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# The Doob transform and the tree behind the forest, with application to near-critical dimers.

Lucas Rey \*

CEREMADE, CNRS, UMR 7534, Université Paris-Dauphine, PSL University,  
75016 Paris, France.

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## Abstract

The Doob transform technique enables the study of a killed random walk (KRW) via a random walk (RW) with transition probabilities tilted by a discrete massive harmonic function. The main contribution of this paper is to transfer this powerful technique to statistical mechanics by relating two models, namely random rooted spanning forests (RSF) and random spanning trees (RST), and provide applications.

More precisely, our first main theorem explicitly relates models on the level of partition functions, and probability measures, in the case of finite and infinite graphs. Then, in the planar case, we also rely on the dimer model: we introduce a killed and a drifted dimer model, extending to this general framework the models introduced in [Chh12, dT20]. Using Temperley’s bijection between RST and dimers, this allows us to relate RSF to dimers and thus extend partially this bijection to RSF. As immediate applications, we give a short and transparent proof of Kenyon’s result stating that the spectral curve of RSF is a Harnack curve, and provide a general setting to relate discrete massive holomorphic and harmonic functions. The other important application consists in proving universality of the convergence of the near-critical loop-erased RW, RST and dimer models by extending the results of [CW21, Chh12, HSB22] from the square lattice to any isoradial graphs: we introduce a loop-erased RW, RST and dimer model on isoradial discretizations of any simply connected domain and prove convergence in the massive scaling limit towards continuous objects described by a massive version of SLE<sub>2</sub>.

## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
1.1	General results . . . . .	3
1.2	Applications . . . . .	5
1.3	Perspectives . . . . .	6
1.4	Organization of the paper . . . . .	6
<b>2</b>	<b>Preliminaries</b>	<b>7</b>
2.1	First definitions . . . . .	7
2.2	Wilson’s algorithm and the transfer current theorem . . . . .	13
2.3	The Doob transform technique . . . . .	16

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\*lucas.rey@dauphine.psl.eu

<b>3</b>	<b>From rooted spanning forests to directed spanning trees</b>	<b>17</b>
3.1	The Doob transform technique for rooted spanning forests . . . . .	17
3.1.1	The infinite case . . . . .	18
3.1.2	The finite case . . . . .	18
3.1.3	Statement and proof of the main result . . . . .	18
3.2	Positive massive harmonic functions . . . . .	19
3.2.1	The finite case . . . . .	19
3.2.2	The infinite case: Martin boundary of a killed random walk . . . . .	20
<b>4</b>	<b>The planar case: from rooted spanning forests to dimers</b>	<b>22</b>
4.1	Preliminary: Temperley’s bijection . . . . .	22
4.2	The drifted and killed dimer models. . . . .	25
4.3	Massive holomorphy and self-duality. . . . .	31
<b>5</b>	<b>Application: rooted spanning forests of <math>\mathbb{Z}^2</math>-periodic graphs</b>	<b>32</b>
5.1	Definitions and first results. . . . .	32
5.2	Massive harmonic functions on $\mathbb{Z}^d$ -periodic graphs . . . . .	33
5.3	Temperley’s bijection for the free energy of $\mathbb{Z}^2$ -periodic models . . . . .	34
<b>6</b>	<b>Application to near-critical dimers</b>	<b>35</b>
6.1	General setting: near-critical drifted trees and dimers on isoradial graphs . . . . .	36
6.1.1	Isoradial graphs, $Z$ -invariant Laplacian and discrete exponential functions. . . . .	36
6.1.2	Isoradial approximation of an open set. . . . .	38
6.1.3	Application of the Doob transform technique and introduction of the model. . . . .	39
6.1.4	Near-critical regime. . . . .	40
6.2	From the killed random walk to the drifted random walk . . . . .	42
6.3	Convergence of the loop-erased killed random walk to massive $SLE_2$ . . . . .	43
6.4	Convergence of the near-critical rooted spanning tree and drifted dimer models . . . . .	45
<b>A</b>	<b>Critical and near-critical isoradial Laplacians</b>	<b>45</b>
A.1	Proof of the asymptotic of the mass in the near-critical regime . . . . .	46
A.2	Proof of the uniform crossing estimate . . . . .	47

# 1 Introduction

A *rooted spanning forest* (RSF) of a graph  $G = (V, E)$  is a set of directed edges  $F \subset \vec{E}$  with no cycles such that every vertex has at most one outgoing edge in  $F$ . The root vertices  $R(F)$  are the vertices with no outgoing edges. A *rooted spanning tree* (RST) with root vertex  $r$  is a RSF with only one root vertex  $r$ . When  $G$  is finite, given a conductance function  $c : \vec{E} \rightarrow \mathbb{R}_{>0}$  and a mass function  $m : V \rightarrow \mathbb{R}_{\geq 0}$  on  $V$ , the *Boltzmann measure* on the set of RST, resp. RSF, is defined by

$$\forall T, \mathbb{P}_{\text{RST}}^r(T) = \frac{\prod_{\vec{e} \in \vec{E}} c_{\vec{e}}}{Z_{\text{RST}}^r(G, c)}, \text{ resp. } \forall F, \mathbb{P}_{\text{RSF}}(F) = \frac{\prod_{\vec{e} \in F} c_{\vec{e}} \prod_{r \in R(F)} m(r)}{Z_{\text{RSF}}(G, c, m)}$$

where the normalizing constant is the *partition function*

$$Z_{\text{RST}}^r(G, c) = \sum_T \prod_{\vec{e} \in \vec{E}} c_{\vec{e}}, \text{ resp. } Z_{\text{RSF}}(G, c, m) = \sum_F \prod_{\vec{e} \in F} c_{\vec{e}} \prod_{r \in R(F)} m(r).$$

