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# The didactical transformation of the concept of variables

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*The adequate use of symbols as variables in mathematics is fundamental to the successful use of mathematics. It is thus of interest how this praxeology  $\wp = [T / \tau / \theta / \Theta]$  in the sense of Anthropological Theory of the Didactic (ATD) is transformed from academic mathematics to school mathematics. It may be supposed that this transposition is mediated through mathematics didactics. The paper analyses in detail the corresponding transpositions as they take place in the German speaking didactic tradition.*

*Keywords: Didactic transposition, praxeology, variables.*

## Introduction

The transposition of knowledge between institutions is a central theme of the theory of the didactics ATD, as coined by Chevallard (2019). An important part of mathematics is the use of symbols as variables and much research has been done on this, e.g., by Küchemann (1979), Epp (2011) and Usiskin (1988). The use of symbols as variables was invented and sharpened in academic mathematics in universities over a long period of time. The results of its didactical transposition to school mathematics will be investigated here. Since the 1970s there is also scientific didactical research on the teaching and learning of algebra and especially on the use of symbols. Broadly speaking, one may group this research in two clusters, where one is mainly empirically describing how students actually use symbols (e.g., Küchemann, 1979) and the other is investigating the issue from a normative point of view, i.e., analysing how students should use symbols (e.g., Freudenthal, 1973). Malle (1993) gave a synthesizing view that was very influential in the German-speaking countries. This paper restricts mainly to his theory, i.e., it defines the knowledge about variables in the institutions of the didactics of mathematics as considered here. Then, we have two more transposition processes, namely from the academic mathematics to the didactics of mathematics, and from the didactics of mathematics to the teaching of variables in schools.

The research questions investigated in this paper are thus:

1. How can the praxeologies of working with variables in academic math, school math and didactics be described?
2. What changes are involved in the didactic transpositions between these praxeologies?

The paper will recall some basics of ATD, then it will investigate the praxeologies and their transpositions. The paper does not, however, investigate the historical development of these praxeologies: Current school mathematics is not just the product of a transposition from academic math but also a product of earlier conceptions of school math.

## ATD

The underlying framework of this research is ATD (Chevallard, 2019). It models knowledge by praxeologies  $\wp = [T / \tau / \theta / \Theta]$  where T denotes a task or a type of tasks,  $\tau$  is a technique to solve it,

$\theta$  is a technology that explains  $\tau$ , and finally  $\Theta$  is a theory that justifies  $\theta$ . I will investigate the knowledge about variables as present in universities, didactics and schools and will use subscript U, D, S to differentiate between them:  $\mathcal{p}_U = [T_U / \tau_U / \theta_U / \Theta_U]$  is the praxeology of the use of variables and symbols in academic university level mathematics. The didactical transpositions to be investigated are thus  $\mathcal{p}_U \rightarrow \mathcal{p}_S, \mathcal{p}_U \rightarrow \mathcal{p}_D, \mathcal{p}_D \rightarrow \mathcal{p}_S$ .

## Symbols and variables at university level

Consider a university praxeology  $\mathcal{p}_U = [T_U / \tau_U / \theta_U / \Theta_U]$ . Most of the arguments given later apply to a large set of tasks, but to be concrete, fix  $T_U$  to be the task to prove that the square root function is monotonically increasing. This choice of task allows to choose  $T_U = T_D = T_S$ , but we will see that the other parts of the praxeology differ. Then  $\tau_U$  consists, among other things, of proposing domains ( $x_1, x_2 \in \mathbb{R}^+$ ), relations such as  $x_1 < x_2$ , writing down implications, applying transformation rules etc.  $\theta_U$  is the calculus (note the double sense of the word) of handling expressions and  $\Theta_U$  is predicate logic combined with set theory. The same  $\tau_U$  may, of course, be applied for other tasks  $T'_U$ .

Most introductory textbooks at university level only give a superficial introduction to  $\tau_U / \theta_U / \Theta_U$ . Many concepts are not defined explicitly but implicitly in the language game exemplified in many examples. A German university textbook that does contain some explicit treatments of the theoretical part of the praxeology is one written by Grieser (2015) but even this book explains variables very quickly in the context of expressions: “A propositional form is an expression that contains one or more variables and that will turn into a proposition when variables are replaced by values.” (Grieser, 2015, p. 31, own translation). Further on, Grieser differentiates between free and bound variables and introduces the semantics of quantification with a single sentence: “The statement  $\forall x \in M: A(x)$  is true if  $A(x)$  is true for all  $x \in M$ .” Other introductory university level textbooks typically devote even less time to make the logic of the use of variables explicit. However, there is of course the established theory  $\Theta_U$  that allows mathematicians to justify  $\theta_U$  and in turn  $\tau_U$ :  $\mathcal{p}_U$  is based on logic and set theory.

