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Relational thinking in Grade 1: A study framed in the new Portuguese mathematics curriculum

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In Portugal, the new mathematics curriculum presents Algebra as a theme since elementary school, for the first time. In the 2021/22 school year, the curriculum was implemented in pilot classes. This communication focuses on the implementation of a task in one of these first-grade classes and intent to discuss how relational thinking emerges in the students' work when they solve a word problem where the known quantities were represented visually (and not in an explicit numerical expression) and the unknown quantities were "hidden". Data were collected by observing the students' work and audio recordings of class discussions. The conclusions indicated that some interesting ideas of relational thinking emerged, and the difficulties expressed by the students showed the importance of teachers recognising opportunities in class to develop students' relational thinking.

Keywords: Relational thinking, equality, unknown quantities, new curriculum implementation.

Introduction

The new Portuguese mathematics curriculum (Canavarro et al., 2021) presents, for the first time, Algebra as a theme in elementary school, from grades 1 to 4. In this new curriculum, arithmetic equalities begin with addition in first grade. The learning objectives involve recognising and completing arithmetic equalities and justifying their correctness. Despite considering Algebra as an isolated theme, it also assumes its transversality and highlights the internal articulation with other mathematical themes, especially with Numbers, emphasizing an approach to generalised arithmetic. To develop algebraic thinking from an early age, some of the major ideas the new curriculum emphasizes are that students should "... understanding of variation in different situations and developing the ability to conjecture, recognize and express relationships and generalisations, numerical and algebraically, through representations appropriate to their ages, which can give meaning to what they think" (Canavarro et al., 2021, p. 10).

In this paper we focus on the implementation of a task, in one of the first-year classes of the new curriculum implementation, and we will discuss how relational thinking emerges in the students' work when they solve a word problem where the known quantities were represented visually (and not in an explicit numerical expression) and the unknown quantities were "hidden".

Relational thinking

Algebraic thinking can be seen "as a process in which students generalise mathematical ideas from a set of particular instances, establish those generalisations through the discourse of argumentation, and express them in increasingly formal and age-appropriate ways" (Blanton & Kaput, 2005, p. 413). One

of the possible approaches for the development of algebraic thinking is based on the potentially algebraic character of arithmetic, in other words, the generalised arithmetic, as considered in the new Portuguese curriculum. This implies the construction of the generalisation through numerical relationships and arithmetic operations and their properties, and also includes the notion of equivalence related to the equal sign. Carpenter et al. (2003) acknowledge these ideas as relational thinking, which refers to the ability to look at expressions or equations from a broader perspective, revealing the existing relationships in those expressions or equations. They described four benchmarks to analyse students' relational thinking: i) students express their interpretation of equal sign; ii) students accept as equal expressions not just those in the form $a + b = c$ (which is strongly related to the operational conception of equal sign); iii) students compare the two sides of an equal expression using calculation; and iv) students compare an equal expression without using calculation. In this last benchmark, according to the authors, the students showed evidence of relational thinking.

Relational thinking was also considered in Blanton et al.'s (2018a) conceptual framework of early algebra, in this way: "We take equivalence, expressions, equations, and inequalities to include developing a relational understanding of the equal sign and generalizing, representing, and reasoning with expressions, equations, and inequalities, including in their symbolic forms" (Blanton et al., 2018a, p. 31).

In this sense, they identify learning goals related to the relational understanding of the equal sign:

- Identify the meaning of '=' as expressing a relationship between quantities.
- Interpret equations written in various formats (e.g., other than $a + b = c$) to correctly assess an equivalence relationship (true/false number sentences).
- Solve missing value problems by reasoning from the structural relationship in the equation (open number sentences). (Blanton et al., 2018a, p. 32-33)

In this study we focused on the first goal presented by Blanton et al. (2018a), using the equality to express the relationship between quantities. As in another study of Blanton et al. (2018b), the quantities were represented visually, the unknown quantities were "hidden", and the equal sign was not present to students. The purpose of the task was to lead students to recognise the relationship between quantities and express it in a numerical expression.

Evidence of students' relational thinking has been presented in several studies, advocating its relevance for the development of algebraic thinking throughout schooling (e.g., Britt & Irwin, 2011; Fujii & Stephens, 2008; Mestre & Oliveira, 2012).

Methods

This study followed a qualitative methodology, with descriptive and interpretative characteristics, conducted with the objective of identifying and analysing how relational thinking emerges in the students' work when they solve a word problem where the known quantities were represented visually and the unknown quantities were "hidden". For that, students must recognise and give meaning to the arithmetic equality that might translate the situation posed in the word problem and must recognise the unknown quantity to complete the equality. The problem involves a visual representation focused on the number five and their multiples and does not explicitly present a numerical expression. For

data collection, we made audio recordings of students' autonomous work moments and collective discussion. We analysed students' written productions, their dialogues when solving the task and their explanations in class discussions.