Kirchhoff’s celebrated matrix-tree theorem [Kir47] computes the partition functions:

$$Z_{\text{RST}}^r(G, c) = \det(\Delta_{|V \setminus \{r\}}) \text{ , resp. } Z_{\text{RSF}}(G, c, m) = \det(\Delta^k),$$

where  $\Delta$  is the discrete Laplacian on  $G$  with conductances  $c$  and  $\Delta^k$  is the discrete massive Laplacian on  $G$  with conductances  $c$  and masses  $m$ . The RST model was later shown to exhibit

many integrable features: RST can be sampled by Wilson’s algorithm [Wil96] using the loop-erasure of the random walk  $S$  associated to  $\Delta$ , the transfer current theorem of Burton and Pemantle [BP93] shows that edge correlations are determinantal, [Pem91] and later [BLPS01] build Gibbs measures on infinite graphs, and more recently the branches of the tree were shown to converge to  $SLE_2$  in the scaling limit in [LSW04]. For specific planar graphs called *Temperleyan graphs*, RST were shown in [TF61, KPW00] to be in bijection with *dimer covers*, exhibiting a fruitful link with another well-studied model of statistical mechanics. Many properties can be transferred from one model to the other by this bijection.

Some of these integrable features also hold for the RSF model, replacing  $\Delta$  by  $\Delta^k$  and the random walk (RW)  $S$  by the killed random walk (KRW)  $S^k$ , but not all of them: in particular no extension of Temperley’s bijection is known to be true. One of the main goals of this paper is to fill in this gap by introducing a general method which enables to relate a RSF on a planar graph with a RST and a dimer model on the same graph. The key underlying idea is to use the *Doob transform technique* (see [KGSK12] for a general introduction, [DHESC20] for a more recent and concise introduction). Given a positive massive harmonic function  $\lambda$ , this technique relates the KRW  $S^k$  to a RW  $\tilde{S}$  with conductances  $\tilde{c}_{(x,y)} = \frac{\lambda(y)}{\lambda(x)}c_{(x,y)}$ , and also the associated massive and non-massive Laplacian operators:

$$\forall x \sim y \in \mathbb{V}, \mathbb{P}_x(\tilde{S}_1 = y) = \frac{\lambda(y)}{\lambda(x)}\mathbb{P}_x(S_1^k = y) \quad ; \quad \tilde{\Delta} = \text{Diag}(\lambda)^{-1}\Delta^k \text{Diag}(\lambda).$$

The question of existence of massive harmonic functions is addressed in Section 3.2.

## 1.1 General results

A version of the transfer current theorem for RSF established by Chang [Cha13] states that

$$\forall \{\vec{e}_1, \dots, \vec{e}_k\} \subset \vec{\mathbb{E}}, \mathbb{P}_{\text{RSF}}(\vec{e}_1, \dots, \vec{e}_k) = \det \left( (H_{\vec{e}_i, \vec{e}_j}^k)_{i,j=1, \dots, k} \right) \prod_{i=1}^k c_{\vec{e}_i}$$

where the *transfer current operator*  $H^k$  is defined in terms of the massive *potential* (or *Green function*): for all  $\vec{e} = (x, y), \vec{f} = (z, w) \in \vec{\mathbb{E}}$ ,

$$H_{\vec{e}, \vec{f}}^k = \frac{V^k(w, y)}{\Delta^k(y, y)} - \frac{V^k(x, y)}{\Delta^k(y, y)} \quad \text{where} \quad \forall x, y \in \mathbb{V}, V^k(x, y) = \sum_{n=0}^{\infty} \mathbb{P}_x(S_n^k = y). \quad (1)$$

Note that the transfer current theorem for RST rooted at  $r$  corresponds to  $m = \delta_r$ . When  $\mathbb{G}$  is infinite,  $\mathbb{P}_{\text{RST}}$  is the *Wilson’s measure* defined in Section 2.2. Our contribution consists in relating a RSF and RST model through the Doob transform of the massive potential (see Theorem 3.1 for a precise statement):

**Theorem 1.1.** *When  $\mathbb{G}$  is finite,*

$$Z_{\text{RSF}}(\mathbb{G}, c, m) = Z_{\text{RST}}^o(\mathbb{G}^o, \tilde{c}).$$

*In the right-hand side, RST are considered as rooted “outside”  $\mathbb{G}$ . Moreover, for finite or infinite graphs  $\mathbb{G}$  with no loop, if the KRW  $S^k$  dies almost surely in finite time, the measure  $\mathbb{P}_{\text{RST}}^o$  on RST of  $\mathbb{G}^o$  weighted by  $\tilde{c}$  rooted “outside” satisfies the transfer current theorem with transfer current operator*

$$\forall \vec{e} = (w, x), \vec{f} = (y, z) \in \vec{\mathbb{E}}, \tilde{H}_{\vec{e}, \vec{f}} = \frac{\lambda(y)}{\lambda(w)} \frac{V^k(w, y)}{\Delta^k(y, y)} - \frac{\lambda(y)}{\lambda(x)} \frac{V^k(x, y)}{\Delta^k(y, y)}.$$

The novelty of this theorem is that it relates a RSF and a RST model, at the level of partition functions when  $\mathbb{G}$  is finite, but also in general at the level of edge probabilities as they are expressed in terms of the massive potential both in the RST model (our Theorem 1.1) and in the RSF model (Equation (1)).