A cornerstone of the modern view of the logic foundations of mathematics was the book by Hilbert and Ackermann (1967). The theory is mainly consistent over the various presentations, e.g., Barnes and Mack (1975), Quine (1973), Rautenberg (2010) and many others. This model theoretic semantic is mainstream in academic math (although there are alternatives, such as proof theoretic semantics). A short recap is the following: A symbol like  $x$  is an atomic language element without meaning in itself. It can be used as a variable (also called individual variable) when it is given a domain  $D$  by saying  $x \in D$ . An assignment is a map that assigns an object of  $D$  to each individual variable and may be written e.g., as a set of mapping pairs  $\{a \rightarrow 1, x \rightarrow 2\}$ . When an assignment is applied, each logical formula of the language turns into a logical truth value and each functional formula turns into an object, especially the expression  $\forall x: A(x)$  is true if it is true for all assignments augmented by  $x \rightarrow \dots$  applied to  $A(x)$ . Based on this, one may define that expressions and formulas (equations) are equivalent if they give the same (truth) value under all assignments. This justifies transformation rules for expressions and formulas transparently. Constructs like quantification and set comprehension introduce bound variables that can be freely renamed.

This very brief presentation is not complete but is sufficient as a basis for the following. In the predicate logics of higher levels, there are not only individual variables, but also variables for

functions and relations. This is, however, irrelevant for the realm of elementary school algebra. That's why we can simply say variables instead of individual variables.

One of the subtle issues hidden in this presentation is whether the application of an assignment to a formula to be understood on the textual level or on the object level. The relevance of this distinction was discussed in the debate between Ruth Marcus (1962) and William van Orman Quine (1976). The point is that for domains with more than countable many objects, not every object can have a textual representation as there are only countable many expressions over any finite alphabet. Thus, an object may be missed if one checks  $\forall x: A(x)$  only for those  $x$  that have a textual description. However, the theorem of Skolem-Löwenheim (see Rautenberg, 2010, p. 112) states that at least for first order logic these two views are equivalent because every theory that has a model (i.e., a domain with assignments that makes it true) has a model with a countable domain as well. However, the equivalence of both interpretations is not clear in general. Quine has never moved away from his rejection of the substitutional view and some theories have been found that are incompatible with substitutional quantification. It will become clear later, why this deep theoretical question is nevertheless relevant for this paper.

The distinction between symbols and variables is not made in all university level textbooks, because it is not relevant in all parts of mathematics. However, it is usually pointed out when dealing with polynomials over finite rings such as  $\mathbb{Z}_2[x]$ . Here, it is important to see that  $x$  in the polynomial ring is a symbol (sometimes called a formal variable or indeterminate). There are two reasons for this: First,  $x$  maybe replaced by objects of any  $\mathbb{Z}_2$  algebra, e.g., the  $2 \times 2$  matrices with entries in  $\mathbb{Z}_2$ . And second, when specifying  $x$  as a variable in  $\mathbb{Z}_2$ , e.g., the polynomial  $0 \neq p = x^2 + x \in \mathbb{Z}[x]$  will be zero, because  $p(0) = p(1) = 0 \in \mathbb{Z}_2$ . Some textbooks take over this careful distinction to linear situations, e.g., the linear algebra textbook by Smith (1998, p. 199) writes down a system of linear equation twice, first with symbols  $x_i$ , then with variables  $s_i \in \mathbb{R}$ .

It is worth noting that the semantic clarification of what variables are holds throughout different pragmatic uses of variables. They may be used in expressions, in quantification to express identities or in equations to be solved for. The semantics is always the same, only the purpose differs. This situation is the same when learning variables in a specific programming language: The language fixes the semantics, but the variables can be used for different purposes.