The results presented are part of the implementation process of the new mathematics curriculum, which has taken place since the 2020/21 school year, in two first-grade classes of experienced teachers who work in public schools in different regions of the country. The four authors of this communication formed a collaborative work team, where the first two supported and monitored the operationalisation of the new curriculum in the classes of the main teachers (the other authors of the paper) and also participated in some of the implemented lessons. The team met systematically to plan different task proposals and reflect on their applications in the classroom.

The task presented was developed in one of the first-year classes of the new curriculum implementation and aimed to promote the development of algebraic thinking, namely relational thinking, to achieve objectives defined in the curriculum for the first grade. The participants of this study were 26 students (14 boys and 12 girls) with an average age of six years old.

This communication focused on evidence from the work conducted by the students on the task proposed in the classroom. The evidence focused, first, on the students' works made autonomously in pairs, and then on different pairs' resolutions presented and discussed collectively in class.

The task

As previously mentioned, the task presented a word problem accompanied by a visual representation of a known quantity (Figure 1). This visual representation uses an arrangement of five and its multiples, which is particularly important for the construction of number sense at this stage of students' learning.

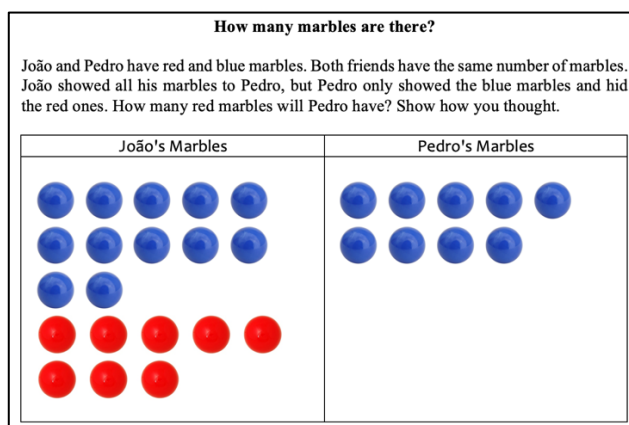


Figure 1: The task presented to students

The task required students to interpret the problem and use a visual representation as a resource to think about the presented situation. It started from an equality — the total number of marbles of two friends — and asked how many red marbles (not visible) of one friend would be needed to verify equality. It is worth mentioning, once again, that the quantities were not expressed numerically and only visually, which required students to interpret the hidden quantities as unknowns and use them to complete the equality.

Results

To present the results, we started by presenting and discussing excerpts from students' dialogues while working autonomously on the task. The excerpts were chosen based on the differences and pertinence that allowed us to obtain a big picture of what happened at those moments. We also present an excerpt from the collective discussion, in which one pair shows how they solve the task and how they think about it. It also included an exploration conducted by the teacher at that time.

Students' work in pairs

In the first excerpt, we can see that this student, Marta, expressed difficulties in realising that the equality refers to all marbles, red and blue. Furthermore, she did not use the structure of five to count and count one by one (Figure 2). Despite the student identifying the difference between the number of blue marbles of the two friends, at this moment, she could not find out the number of marbles (the red ones) that were missing.

- Marta: [counting João's red marbles] One, two, three, four, five, six, seven, eight. Therefore, if Pedro has the same number, he must have eight.
- Teacher: However, they had the same total number of marbles. Are Pedro's blue marbles the same number as João's?
- Marta: No. Pedro has hidden these three [comparing the number of blue marbles and identifying that Pedro is missing three blue marbles to have the same number of blue marbles as João]. He hid all this [rounding João's red marbles].
- Teacher: Can you find the number of red marbles Pedro hides?
- Marta: Uhmmm.... No.

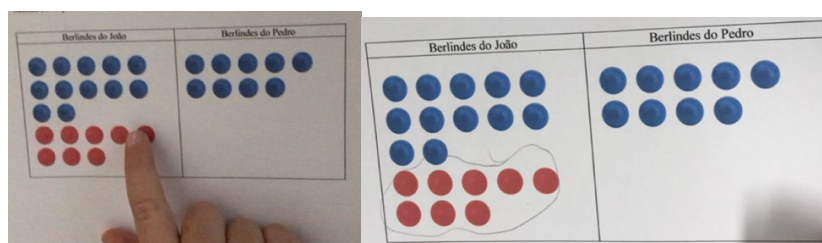


Figure 2: Marta used fingers movements and drawing in the visual representation

The second excerpt also shows that another student, João, counted two by two without using the visual representation of the structure of five. In addition, João experienced difficulties in understanding what was equal in the problem: the total number of marbles.