When  $G$  is furthermore planar, we relate RSF to dimers using the intermediate RST model. This can be seen as an extension of Temperley’s bijection to RSF although there is no one-to-one mapping between configurations, as we relate the partition functions, Laplacian and Kasteleyn matrix of a RSF and dimer models. More precisely, assuming that the conductance function  $c$  is symmetric, given a *massive harmonic* function  $\lambda : V \rightarrow \mathbb{R}_{>0}$  and an arbitrary function  $\lambda^* : V^* \rightarrow \mathbb{R}_{>0}$  on vertices of the dual, we define in Figure 1 two dimer models on the double graph  $\underline{G}^D$  called the *killed* and *drifted* dimer model (see Figure 4 for the definition of the double graph). The drifted dimer model is defined such that when  $G$  is finite, by Temperley’s bijection

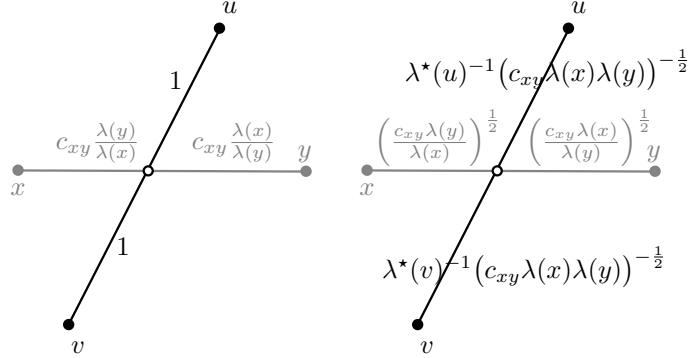


Figure 1: The local weights  $\nu^d$  for the drifted dimer model (on the left) and the local weights  $\nu^m$  for the killed dimer model (on the right). The primal graph is in gray, the dual graph in black.

[KPW00], dimer configurations of the double graph are in weight-preserving bijection with RST of  $G^o$  (that is RST of  $G$  rooted outside) weighted by  $\tilde{c}$ . We state the following result, which is a summarized and simplified combination of Proposition 4.5, Corollary 4.6 and Proposition 4.7 (when  $G$  is finite, the statements have to be refined to take into account boundary effects):

**Theorem 1.2.**

- (i) When  $G$  is finite,  $Z_{\text{RSF}}(G, c, m) = Z_{\text{dim}}(\underline{G}^D, \nu^d)$ , where the right-hand side is the partition function of the drifted dimer model on  $\underline{G}^D$  with weights  $\nu^d$ .
- (ii) The drifted and killed dimer model are gauge equivalent: their respective Kasteleyn matrices  $K^d$  and  $K^k$  satisfy  $K^k = \Phi K^d \Psi$  for some diagonal matrices  $\Phi, \Psi$ .
- (iii) There exists an operator  $\tilde{\Delta}^*$  on the dual graph such that the Kasteleyn matrix of the killed dimer model satisfies

$$(K^k)^\dagger K^k = \begin{pmatrix} \Delta^k & 0 \\ 0 & \tilde{\Delta}^* \end{pmatrix}.$$

*Remark 1.3.*

- When  $c$  is symmetric, this gives an alternative proof of our Theorem 1.1 using the local statistics formula for edge probabilities in the dimer model (see the discussion after Proposition 4.7).
- (iii) implies a natural definition of *discrete massive holomorphic* function as the functions  $f$  in the kernel of  $K^d$ . For isoradial graphs with critical or  $Z$ -invariant weights, it generalizes the discrete holomorphy theory of [Ken02, CS11] (non-massive case) and [MS09, dT20, Par22] (massive case). This is discussed in Section 4.3.

This result was previously shown in some specific cases: our drifted and killed dimer models can be seen as generalizations of the drifted and killed dimer models (on  $\mathbb{Z}^2$ , with specific weights) of [Chh12] which are also central to the subsequent article [HSB22]. These models were themselves related to a drifted and killed Laplacian operator by ad hoc computations. The idea of relating a

massive Laplacian with a dimer model through an intermediate RST model was already used in the special case of isoradial graphs in [dT20] (see in particular Theorems 19,21), with the purpose of relating the massive Laplacian with a  $Z$ -invariant massive Dirac operator. Let us also mention a related work in a forthcoming paper of Ballu, Boutillier, Mkrtchyan and Raschel of which we are aware.