## **Symbols and variables as seen from the didactics of mathematics**

A lot of work has been done to clarify variables, their use in school mathematics and the obstacles they present to learners. Freudenthal (1973, p. 262) was very influential with his conceptual distinction between unknown, indefinite, and variable. Küchemann (1979), on the other hand, takes an empirical approach and finds six different ways learners use letters. Usiskin (1988) incorporates the different types of variables into four conceptions of algebra. Mason and Sutherland (2002) give four uses: unknown, indeterminate, variable, and parameter. Some authors have also considered the relationship to the use of variables in programming or more generally in computer science (Arcavi, 1994). However, the classification of variable aspects according to Malle (1993) is probably the most influential at least in German-speaking countries. It can be found in many scientific publications up

to the present day. Notions coined by Malle are also contained in the German standards for school mathematics. Thus, this paper restricts itself to Malle's theory.

Malle distinguishes between "object aspect", "insertion aspect" (placeholder aspect) and "calculus aspect" as possible views of one and the same variable. Regarding functions, he furthermore differentiates the "object aspect" into the "single number aspect" and the "range aspect," whereby the latter is again differentiated into the "variation aspect" and the "simultaneous aspect." Since this will be questioned in the further course, here are the definition-like characterizations by Malle in the original wording (Malle, 1993, p. 46, p. 80, own translation):

1. Object aspect: Variable is seen as an unknown or unspecified number (more generally, as an unknown or unspecified object of thought).
  1. Single number aspect: Variable as any but fixed number from the relevant range. Only one number from the range is represented.
  2. Range aspect: Variable as any number from that range, representing any number in the range. This aspect again occurs in two forms:
    1. Simultaneous aspect: All numbers from the relevant range are represented simultaneously.
    2. Variation aspect: All numbers from the range in question are represented in chronological order (traversing the range in a certain way).
2. Insertion aspect: Variable is seen as a placeholder for numbers or spaces in which numbers (more precisely: number names) may be inserted.
3. Calculus aspect: Variable as a meaningless symbol manipulated according to rules.

This classification has found its way into many mathematics didactic textbooks, lectures, and publications. Obviously, many of Malle's aspects can be matched with ideas of other authors, e.g., Schoenfield and Arcavi (1988) list a large variety of meanings attributed to variables and try to systemize them into two conceptions: "polyvalent names" which matches Malle's single number aspect, and "variable objects" which matches Malle's variation aspect. A full discussion of this is beyond the scope of this paper, but note that Malle (p. 47) stresses that none of the aspects can be reduced to the others, while Epp (2011) argues that the placeholder aspect is fundamental. Linchevski, on the other hand, sees a version of the "object aspect" to be central when she proclaims that "Operating on and with the unknown implies understanding that the letter is a number. It does not only symbolize a number, stand for a number ..." (Linchevski, 2001, p. 143).

However, these differences shall be put aside and the didactical praxeology  $\mathcal{p}_D = [T_D / \tau_D / \theta_D / \Theta_D]$  chosen according to Malle's theory. The technology  $\theta_D$  and the techniques  $\tau_D$  are not given explicitly in Malle, but in my understanding it may be reconstructed to be composed of sub-technologies for every aspect defined in  $\Theta_D$ . E.g., in solving a simple equation like  $2x + 1 = 7$  one may apply the technique of inserting numbers to try to find a solution. This technique is based on the technology of placeholder, which is part of the theory. Alternatively, one may apply techniques based on technologies of the "calculus aspect" (equivalence transformations).

## **Symbols and variables in German school mathematics**

Symbols and variables are not differentiated in German schoolbooks. There is no underlying theory given, hence  $\Theta_s = \emptyset$ . There are, however, some technologies explained. E.g., the textbook by Freytag et al. (2013) for grade 7 explains (p. 51) that sometimes one wants to state general propositions that

hold not only for a specific number. In such situations, the textbook explains “you should use placeholders, also called variables. You can use signs like  $\nabla, \otimes, \square$  or letters. Always think about what numbers can be inserted.” (Freytag et al., 2013, p. 51, own translation). Part of the technology  $\theta_s$  is the equivalence of expressions, which is defined as follows: “Two expressions are called equivalent when they give the same result for every possible insertion.” (Freytag et al., 2013, p. 88, own translation). Regarding equations it is stated (p. 119) that “Two equations are called equivalent if they have the same solutions.” (own translation). It is noteworthy that this textbook is very similar with regards to variables, expressions, and equations to most other German textbooks for schools. Many of them contain phrases like “Equations are equivalent, if they have the same set of solutions.”

