceived growth in the number of blue beads, from term to term, the PTs conclude that to know the total number of beads in any necklace they will have to multiply the number of the figure by three.

- Júlia: It is three times n . . . n being the number of the figure.
Beatriz: Because it is one more in every portion . . . You will notice that the number of the figure is . . .
Júlia: The number of blue beads . . . What do we say is the [number] three . . . the number of times it repeats?
Beatriz: It is the number of beads that are added. You always add three . . . You always add three beads. The third necklace is the second necklace plus three. And two is the number of red beads.

Although the PTs recognise that the number of red beads is always two and that there is a relationship between the necklace number and the number of blue beads, they did not establish immediately the relationship between the necklace number and the total number of beads. They had to try some substitutions from the numerical pattern to confirm the way they can obtain the total number of beads from the necklace's number. Regarding the task's first question that required the determination of the number of beads in the 4th necklace, the PTs wrote that "Three is the number of beads that are added and two is the number of red beads" (Beatriz and Júlia's written production), and thus not referring to the fact that three is also the number of blue beads groups in each figure.

By analysing different groups of 4th graders' solving strategies of this task to characterise their algebraic thinking, Beatriz and Júlia end up recognising different aspects relating to the pattern that, apparently, they had not perceived when solving the mathematical knowledge task by themselves. Regarding one of the groups that express some difficulties, the PTs refer that "Group A is not able to isolate what is constant (number of red beads) from what varies (number of blue beads)" (Beatriz and Júlia's written production). The understanding of the constant and varying elements of the pictorial pattern had not been expressed by the PTs before and seem to emerge from the analysis of student's work. When analysing another group's work, they have the following conversation:

- Beatriz: They are saying it well, but three times is not the number of times that the necklace number appears. It's the number of the little portions of the necklace.
Júlia: Of the groups. It is the number of groups of blue beads... I also think that she wants to say that the necklace number is the number of blue beads that each group has.

Referring still to another group, the PTs discuss again the meaning of the three in the rule that allows them to determine the total number of beads:

- Beatriz: Yes, three is the number of beads that have been added.
Júlia: Is three really the number of beads that have been added or is the number of groups?
Beatriz: It's the same, if you have three groups and you add one bead per group three is the number of groups and it is also the number of beads that add up.

The presented excerpts show that the PTs deepened their perception about the structural component of the pattern. In particular, Beatriz's last quote shows that she is able to connect both structural and numerical components of the pattern under study.

Regarding the category *generalising*, PTs' solution of "The necklaces" task shows that they were successful in establishing the general rule that relates the necklace's number with the total number of beads, as evidenced by the way they find how many beads are there in the 4th necklace:

The 4th necklace will have 14 beads because if we apply the rule $3n + 2$, being n the figure number, we can know the number of beads in each necklace. Being 3 the number of beads that are added [between terms] and 2 is the number of red beads. (Beatriz and Júlia's written production)

The way the pair refers to three shows that they recognise the coefficient of a linear function as the difference between two consecutive terms. Although this is an important fact to be known, the PTs' focus on numerical aspects seems to prevent them from realising the meaning of this value when they invert the rule, as observed in their explanation:

The rule to know the number of the necklace in any case (knowing the number of beads) is $\frac{c-2}{3}$, being c the number of beads, 2 the number of red beads and 3 the number of blue beads that had been added. (Beatriz and Júlia's written production)

After that, they analysed a video from a classroom where a student seeks to explain how he can find the necklace's number when he knows the total number of beads. Beatriz then explains Júlia what she understood:

Beatriz: He's got the total number of beads, right? And what he wants to find out is the number of blue ones. Imagine you don't have the number of the figure and you have the total number of beads. So, he does the total number of beads minus the red beads and gets just the blue ones. And now he knows there's always three sets. That's why he divides the total number of blue ones by three, because there are three sets. The result will give the number of blue beads in each set.

Beatriz seems to understand the meaning of division by three, considering the structural component of the pattern, which is particularly relevant because this PTs was always very focused in the numeric aspects of the pattern.

Discussion

The PTs' analysis of children's productions and discourse seems to impact in their own algebraic thinking, as it allows them to deepen their knowledge in a compatible way with the work that should be realised with early years' students. In the scope of relational thinking, when contacting with student's work, the PTs became more aware of different aspects such as the numeric relationships between the values on the two sides of an equation. Regarding the functional thinking, the analysis of different student's productions and one video seems to have contributed for PTs to attend in a more efficient way to the connection between the structural and the numerical component of a pictorial growing pattern and their relationship with the algebraic expression of generalisation.

The presented data shows that analysing student's work seems to make the PTs more conscious of their own mathematical knowledge, as referred by other studies (Appova & Taylor, 2019; Depaepe

et al., 2018). The way that the teaching experiment was conceived, with the integration of aspects related to the analysis of student's work in a mathematical content course, seems to benefit both the deepening of PTs' mathematical knowledge in an early algebra perspective and their ability to notice children's thinking. When focusing their attention on children's algebraic thinking, the PTs deepen mathematical knowledge dimensions that are compatible with an early algebra approach that we expect they will mobilise in their future practice. This study also has the particularity of not focusing only on a specific context of algebraic thinking, allowing an integrated vision of PTs' relational and functional thinking development, as well as addressing their noticing skill. Being important that teacher educators find new ways to help teachers to develop their noticing skills (Jacobs et al., 2018), this study gives a glimpse of the possibility to combine, in a content course, aspects of noticing and algebraic thinking.

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