Every time it appeared, this idea was model-specific: our approach unveils that *whenever we can find “nice” massive harmonic functions on a graph, the RSF model can be related to a RST and dimer model*. It has the advantage of using the well-developed Doob transform technique, which is used to study RW or KRW conditioned on particular events. The link with *quasi-stationary distribution* might also provide new and interesting applications.

## 1.2 Applications

We provide applications of our general technique to at least two settings.

**$\mathbb{Z}^2$ -periodic graphs** On  $\mathbb{Z}^2$ -periodic graphs, a family of Gibbs measures on the infinite graph is obtained in [Ken19] as the weak limit of multi-type spanning forests (MTSF) on a toroidal exhaustion. A MTSF is a spanning collection of rooted trees and cycle-rooted trees. In the limit, it becomes a RSF of the infinite graph with infinite unrooted tree components of a given slope and rooted tree components. We build “nice”  $\mathbb{Z}^2$ -periodic positive massive harmonic functions, and using the Doob transform technique, we find that the characteristic polynomial of the MTSF model is equal up to a change of variable to the characteristic polynomial of a drifted dimer model with explicit  $\mathbb{Z}^2$ -periodic weights: denoting by  $P^k$  and  $P_{\text{dim}}^d$  the characteristic polynomials of the RSF and drifted dimer models, Proposition 5.4 states that

**Proposition 1.4.** *There exists  $(z_0, w_0) \in (\mathbb{R}_{>0})^2$  such that*

$$\forall (z, w) \in \mathbb{C}^2, P^k(z, w) = P_{\text{dim}}^d \left( \frac{z}{z_0}, \frac{w}{w_0} \right).$$

This provides an alternative and direct proof of the fact (stated among other things in Theorem 1.4 of [Ken19]) that the spectral curve of the MTSF model is a Harnack curve, see our Theorem 5.5.

**Near-critical dimers and trees.** Our second and more detailed application is to extend the results of [Chh12] and [HSB22] to isoradial graphs: we prove universality of the convergence of the RST and drifted dimer model in the *near-critical regime* or *massive scaling limit*. We use the *discrete exponential function* which is massive harmonic for the  *$Z$ -invariant massive Laplacian* on isoradial graphs (these objects were introduced and studied in [BdTR17]) to relate a RSF model with a killed and drifted dimer model. In the critical case (corresponding to a vanishing mass  $m = 0$ ), our model corresponds to the critical RST model on isoradial graph whose branches are known to converge to  $SLE_2$  by combining [LSW04] and [CS11]. The associated killed and drifted dimer models coincide with the critical dimer model on isoradial graphs whose height function is known to converge to the gaussian free field [Ken00, Ken01, dT07, Li17]. In [HSB22], they consider a dimer model on  $\mathbb{Z}^2$  with specific weights in the *near-critical regime* which corresponds to scaling appropriately the mass with the mesh, and show convergence of the branches of the RST model towards *massive  $SLE_2$*  building on the proof of convergence of the loop-erased KRW towards massive  $SLE_2$  by Chelkak and Wan [CW21]. The height function of the near-critical dimer model also converges and the limit exhibits *conformal covariant properties*.

In this paper, we show *universality* of the convergence of the near-critical RST and dimer model by extending the results of [HSB22] to any isoradial graph, answering in particular (iv) of their open questions section (1.7). Let  $G_\delta = (V_\delta, E_\delta)$  be an isoradial approximation of a simply connected open set  $\Omega$  with smooth boundary. The near-critical regime corresponds to  $m(x) \approx M^2 \delta^2$  for all  $x \in V_\delta$  where  $M > 0$  is a fixed *mass parameter* and  $\delta$  is the *mesh*. The precise scaling of the mass with the mesh is discussed in Section 6.1 and Remark 6.5 in relation with conformal covariance.

To each drift parameter  $\bar{u} \in (0, 2\pi)$  corresponds a discrete massive exponential function  $\mathbf{e}$  and hence a RST and drifted dimer models by the Doob transform technique. A precise statement of the following result can be found in Theorems 6.12 and 6.13.

**Theorem 1.5.** *When  $\delta \rightarrow 0$ , the random RST  $\mathcal{T}_\delta^{M, \bar{u}}$  on  $\mathbf{G}_\delta$  associated with the drifted dimer model with mass parameter  $M > 0$  and drift parameter  $\bar{u} \in \mathbb{R}$  converges in the Schramm sense to a continuum limit tree  $\mathcal{T}^{M, \bar{u}}$ . Conditionally on the endpoint  $y \in \partial\Omega$ , a branch of the tree from a point  $x \in \Omega$  has the law of massive radial  $SLE_2$  in  $\Omega$  with mass parameter  $\sqrt{2}M$  from  $x$  to  $y$ . The law of the endpoint of the branch started at  $x$  is the exit law from  $\Omega$  of the brownian motion with drift vector  $\sqrt{2}Me^{i\bar{u}}$ . Moreover, the centered height function  $h_\delta^{M, u} - \mathbb{E}[h_\delta^{M, u}]$  of the drifted dimer model converges in law to a limit.*

Our main contribution is to introduce the relevant RST and dimer models. In order to apply a general theorem of [HSB22] to obtain convergence of the loop-erasure of the KRW towards massive  $SLE_2$ , we prove a *uniform crossing estimate* and convergence of the KRW towards the *killed brownian motion*. This is done by applying to the near-critical regime some results of discrete harmonic analysis on isoradial graphs developed in [CS11]. It is not the first time that universality of convergence in the near-critical regime is proved for a model of statistical mechanics: [Par22] and [CIM21] show universality of the convergence of correlations in the massive scaling limit for respectively the random cluster model and the Ising model on isoradial graphs.