### **The transposition $p_U \rightarrow p_s$**

For a long time, in fact, for longer than research in mathematics education exists, school textbooks have established  $p_s$  similar to what has been given above using the example of one typical book. This transposition is mostly characterized by avoiding explicitly dealing with a theory and rather focusing on correct execution of procedures. Regarding the level of technology, the notion of equivalence is dealt very similarly in  $p_U$  and  $p_s$ . However, when it comes to variables in equations, the school technology differs from that of academic math: The two equations  $x = 0, y = 0$  are not equivalent in  $p_U$  because they differ in their truth value e.g. under the assignment  $\{x \rightarrow 1, y \rightarrow 0\}$  but they are equivalent in  $p_s$  as they have the same solutions, namely the numbers in the solution set  $\{0\}$ . Making the use of assignments explicit in school math could clarify such situations consistently.

### **The transposition $p_U \rightarrow p_D$**

This didactical transposition is surprisingly complex. We have seen that  $p_U \rightarrow p_s$  mostly consists of omitting theory (at the cost of having some things taught without justification) and neglecting some subtle differences. Thus, one may expect this transposition to simply factor over  $D$ , such that  $p_U \rightarrow p_D \rightarrow p_s$  gives the same transposition. However, this is not the case as, will be shown now.

First, one may wonder that in didactics there seems to be much more to be said about variables than in academic mathematics. To understand this phenomenon, I will look at Malle’s aspects from various points of view. He makes two important claims about the aspects:

1. Every variable can be seen under every aspect (p. 50).
2. No aspect is superfluous and can be reduced to another aspect (p. 47)

He underpins these two claims with many examples that shall support his conviction: “It is characteristic of doing mathematics that one needs to switch between aspects all the time, and sometimes one even has to keep several aspects in mind simultaneously.” (Malle, 1993, p. 48).

For some of his aspects, Malle refers to the literature, for others he doesn’t. Regarding the distinction between the “object aspect” and the “insertion aspect” he refers to Quine (1976), without giving page numbers in this book. One may suspect, however, that he refers to paragraph 26 where Quine discusses quantification and the distinction between substitutional and referential meaning of the quantified variable. Quine argues against Ruth Marcus for the position that the referential meaning is more powerful and more elaborated. In his view, substitutional semantics is typical for a stage in the learning process but that it should be overcome in mastering variables fully (Quine, 1976, p. 141).

Hence, for Quine these are not equivalent aspects but rivalling interpretations. While for academic purposes it is essential to find out if there is one single universal understanding of variables, this needs not be the case for learners. Quine's suggestion that the placeholder view is a step in the learning process may indeed be an answer, why didactics should have distinctions that are not so much relevant in university mathematics. Now, I look at other aspects in detail.

### **The variation aspect**

Recall that the variation aspect says that all numbers from the range in question are represented in chronological order (traversing the range in a certain way). This makes it possible to say that a function  $f$  (with equation  $y = f(x)$ ) is increasing over an interval, if for increasing  $x$ ,  $y$  increases as well. While this kind of speaking is common, especially in the natural sciences, it should be recalled that it is incompatible with  $\mathcal{P}_U$ . In calculus the fact that a real function  $f$  is monotonically increasing on an interval  $[a, b] \subset \mathbb{R}$  is expressed by  $\forall x_1, x_2 \in [a, b]: x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$ . Obviously, this is clumsier than "if  $x$  increases, then  $y$  increases". However, variation of values cannot be modelled within an assignment (because an assignment gives exactly one value for each variable), it can only be used sensibly by comparing several assignments. To bring it into a single formula for one assignment, one needs to "double" the variable  $x$  to get the pair  $x_1, x_2$  that can be compared within the same assignment. The proposition "If  $x$  grows,  $f(x)$  grows" would under the assignment  $\{x \rightarrow 5\}$  give the meaningless statement "if 5 grows,  $f(5)$  grows". Already Quine (1960) has expressed this very clearly: "As  $x$  increases, we are told,  $2/x$  decreases. Since numbers never increase or decrease, such talk of variables must be taken metaphorically. The meaning of this example is, of course, simply the general statement that if  $x > y$  then  $2/x < 2/y$ . Indeed, logicians and mathematicians nowadays use the word 'variable' mostly without regard to its etymological metaphor." (Quine, 1960).

The aspect of variation is thus not necessary for  $\mathcal{P}_U$  and, moreover, it presents problems as it goes beyond the established semantics of predicate logic. Knowing how to translate metaphorical statements into formally correct logical formulations is thus essential for  $\mathcal{P}_U$ .

