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Supporting students' comparisons of different conceptions of variables by telling stories – Insights from a design research study

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Despite instructional efforts to help students understand variables as generalizers (e.g., in generalization activities), they often confound this meaning of the variable with a second meaning, the variable as an unknown (e.g., in equation-solving activities). In the present design research study, a learning environment was developed and investigated in which both meanings of the variable can be constructed and compared to each other. For such comparisons, students are engaged in constructing mathematical narratives. The design experiments with seventh and eighth graders indicate that such narratives are a suitable means to initiate in-depth comparisons, because the possibilities of a narrative allow students to reflect on the epistemic specificities of the two meanings of variables. In addition to this potential, three challenges to students' narrative construction are empirically identified.

Keywords: Variables as generalizers, variables as unknowns, mathematical narratives, learning from comparison, design research.

Theoretical background

Encouraging students to construct mathematical narratives is an approach suggested by Nemirovsky (1996) to help students make sense of algebraic concepts by linking them to “contexts about which students have expertise, expectations, and ways of thinking” (p. 220). *Narratives* are texts that tell a story about a “succession of events, real or fictitious” (Genette, 1972/1980, p. 25); and *mathematical narratives*, in particular, link aspects of events and situations to mathematical symbolism (Nemirovsky, 1996).

This paper reports on parts of a design research study that focuses on mathematical narratives that students create to compare different meanings of the variable (Usiskin, 1988; Malisani & Spagnolo, 2009). The research question of this paper is: *What potentials and challenges does the construction of mathematical narratives hold for students' understanding of the differences between variables as generalizers and as unknowns?* The paper first presents the theoretical background of the different meanings of the variable, the design principle of learning from comparison, and students' construction of mathematical narratives. After outlining the methodological framework, empirical snapshots of seventh and eighth graders' narrative construction are presented and discussed.

Understanding the variable as a generalizer and as an unknown by learning from comparison

The concept of variable is one of the central concepts in algebra, yet several studies show that many students have difficulty understanding this concept in all its meanings (e.g., Küchemann, 1981; Malisani & Spagnolo, 2009). While students tend to understand *variables as unknowns* more easily (i.e., a fixed set of hidden numbers to be revealed in equation-solving activities), the concept of *variables as generalizers* poses a significant challenge for many students to understand (Bardini et al., 2005; Küchemann, 1981; Malisani & Spagnolo, 2009; Prediger & Krägeloh, 2016). Variables as

generalizers derive their significance from the use of variables in generalization activities, where they are used to express general relationships without specifying specific numbers (Usiskin, 1988).

Although various instructional approaches have been established to improve students' understanding of variables as generalizers, students still tend to confound the two meanings of variables even after going through these approaches (e.g., Bardini et al., 2005; Prediger & Krägeloh, 2016). There are at least three reasons for this confounding: 1) Variables as generalizers and variables as unknowns share crucial similarities on the surface, because both represent numbers, are denoted by letters, and are subject to the same syntactical operations. 2) Students use imprecise language to express their ideas about variables (Prediger & Krägeloh, 2016). 3) Students are not aware of the different types of algebraic activities in which variables are used (Korntruff & Prediger, 2022). Many students naively assume that all variables are used for problem solving, where they represent hidden numbers that must be revealed by solving an equation. By extending this understanding also to variables in generalization activities, students are found to interpret a variable as a "*temporally indeterminate number whose fate is to become determinate at a certain point*" (Bardini et al., 2005, p. 129, emphasis in original).