### 1.3 Perspectives

This paper brings new perspectives in the study of RSF models by introducing a generic way to relate them to a RST and dimer model for which more tools are available. As soon as a “nice” massive harmonic function is defined on a graph, new links can be made: we applied this technique to the two settings we were aware of, but there are probably many others.

- For the model of [Ken19] on  $\mathbb{Z}^2$ -periodic graphs, we believe that our technique can be used to show that the law of the infinite branches of the Gibbs measure is related to a drifted random walk, and that the law of these infinite branches for a RSF model coincides with the law of these branches for a “drifted” RST model with no mass, and can hence be related to the dimer model of [Sun16].
- For the near-critical dimer model, new results might be obtained by using the comprehensive analysis of the  $Z$ -invariant massive Laplacian performed in [BdTR17]. In particular, the explicit asymptotic of the Green function might provide new information. We mention that on the square lattice, the recent work of Mason [Mas23] proves convergence of the two point correlation functions of the near-critical dimer model towards those of the sine-Gordon field.

### 1.4 Organization of the paper

The first sections present the general theory.

- In Section 2, we give preliminaries on RST and RSF models, (massive) Laplacians and (massive) random walks, and state useful results on the Doob transform that we will apply all along the article.
- In Section 3, we develop general theorems to transfer results from RST models to RSF models by using the Doob transform technique. We also explain why it is always possible to find massive harmonic functions on infinite graphs and introduce a notion of “killed Martin boundary”, justifying that in principle our results apply to any conductance and mass functions and any graph.
- In Section 4, we focus on planar graphs: we define the killed and drifted dimer models and show how they are related to the RST and RSF model.

The last sections are devoted to applications.

- Section 5 shows some applications to  $\mathbb{Z}^2$ -periodic graphs, explaining in particular how to find explicit massive harmonic functions. For this section, we only need the preliminary results of Section 2 on the Doob transform.
- Section 6 shows applications to the near-critical dimer and RST models. We recall the setting of isoradial graphs and the results of [BdTR17], present our model and prove Theorem 1.5. For this section, we need the preliminary results of Section 2, the setting of Paragraph 3.1.2 and the definition and basic properties of the drifted dimer model of Section 4.
- Appendix A contains the proof of some technical results on the  $Z$ -invariant massive Laplacian in the near-critical regime used in Section 6. It uses results from [CS11] on the critical Laplacian.

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## 2 Preliminaries

### 2.1 First definitions

Consider a connected locally finite countable graph  $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ . A *loop* is an edge  $e = xx$  with  $x \in \mathbf{V}$ . The undirected edge set  $\mathbf{E}$  induces a set of directed edges  $\vec{\mathbf{E}}$ : to every undirected edge  $e = xy \in \mathbf{E}$  are associated the two directed edges  $(x, y)$  and  $(y, x)$ . We say that two vertices of  $\mathbf{G}$  are *neighbours* and we write  $x \sim y$  (or  $x \stackrel{\mathcal{C}}{\sim} y$  when there is a risk of confusion) if  $xy \in \mathbf{E}$ . A *path*  $\gamma$  is a set of directed edges  $\vec{e}_1 = (x_0, x_1), \dots, \vec{e}_n = (x_{n-1}, x_n)$  with  $\vec{e}_i \in \vec{\mathbf{E}}$  for all  $1 \leq i \leq n$ . It is often identified with the sequence of its vertices  $(x_0, x_1, x_2, \dots, x_n)$ . The length of the path is  $|\gamma| = n$  and for all  $0 \leq i \leq |\gamma|$ ,  $\gamma(i) = x_i$ . Such a path *connects* the vertices  $x_0$  and  $x_n$ . A *cycle* is a path returning to its starting point:  $x_n = x_0$ .

Consider a positive *conductance function*  $c : \vec{\mathbf{E}} \rightarrow \mathbb{R}_{>0}$  which assigns to every directed edge a conductance  $c_{(x,y)}$ . We sometimes extend  $c$  to all pairs of vertices by  $c_{(x,y)} = 0$  if  $xy \notin \mathbf{E}$ . Define the total conductance at a vertex  $c : \mathbf{V} \rightarrow \mathbb{R}_{>0}$  by  $c(x) = \sum_{y \sim x} c_{(x,y)}$ . When  $\mathbf{G}$  is infinite, we always assume that the total conductances are uniformly bounded: there exists  $C > 0$  such that for all  $x \in \mathbf{V}$ ,  $0 < c(x) \leq C$  (this is trivial when  $\mathbf{G}$  is finite). If for all  $x \sim y \in \mathbf{G}$ ,  $c_{(x,y)} = c_{(y,x)}$ , we say that  $c$  is symmetric and we write  $c_{xy} = c_{(x,y)}$ .