- João: There are 9 [pointing to Pedro's blue marbles] and here there are 12 [pointing to João's blue marbles].
- Teacher: How do you know it's 9?
- João: I counted.
- Teacher: How?
- João: 2, 4, 6, 8, 10. That's 10.
- Teacher: 10 are Pedro's marbles?
- João: No, it's 9.

At this moment, another student, Rafael, used an interesting idea of “tying” the amounts of red and blues marbles. After a while, he understood the problem and used emergent relational thinking ideas to find the answer to the problem when he realised that the total amount of marbles must be the same. The idea of “tie” illustrates the idea of balance which is emerging in the student's explanation,

although he mentioned 10 red marbles instead of the 11 that would be the correct number of red marbles.

- Rafael: Here, we do the same [referring to Pedro's marbles and pointing out João's]. We must tie the two.
- Teacher: What colour are you put to tie?
- Rafael: Both.
- Teacher: No. Blue marbles were already present. Which ones are missing?
- Rafael: Reds. We must place 8 [pointing to João's eight red marbles] plus 2 [pointing to João's two blue marbles in the 3rd row] and makes 10. We need to place 10 red marbles.
- Teacher: Do you think it takes 10 reds?
- Rafael: Yes, because we must [complete the] drawing with the reds [marbles].
- Teacher: If you give Pedro 10 reds, is it tied?
- Rafael: Yes.

The next excerpt also shows an interesting idea of “imagine” the red marbles that were hidden (Figure 3). This student, Maria, finds the answer to the problem relating the quantities, showing the emergence of relational thinking because she compares both amounts of blue marbles and, without counting them, she realises how many red marbles are missing.

- Maria: There are 11 red marbles.
- Teacher: How did you do that?
- Maria: I took the ones that I imagined that were red, and I counted them, and it was 11.
- Teacher: Did you take the ones you imagine red?
- Maria: Yes.
- Teacher: Explain it to me better because I do not understand it.
- Maria: So, if there were so many here [pointing to João's blue marbles], I drew one to make it the same [pointing to the missing space of Pedro's blue marbles in the second line].
- Teacher: So do the blue ones remain the same, right? Did you take 1?
- Maria: Yes.
- Teacher: How many blue ones were left behind?
- Maria: 10.
- Teacher: And then, what did you do?
- Maria: I counted the rest that were missing.
- Teacher: Did you discover how many red marbles Pedro was missing? That was the question.
- Maria: 11.

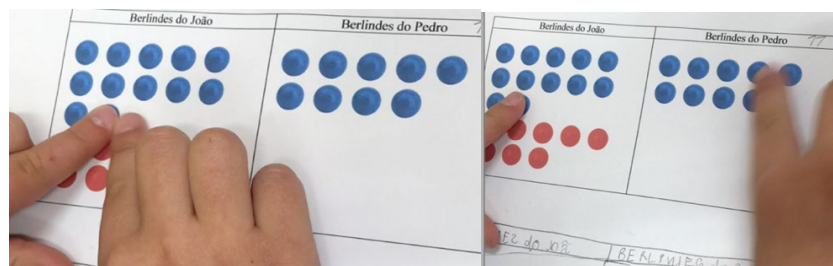


Figure 3: Maria also used finger movements on the visual representation

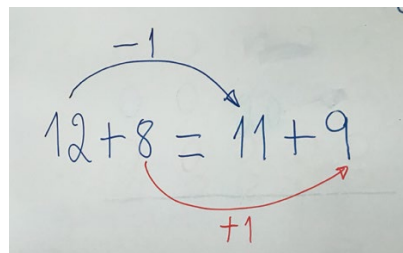
Collective discussion

During the collective discussion, one student, Carlos, explained how his pair thought to find the number of red marbles Pedro had and wrote $11 + 9 = 20$ on the board. His partner, Raquel, also

explained how they wrote the number of João marbles. The teacher posed questions asking the pair to explain their thoughts more clearly and leading the class to understand their reasoning.

- Teacher: When you wrote that $11+9$ equals 20, were you thinking about João or Pedro's marbles?
Carlos: Pedro.
Teacher: What is this 9? Is it the red marbles or the blue marbles?
Carlos: Blues.
Teacher: What is this 11?
Carlos: The red ones.
Teacher: When I think about João's marbles, am I going to write the same thing?
Carlos: No.
Raquel: We write 12 plus 8 equals 20.
Teacher: How is it that 12 plus 8 equals 20 and 11 plus 9 also equals 20?
Raquel: 8 is equal to 9 plus 1 and 9 are added to 11. Twelve is like taking one from the 9 and putting it in the 11, and then the two are the same.