A promising design principle to assist students distinguish the generalizer from the unknown is *learning from comparison*, as it helps students identify relevant structural features and distinguish them from irrelevant surface features. Learning from comparison has been found to be effective in improving students' understanding of concepts and procedures in various subject areas (Alfieri et al., 2013). However, the adaptation of the learning from comparison approach to the different meanings of the variable is not self-evident, because these meanings are grounded in different algebraic activities and therefore lie particularly deep. Hence, the *meaning* of the generalizer and the unknown need to be addressed in these comparisons (Bardini et al., 2005; Prediger & Krägeloh, 2016): The generalizer stands for *all possible numbers* of a relevant domain, whereas the unknown stands for *a (concrete set of) hidden number(s)* of a relevant domain *that we are looking for* (Usiskin, 1988). Moreover, the comparisons need to reflect the different *epistemic purposes* (Korntruff & Prediger, 2022) of the two variable meanings that originate from their respective algebraic activities: By using an unknown in problem-solving activities, one expresses *that something is not yet known, but shall be known in the future*; whereas by using the generalizer in generalization activities, one expresses that no further determination of an algebraic statement is needed: *there is nothing more to know* (Bardini et al., 2005). Thus, these different meanings of the variable are used for different purposes, which are associated with different epistemic intentions (one wants to know something vs. there is nothing more to know).

Since the desired comparisons of the unknown and the generalizer deal with particularly complex abstract ideas, it seems to be necessary to connect them appropriately to students' experiential contexts. As a promising way to achieve this connection, this paper explores the potentials and challenges of students' construction of mathematical narratives for such deep comparisons.

Inventing mathematical narratives as a way to connect abstract ideas to experiential contexts

Nemirovsky (1996) used students' narrative construction in algebra education as a "meaning-making activity" (p. 197) in which students "[fuse] events and situations with properties of symbols and

notations” (p. 220). Insofar as narratives present a series of real or fictional, logically and chronologically related events, they have two levels of chronology and time (Genette, 1972/1980, p. 33): the chronology of the story presented (*story time*) and the chronology in which the story is told (*narrative time*), which may differ, e.g., when flashbacks are used in the narrative. Mathematical narratives, in turn, tell stories about mathematical objects and relationships, and therefore their stories are primarily structured by logical rather than chronological connections (Dietiker, 2013), e.g., in $3x - 1 = 4$ the relationship between $3x - 1$ and 4 is not chronological, but logical. Thus, the story time of a mathematical narrative stands still with respect to the mathematical content, and story chronology is introduced only by reference to everyday situations (ibid.). A typical means of slowing down or even pausing the story time in narratives are descriptions of characters or situations that help to flesh out the imaginary world (Genette, 1972/1980). In order to fuse the different epistemic purposes of the variable with students’ experiential contexts, it might be sufficient for the students to narrate only situational descriptions. However, students introduce story chronology into their narratives to reconstruct the meaning of logical relationships, as later analysis of their narrations will show.

In many of the students’ narratives, *focalization* plays an important role, i.e., the point of view from which the story is told (Genette, 1972/1980, p. 189). *Internal focalization* describes the fact that the narrator says only what a character in the story knows – nothing more (*zero focalization*) or less (*external focalization*) than the character knows (ibid.). If a narrative section uses internal focalization, it must be possible to rewrite that section into the first person without changing or damaging its meaning (ibid.). For example, while “Anna Karenina was panic-stricken and blissful at it” can be translated into the first person (“I was...”), “Sherlock Holmes *appeared* to be in a state of nervous exaltation” cannot be translated into the first person without damage to semantic consistency.

Methodological framework of the design research study

Design experiments as method of data gathering

This paper, which stems from a larger design research project, focuses on snapshots of students’ *construction of mathematical narratives* in a teaching-learning arrangement that first engages students in generalization activities and informal equation-solving activities, and then explicitly contrasts these activities and their respective conceptions of the variable. While Design Experiment Cycle 1 was analysed with another focus (Korntreff & Prediger, 2022), this paper reports on Cycle 2 in which the material was refined to encourage students to make deeper comparisons by constructing mathematical narratives. In 2022, the design experiments were conducted by the author of this paper in Germany with six pairs of students ($n = 12$) and included 6-8 sessions of 90 minutes each, covering the entire learning environment. In total, 27 hours of video were recorded and partially transcribed. This paper focuses on transcribed video data from the final session of two pairs: Annika and Mara, two 7th grade grammar school students, and Rafik and Nadim, two 8th grade secondary school students. Prior to working on this learning environment, both groups had only very limited experience with variables represented by letters in their formal education (i.e., simple problem-solving activities).