Finally consider a *mass function*  $m : \mathbf{V} \rightarrow \mathbb{R}_{\geq 0}$  assigning a non-negative mass  $m(x)$  to each vertex  $x$  of  $\mathbf{V}$ . We will write  $m \neq 0$  when  $m$  takes at least one positive value and  $m = 0$  otherwise.

**Matrix and linear operator notations.** A matrix  $L$  with rows and columns indexed by  $\mathbf{V}$  and coefficients  $L_{x,y} = L(x, y)$  such that  $L_{xy} = 0$  except when  $x = y$  or  $x \sim y$  is identified with the *linear operator* also denoted by  $L$  acting on functions  $f : \mathbf{V} \rightarrow \mathbb{C}$ :

$$\forall f \in \mathbf{V}^{\mathbb{C}}, \forall x \in \mathbf{V}, (Lf)(x) = \sum_{y \sim x} L(x, y)f(y).$$

For  $\mathbf{U} \subset \mathbf{V}$ , we denote by  $L_{\mathbf{U}}$  the restriction of the matrix  $L$  to the subset  $\mathbf{U}$ . For a function  $\phi : \mathbf{V} \rightarrow \mathbb{C}$ , we denote by  $D(\phi)$  (or sometimes  $\Phi$  to lighten notation) the diagonal matrix with entries  $\phi$  on the diagonal.



**Spanning trees and forests.** A *directed rooted spanning forest* (directed RSF)  $F$  of  $\mathbf{G}$  is a subset of directed edges  $F \subset \vec{\mathbf{E}}$  such that each vertex has at most one outgoing edge in  $F$  and there is no cycle constituted of edges of  $F$ . Denote by  $\mathcal{F}(\mathbf{G})$  the set of RSF of  $\mathbf{G}$ . Consider  $F \in \mathcal{F}(\mathbf{G})$ . A vertex  $x \in \mathbf{V}$  is a *root* of  $F$  if it has no outgoing edge in  $F$ : we denote by  $R(F) \subset \mathbf{V}$  the set of root vertices of  $F$ . Given a distinguished vertex  $r \in \mathbf{V}$ , a *directed rooted spanning tree* (directed RST)  $T$  of  $\mathbf{G}$  rooted at  $r$  is a RSF of  $\mathbf{G}$  satisfying  $r(T) = \{r\}$ : it is a forest with only one root which is  $r$ . Denote by  $\mathcal{T}^r(\mathbf{G})$  the set of directed RST of  $\mathbf{G}$  rooted at  $r$ .

*Remark 2.1.* Observe that with this definition, when  $\mathbf{G}$  is infinite, a RST is not necessarily connected: it can have a rooted connected component and several unrooted infinite components. Similarly, a RSF can have rooted connected components and infinite unrooted components. To align with previous notation, we should say “spanning forest” instead of RSF and “spanning forest with exactly one root” for RST.

When  $\mathbf{G}$  is finite, the conductance function  $c$  induces a weight function on  $\mathcal{T}^r(\mathbf{G})$ :

$$\forall T \in \mathcal{T}^r(\mathbf{G}), \nu_{\text{RST}}^r(T) = \prod_{\vec{e} \in T} c_{\vec{e}}. \quad (2)$$

The *partition function* is the weighted sum of RST rooted at  $r$ :

$$Z_{\text{RST}}^r(\mathbf{G}, c) = \sum_{T \in \mathcal{T}^r(\mathbf{G})} \nu_{\text{RST}}^r(T).$$

Since  $\mathbf{G}$  is finite and connected,  $0 < Z_{\text{RST}}^r(\mathbf{G}, c) < \infty$  and the weight function induces a probability measure on  $\mathcal{T}^r(\mathbf{G})$ :

$$\forall T \in \mathcal{T}^r(\mathbf{G}), \mathbb{P}_{\text{RST}}^r(T) = \frac{\nu_{\text{RST}}^r(T)}{Z_{\text{RST}}^r(\mathbf{G}, c)}. \quad (3)$$

When  $\mathbf{G}$  is finite, the conductance and mass functions  $(c, m)$  also induce a weight function on  $\mathcal{F}(\mathbf{G})$ :

$$\forall F \in \mathcal{F}(\mathbf{G}), \nu_{\text{RSF}}(F) = \prod_{\vec{e} \in F} c_{\vec{e}} \prod_{x \in R(F)} m(x).$$

The partition function is the weighted sum

$$Z_{\text{RSF}}(\mathbf{G}, c, m) = \sum_{F \in \mathcal{F}(\mathbf{G})} \nu_{\text{RSF}}(F).$$

When  $m \neq 0$ ,  $0 < Z_{\text{RSF}}(\mathbf{G}, c, m) < \infty$ , the conductance and mass functions induce a probability measure on  $\mathcal{F}(\mathbf{G})$ :

$$\forall F \in \mathcal{F}(\mathbf{G}), \mathbb{P}_{\text{RSF}}(F) = \frac{\nu_{\text{RSF}}(F)}{Z_{\text{RSF}}(\mathbf{G}, c, m)}.$$

This measure is supported on RSF with root vertices in the support of the mass function i.e.  $R(F) \subset \{x \in \mathbf{V} : m(x) > 0\}$ . The probability measures  $\mathbb{P}_{\text{RST}}^r$  and  $\mathbb{P}_{\text{RSF}}$  coincide when  $m = \delta_r$ . For a subset of directed edges  $\{\vec{e}_1, \dots, \vec{e}_n\} \subset \vec{\mathbf{E}}$  we denote by  $\mathbb{P}_{\text{RST}}^r(\vec{e}_1, \dots, \vec{e}_n)$ , resp.  $\mathbb{P}_{\text{RSF}}(\vec{e}_1, \dots, \vec{e}_n)$ , the measure of the set of all trees, resp. forests, containing these edges.