After this moment, the teacher wrote arrows on the board that illustrated what the pair said (Figure 4). Then, she asked about the relationships between the parts involved in the equality and illustrated this procedure using the context of the marbles by saying: “João had 12 blue marbles, one more than Pedro and, as they had the same number of marbles, 20, Pedro had one more red marble than João”.



The image shows a handwritten diagram on a board. It features the equation $12 + 8 = 11 + 9$. A blue curved arrow labeled -1 points from the number 12 on the left to the number 11 on the right. A red curved arrow labeled $+1$ points from the number 8 on the left to the number 9 on the right.

Figure 4: Visual representation made by the teacher to show the pair’s explanations.

As we can see, this did not translate the concrete situation presented by the visual problem. In both situations, the visual representation and oral explanation, the teacher did not directly translate the most appropriate order of the quantities related to the colours of the marbles, which should be $12 + 8 = 9 + 11$. As Carlos wrote $11 + 9$ and Raquel said “12 plus 8 equals 20”, the teacher related the two expressions and illustrated the compensation relationship (plus one and minus one) without considering the context. At that moment, students appeared to move from the real context to focus only on numerical expressions. Therefore, at that point, the teacher proposed the resolution of the equality $13 + \underline{\quad} = 12 + 8$, and another student immediately replied that 7 should be written in the space and explained her reasoning as follows:

- Sandra: It is seven.
Teacher: And why? What did you think?
Sandra: Because you take 1 of 8 to join the 12.

As shown in this excerpt, Sandra did not use calculation but rather considered the relationship between the parcels to complete equality, using the concept of balance, which can be a sign of the emergence of relational thinking.

Final remarks

The results indicated that the task was challenging for students and for teachers to use it in the classroom to promote relational thinking. The students experienced some difficulties in understanding the meaning of equal quantities in the proposed problem, mostly because the quantities were expressed visually, and they needed to determine what equal quantities meant in the problem. Initially, they did not realise that the equality of the total number of marbles was the sum of the number of red and blue marbles. The equality expressed by the problem was not explicit numerically, and involved an unknown quantity that was “hidden” which they must find out to complete the equality. Even if the numerical expression had been posted explicitly, several studies have pointed out (e.g. Carpenter et al., 2003) that this is not familiar to students because the expressions that are usually presented in class are mostly in the form $a + b = c$. As the expression was not explicit in the problem, students had to interpret the situation and be able to write the expression or think in those terms, so that means that the task was even more challenging. The excerpts presented show that it was not easy for most students in the class to embrace the task. However, at the same time, the challenge was not a reason for students to give up on or not invest in the resolution. They were involved in discussions in pairs and collectively in class. In this way, after understanding what quantities are equal, they used some interesting ideas such as “tie the amounts” of marbles or “imagine” the red marbles that are hidden. In addition, another aspect concerns the structure of five and its importance to students’ development number sense at this age. The way the task was designed assumed that students used that visual representations to recognise the five structure: the marbles were posed in rows of five and, in this class, the students already worked with visual models such as necklaces with the 20 beads arranged in groups of five, by colours (for example, five red beads, then five green beads, then another group of five red beads and another group of five green beads). However, the students did not mobilise this experience to solve the problem. Therefore, this study also shows that the trend of counting one by one is strong and requires time to move to another and a more efficient counting method.

In addition to the difficulty of understanding what was equal in the problem, students also expressed difficulty in translating the situation in a numerical expression. They began to express the amounts of marbles of João and Pedro separately. One pair of students wrote $11 + 9 = 20$ concerning Pedro marbles and said that $12 + 8 = 20$ expressed João marbles. Therefore, they did not represent the situation in an expression of the form $a + b = c + d$. When the teacher tried to use this kind of equality expression, she did not express it in the most appropriate order, which should be $12 + 8 = 9 + ___$, and she did not use the visual structure of the initial problem that was proposed. After that, students focused their attention only on the numerical expression and used relationships between numbers. This experience enabled them to employ the same strategy using a similar numerical expression. In this sense, students recognised the equality using a compensation relationship, but if the teacher had noticed the importance of relating what students did with the most correct expression, this task could be more efficient in improving the understanding of equality using a concrete situation to give meaning to equality relationships. Thus, we conclude that this kind of work requires time and intentional work in relational thinking from early years of schooling. It is relevant to acknowledge that this is not an easy feat for most teachers, as demonstrated by this episode, even experienced

teachers may miss significant opportunities to enhance their students' understanding of the concept of equivalence.

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