Methods of data analysis

In Step 1 of the qualitative analysis, all segments of the transcripts were selected in which students develop or tell mathematical narratives. In Step 2, students’ utterances were analysed with respect to

their ideas about the variable as *generalizer* and *unknown* (all possible numbers vs. an unknown number) and their respective *epistemic purposes* (there is nothing more to know vs. there is something to be searched for). In step 3, the narratological concepts of *story time* and *focalization* were used as sensitizing concepts that “suggest” narratological “directions along which to look” (Blumer, 1954).

Empirical insights into students’ construction of mathematical narratives

Annika and Mara’s construction of a narrative for the variable as an unknown

Annika and Mara have worked on generalization and informal problem-solving activities in seven sessions over the past weeks and are familiar with the variable as a generalizer and as an unknown. Moreover, they have already started to contrast the relevant differences in meaning, e.g., by comparing typical language means such as “ x is a fixed number that we do not know yet” or “You use the variable to describe calculations of all possible numbers.” In a first activity involving mathematical narratives, the design experiment leader asks Annika and Mara to make an informed decision about whether they would assign the given narrative “Lara’s Training” (Figure 1) to the generalizer or to the unknown. Annika and Mara quickly decide that in this story the variable must be used as a generalizer:

- 9 Mara: [reads from the given story] Then she runs an arbitrary [*stressed*] number of laps around the sports field.
- 10 Annika: As much as she wants to run.
- 11 Mara: There is no specific-
- 12a Annika: is no specific number. So, she doesn’t have a specific number of laps she always runs.
- 12b Maybe she chooses it beforehand or something.

Mara begins to justify the pairs’ decision by referring to the given text (Turn 9). Here she is able to identify a typical language means (“an arbitrary number”) that is associated with the generalizer. Annika extends this justification by first interpreting the given narrative in her own words (Turn 10 and 12a), and then (Turn 12b) goes beyond the given text by making her interpretation plausible from an everyday perspective: Lara chooses (every day anew) how many laps she will run. Thus, Annika links the meaning of the generalizer to the specific situation described in the story, and makes implicitly clear that the purpose of the generalizer is to allow Lara to choose a new number of laps every day. Annika’s “maybe” in Turn 12b suggests an external focalization. That is, she only speculates about the Lara’s mental state (“she chooses it”).

Lara’s Training

Lara is training for a half marathon. For each training session, she first runs 1.5 km to warm up. Then she runs an arbitrary number of laps around the sports field. One lap is 400 m long.
Which expression can be used to describe the overall distance for every arbitrary trainings session of Lara?

Figure 1: Narrative “Lara’s Training”

The design experiment leader (DE leader) then asks the students, how to change the story so that it is about the variable as an unknown:

- 14 Annika: Um then she runs-
- 15 Mara: There has to be a number instead of “arbitrary” [*in the story*].
- 16 Annika: Yes, a number

- 17 Mara: Three laps, four laps
 18 DE leader: Yes? Can you write down for me the equation for three laps?
 19 Mara: [*writes down the expression $1.5 \text{ km} + 400 \text{ m} \times 3$*]
 20 DE leader: This is an equation in which the unknown appears?
 21-23 [*Annika points to the missing solution; Mara calculates the solution 3.7 km and adds $= 3.7 \text{ km}$ to the expression; Annika nods in agreement*]