*Remark 2.2.* There is no obvious extension of these probability laws to the case of infinite graphs. This issue is common to many models of statistical physics and we will address it in Section 2.2.

**The forest-tree bijection.** Adding a cemetery vertex to  $\mathbf{G}$  allows to connect RSF of  $\mathbf{G}$  and RST of a modified graph  $\mathbf{G}^\rho$  as we now explain. Define a new graph  $\mathbf{G}^\rho = (\mathbf{V}^\rho, \mathbf{E}^\rho)$  by adding a vertex called the cemetery state:  $\mathbf{V}^\rho = \mathbf{V} \cup \{\rho\}$ , and  $\mathbf{E}^\rho$  is the set  $\mathbf{E}$  plus an edge  $x\rho$  for each  $x$  such that  $m(x) > 0$ . When  $m = 0$ , we define  $\mathbf{G}^\rho = \mathbf{G}$ , so the graph  $\mathbf{G}^\rho$  is always connected. Define a new positive conductance function  $c^k : \vec{\mathbf{E}}^\rho \rightarrow \mathbb{R}_{>0}$  by

$$c_{(x,y)}^k = \begin{cases} c_{(x,y)} & \text{if } (x,y) \in \vec{\mathbf{E}} \\ m(x) & \text{if } m(x) > 0, y = \rho \\ m(y) & \text{if } m(y) > 0, x = \rho \\ 0 & \text{otherwise.} \end{cases}$$

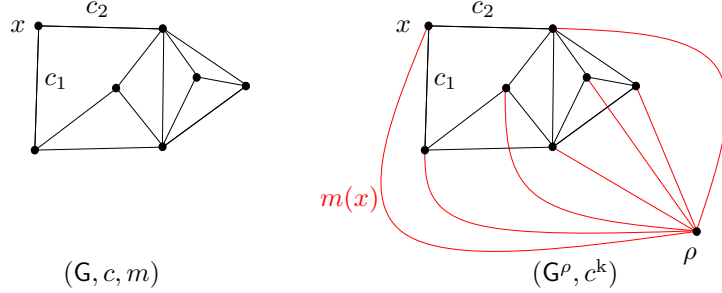


Figure 2: On the left: a graph  $G$  with conductance and mass functions  $(c, m)$ . On the right: the associated graph  $G^\rho$  with conductance function  $c^k$ , assuming that  $m(y) > 0 \forall y \in V$ . Edges are labelled by their conductances.

An example of the graph  $G^\rho$  with conductance function  $c^k$  is drawn on Figure 2. Observe that when  $c$  is symmetric,  $c^k$  also is. The total conductance at a point  $x \in V$  is  $c^k(x) = c(x) + m(x)$ .

**Definition 2.3.** RSF of  $G$  with roots in  $\{x : m(x) > 0\}$  are in bijection with RST of  $G^\rho$  rooted at  $\rho$ : to  $F \in \mathcal{F}(G)$  we associate  $T \in \mathcal{T}^\rho(G)$  obtained by adding an edge  $(x, \rho)$  for each root vertex  $x \in R(F)$ . When  $G$  is finite, this is a weight-preserving bijection called the *forest-tree bijection*:

$$\nu_{\text{RSF}}(F) = \prod_{\bar{e} \in F} c_{\bar{e}} \prod_{x \in R(F)} m(x) = \prod_{\bar{e} \in F} c_{\bar{e}}^k \prod_{x \in R(F)} c_{(x, \rho)}^k = \prod_{\bar{e} \in T} c_{\bar{e}}^k = \nu_{\text{RST}}^\rho(T).$$

In particular,  $Z_{\text{RSF}}(G, c, m) = Z_{\text{RST}}^\rho(G^\rho, c^k)$  and when  $m \neq 0$ ,  $\mathbb{P}_{\text{RSF}}(F) = \mathbb{P}_{\text{RST}}^\rho(T)$ .

*Remark 2.4.* When  $m = 0$  and  $G$  is infinite, this bijection is simply the identity: since  $G^\rho = G$ , we can use the convention that a RST of  $G^\rho$  rooted at  $\rho$  is a RSF of  $G$  with no root. This is called an *essential spanning forest* (ESF) of  $G$ , and it is a collection of infinite connected components spanning  $V$ , with no cycle.