### **The simultaneous aspect**

According to Malle, the simultaneous aspect means that a variable represents all numbers from the relevant range at the same time. What exactly does that mean? Malle explains: "The simultaneous aspect occurs predominantly when a variable is bound by an universal quantification. But even unbound variables can be considered from this aspect." (Malle, 1993, p. 81, own translation). Are there any useful examples of this aspect? Malle analyses a mathematical convergence proof to illustrate the role of the aspects. He considers a line of the proof (2), in which the constraint is formulated that a free variable should be in the unit interval, i.e.  $0 \leq x \leq 1$ , and then Malle assigns the simultaneous aspect to it: "In line (2),  $x$  is any number of  $[0; 1]$  (simultaneous aspect)" (p. 84, own translation). I see this as a contradiction to his own statement that  $x$  considered under the simultaneous aspect stands for "an arbitrary number" (underline by RO), whereby the underlined "any" indicates a singular contradicting the simultaneous aspect, which in my opinion necessarily requires the plural. In addition, an "arbitrary number" is exactly the formulation with which Malle also defines the single number aspect! Obviously, he failed in separating the aspects here.

In the following, I argue that the simultaneous aspect cannot be defined without contradictions. My argument is made up of various considerations and examples – often in the form of questions to which there is probably no answer: First, it should be noted that the simultaneous aspect is incompatible with the structure of predicate logic, as shown above. There, assignments always assign exactly one object. They may assign a set of objects to a variable, but then it is not a real variable, but a set variable. Consider this in more detail: For example,  $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = \sqrt{x}$ . If  $x \in \mathbb{R}^+$ , then it might be useful to say that one considers  $x$  simultaneously for all values and then find that  $f(x)$  stands simultaneously for  $\mathbb{R}_0^+$ . But isn't it much clearer to say that one applies a function  $f$  to every element of the set  $\mathbb{R}_0^+$  according to the definition  $f(S) := \{f(x) | x \in S\}$  applied to the set  $S = \mathbb{R}_0^+$ ? However, here, the variable  $x$  stands for positive real numbers. The simultaneous aspect would then imply that one does not need the distinction between a number and a set that contains the. In the simultaneous aspect of real variables, there is no difference between  $x \in \mathbb{R}$  and  $x^3 \in \mathbb{R}$  because both stand for the whole of  $\mathbb{R}$ . If  $n$  simultaneously stands for all natural numbers, then the divisibility statement  $2|n$  is simultaneously true and false. What does that mean? If  $x, y$  both stand simultaneously for whole  $\mathbb{R}$  (or another set), is there any difference between  $x$  and  $y$ ? In summary, all these points reveal substantial difficulties of the simultaneous aspect. To sum up, the transposition  $p_U \rightarrow p_D$  is mainly characterized by additions that are problematic from the point of view of  $p_U$ .

### **The transposition $p_D \rightarrow p_S$**

One might expect that the praxeology in schools is mostly influenced by the praxeology of the didactics. Most aspects are not touched upon in textbooks for schools (own analysis of grade 5-12 books). None mentions the simultaneous aspect, and the variation aspect is touched only slightly, e.g. when it is said that the derivative measures how a function varies or that a slider in a dynamic math program is used to explore how a graph varies when a parameter is changed. When dealing with monotony of functions at grade 11, schoolbooks typically don't talk about variation but use (without any explanation) a definition that is compatible with academic math, e.g., “A function grows monotonously on an interval if  $x_1 < x_2$  implies  $f(x_1) \leq f(x_2)$ “ (Jahnke & Scholz, 2009).

### **Conclusion**

The above presentation elaborated on the praxeologies of the use of variables in elementary algebra in three institutions. It was demonstrated that the didactical transpositions that occur are non-trivial, and this may produce obstacles for learners. For example, a mathematics teacher student who is enculturated in  $p_D$  maybe be puzzled if she takes a course in computer supported theorem proving and reads the explanation: “[...] Lean [...] works the same way as mathematical expressions. Once given a value, variables cannot be reassigned.” This sentence is meaningful only within  $p_U$ .

As a result of this work, one could start a project to elaborate how school algebra can be taught and fit into the didactical discourse without that many inconsistencies. Moreover, one should discuss, if the complexity of variables in  $p_D$  compared to  $p_U$  and  $p_S$  is beneficial or if simplification is possible.

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