In this scene, we can see that there seems to be a misunderstanding about the concept of the unknown. The students seem to know, that they need to find a complete equation to express the unknown (Turn 21), and that the variable as an unknown number is a specific number (Turns 15-17). However, the students do not seem to be aware that the unknown expresses the epistemic purpose that this particular number is to be sought. There are at least two explanations for the students' blind spot and their misunderstanding in this scene. First, so far it has been possible for the students to contrast the generalizer and the unknown by checking whether all possible (relevant) numbers are denoted by the variable or only one specific number, because these students were working only with linear equations of the form $ax + b = c$. Thus, limiting the set of numbers that can be expressed by a variable to one "specific" number may have been sufficient for the students to contrast the unknown with the generalizer. Second, a *cognitive challenge* can be observed: Usually, students do not hide unknown numbers for themselves, but only have to find them. This means that the procedure of 1) choosing a number that will later become an unknown, 2) using this chosen number to create an equation with a given solution, and 3) finally hiding the chosen number "under a letter" is quite unfamiliar to the students. Moreover, this procedure is cognitively very complex, because the students have to perceive a number that they have chosen and that they therefore already know as something unknown (to someone else). Therefore, they have to pretend not to know something they already know.

The scene continues with a response from the DE leader in which he reminds the two girls of a statement they made just before this activity:

- 24 DE leader: But you just said that it [*the unknown*] is always about looking for or finding such an unknown number. Look [*points to the equation $4 + 1.5x + 2.5 = 13$ that Annika and Mara previously assigned to the variable as an unknown*], and this equation somehow looks a bit different here.
 25 Mara: [*crosses out 3 in $1.5 \text{ km} + 400 \text{ m} \times 3 = 3.7 \text{ km}$ and writes down x*] [...]
 26 DE leader: Okay this is different from what you said before. Then tell me: What would the story be like if it were about the unknown number?
 [...]
 28 Annika: Lara warms up one- 1.5 kilometres- um one and a half kilometres
 29 Mara: That [*the 1.5 km*] is unchangeable.
 30 Annika: And then she just runs- and then one must- um how many um [*moans because the explanation is not easy for her*]. One round that she per- after- um after um [*moans again*]
 31 [*everyone laughs together*]
 32 Annika: Explaining doesn't work.
 [...]
 35 Annika: Okay, um [*looks at the given story "Lara's training"*] um. [*Using the given story:*] Then she runs an unknown number of laps around the sports field. One lap is 400 meters long. How many laps does Lara run if the result is 3.7 kilometres?
 [...]
 37 Mara: Or she just wants to run a certain number of kilometres and first writes down the result and then she calculates how many laps she has to run.

Two observations can be made from this scene: First, it is obvious that constructing a mathematical narrative places a high discursive demand on Annika (Turn 30, 32). She meets this *discursive challenge* by using the given story to express her conceptual ideas about the unknown (Turn 35). Second, after Annika's explanation, Mara is able to go beyond the given narrative by attributing to the character Lara the desire to run a certain number of kilometres (Turn 37) and to perform certain actions, namely writing something down and calculating. Thus, Mara chooses an internal focalization ("she just wants") to extend the story, and introduces a chronology that was not present in the given story: "*first* writes down the result *and then* she calculates". Mara's utterance is remarkable: Not only does she link the equation ($1.5 \text{ km} + 400 \text{ m} \times x = 3.7 \text{ km}$) to the story context, but she also explains in the story the epistemic order in which the equation has to be read: The total number of kilometres comes first (as an already known quantity), and then one has to calculate the corresponding number of laps (by which she expresses the unknown). In addition, the internal focalization of Mara's extended story allows an access to Lara's epistemic position: What does Lara want and what does she know? In this way, Mara extends Annika's story in such a way that it makes sense (from Lara's perspective) to search for an unknown number of laps. Thus, she implicitly expresses in her story the epistemic purpose of the unknown, namely to describe a number that is not yet known.

Rafik and Nadim's narrative challenge of constructing a sense-making context

Similar to Annika and Mara, Rafik and Nadim are familiar with the generalizer and the unknown, having worked on generalization and informal problem-solving activities in six sessions over the past few weeks. In the following scene, Rafik and Nadim are asked to construct a situation with a variable n that describes a number of laps run on a sports field.

- Nadim: This is difficult.
 Rafik: It is more difficult than the first [*a taxi context, regularly mentioned in the last six sessions, where the variable represented a number of kilometres*].
 Nadim: [*After thinking for about 6 seconds*] The thing is, I don't know what else we could use from here [*points to description of quantities from the taxi context, i.e., basic fee, price per kilometre, tip*], something like that.
 Rafik: I would: n equals. And then I don't know anymore.

This scene illustrates a *narrative challenge*: It is not at all easy for the students to create a meaningful sense-making context around a given interpretation of a variable (n is a number of laps run on a sports field). This challenge continues after the DE leader introduces the story frame of a charity run where a certain amount of money is collected for each lap run on the sports field. The students create the equation $2.00 \times n + 1.00 = 10.00 \text{ €}$ and explain it as follows:

- Rafik: 1 euro for the entry fee and then we have another 2 € for each completed lap. So, I run a lap, get 1 Euro - um 2 €.
 Nadim: Not get. Pay.
 Rafik: Pay? [*looks confused*] I thought one gets it. But whatever. One then pays 2 €.

This scene suggests that the story frame of a charity run is not something these students have had enough experience with (usually a sponsor gives a certain amount of money for each lap a student runs on the sports field). So they end up with a story where someone pays a basic fee of 1 € and on top of this donates 2 € for each lap run. Even if this narrative is not very realistic, it still allows the students to reflect about the epistemic purpose of the variable as an unknown:

Nadim: If, for example, you only- So if you only have 10 euros, then you say- um you plan um to just run 5 times [*shows circular motion with his hand*]. So 11 euro plus the entry fee. Then you just plan to run um here [*shows a circular motion with his hand*] 5 times. And um, once you have seen that you only have 5 euros, then you can- um 11 euro, then you cannot um- So you can um- How should I say? This- So that you know how much you can run. So that you do not run more or less.

In this remarkable utterance, we can once again observe the *high discursive demand* that the articulation of a narrative, in which the use of the variable as an unknown makes sense, poses for the students. However, Nadim overcomes this challenge and extends the narrative of a somewhat strange charity run in such a way that the epistemic purpose of the unknown becomes visible in his narrative context: Because you want to pay exactly 11 €, you have to find out how many laps you can run. In addition, we can see in Nadim's utterance that he overcomes the *cognitive challenge* of constructing conditions for the unknown number that we have already observed in Annika and Mara's learning process. He manages to construct a reasonable solution (11 €) from a chosen number (5 laps) and is then still able to express that one wants to know how many laps one can run (n laps). Similarly to Mara, Nadim uses internal focalization in his narration ("you plan", "you have seen", "you know") and introduces a chronology into his story: "once you have seen" how much money you have, you can "know how much you can run," so that later on "you do not run more or less".

Discussion

This paper explored the potentials and challenges of constructing mathematical narratives for students' understanding of the differences between variables as generalizers and as unknowns. The empirical insight into the students' construction of narratives indicate that narratives can be a fruitful way to engage students in the comparison of the unknown and the generalizer, because they allow the students to connect these abstract concepts to everyday contexts. Moreover, the genre of mathematical narratives allowed the students to express the deep laying differences between the *epistemic purposes* of the unknown and the generalizer in form of variations on a narrative. Specifically in case of the unknown, the students used two typical narrative devices: 1) They *introduced a story chronology* into the timeless structure of an equation, e.g., $2n + 1 = 10$ €, in order to reinvent the epistemic logic of this equation: One must *first* know the 10 €, and only *then* can one search for the unknown number. 2) They *used internal focalization* to identify (with) an epistemic standpoint from which it makes sense to search for an unknown number (because you only want to spend a certain amount of money). These findings again suggest that a reflection on the deep laying epistemic purpose is particularly important for students to construct a proper understanding of the difference between the generalizer and the unknown. However, the empirical snapshots also revealed three challenges that students face in constructing narratives: A discursive challenge of articulating narratives, a cognitive challenge of constructing conditions for the unknown number, and a narrative challenge of constructing a sense-making context for their narrative.

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