**Laplacians and Green functions.** A (discrete) Laplacian operator  $\Delta : \mathbb{C}^V \rightarrow \mathbb{C}^V$  is naturally associated with the conductance function  $c$ :

$$\forall f : V \rightarrow \mathbb{C}, \forall x \in V, \Delta f(x) = \sum_{y \sim x} c_{(x, y)} (f(x) - f(y)).$$

A function  $f : V \rightarrow \mathbb{C}$  such that  $\Delta f = 0$  is called *harmonic*. A (discrete) massive Laplacian operator  $\Delta^k : \mathbb{C}^V \rightarrow \mathbb{C}^V$  is naturally associated to the conductance and mass functions  $(c, m)$ :

$$\forall f : V \rightarrow \mathbb{C}, \forall x \in V, \Delta^k f(x) = m(x)f(x) + \sum_{y \sim x} c_{(x, y)} (f(x) - f(y)).$$

A function  $f : V \rightarrow \mathbb{C}$  such that  $\Delta^k f = 0$  is called *massive harmonic*. In the canonical base of  $\mathbb{C}^V$ , these Laplacians are represented by square matrices whose rows and columns are indexed by  $V$ :

$$\Delta_{x, y} = \begin{cases} -c_{(x, y)} & \text{if } x \sim y, x \neq y \\ c(x) - c_{(x, x)} & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}, \Delta^k_{x, y} = \begin{cases} -c_{(x, y)} & \text{if } x \sim y, x \neq y \\ m(x) + c(x) - c_{(x, x)} & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}.$$

Observe that the massive Laplacian  $\Delta^k$  on  $G$  coincides with the restriction to  $V$  of the (non-massive) Laplacian on the graph  $G^\rho$  with conductance function  $c^k$ . When  $G$  is finite and  $m \neq 0$ , the massive Laplacian is invertible by the Perron-Frobenius theorem: its inverse is the *massive Green function*  $G^k : V \times V \rightarrow \mathbb{R}$ . In matrix notation, it satisfies

$$G^k = (\Delta^k)^{-1}. \quad (4)$$

In other words

$$\forall x, y \in \mathbf{V}, \mathbb{1}_{\{x=y\}} = (\Delta^k G^k)_{x,y} = m(x)G^k(x, y) + \sum_{z \sim x} c_{(x,z)}(G^k(z, y) - G^k(x, y)).$$

*Remark 2.5.* This definition of the Green function can be extended if  $\mathbf{G}$  is infinite under additional conditions. For example if the masses are uniformly bounded away from 0 and  $\infty$ , we can still make sense of  $G^k = (\Delta^k)^{-1}$  by inverting the restriction of the operator  $\Delta^k$  to  $L^2(\mathbf{V})$ , see for example Section 4.1 of [BdTR17].

**Matrix-tree theorem.** In this paragraph, we assume that  $\mathbf{G}$  is finite. The objects introduced in the first two paragraphs (the RST/RSF and the Laplacians) are naturally related. Let  $r \in \mathbf{V}$  be fixed. The matrix-tree theorem of [Kir47] (and its extension to directed weighted graphs, which can be found for example in [Cha82]) relates the determinant of the (non-massive) Laplacian with the weighted sum of RST rooted at  $r$ :

$$Z_{\text{RST}}^r = \det \Delta_{\mathbf{V} \setminus \{r\}}.$$

Applying this theorem to  $(\mathbf{G}^\rho, c^k)$  with distinguished vertex  $\rho$  and using the forest-tree bijection gives

$$Z_{\text{RSF}}(\mathbf{G}, c, m) = Z_{\text{RST}}^\rho(\mathbf{G}^\rho, c^k) = \det \Delta^k \quad (5)$$

since, as we already observed, the restriction to  $\mathbf{V}$  of the Laplacian on  $\mathbf{V}^\rho$  with conductance function  $c^k$  is  $\Delta^k$ . This identity is sometimes called the *matrix-forest theorem*.

**Random walks and potentials.** To the conductance function  $c$  is associated a discrete time Markov chain  $S$  on  $\mathbf{V}$  called the *random walk* (RW) with conductances  $c$ . Given  $x_0 \in \mathbf{V}$ , the RW with initial state  $x_0$  is defined by  $S_0 = x_0$  and transition probabilities

$$\forall i \in \mathbb{Z}_{\geq 0}, \forall x, y \in \mathbf{V}, Q(x, y) = \mathbb{P}_{x_0}(S_{i+1} = y | S_i = x) = \begin{cases} c_{(x,y)}/c(x) & \text{if } y \sim x \\ 0 & \text{otherwise.} \end{cases}$$

Since  $\mathbf{G}$  is connected and  $c$  is positive, this Markov chain is irreducible. Recall that  $D(c)$  denotes the diagonal matrix with entries  $c(x)$ . The transition kernel is related to the Laplacian:

$$Q = I - D(c)^{-1} \Delta. \quad (6)$$

For  $\mathbf{U} \subset \mathbf{V}$ , we denote by  $\tau(\mathbf{U})$ , resp.  $\tau^+(\mathbf{U})$ , the first time  $i \geq 0$ , resp.  $i > 0$ , at which the RW enters the set  $\mathbf{U}$ :

$$\tau(\mathbf{U}) = \inf\{i \geq 0, S_i \in \mathbf{U}\}, \quad \tau^+(\mathbf{U}) = \inf\{i > 0, S_i \in \mathbf{U}\}.$$

If  $\mathbf{U} = \{x\}$  is reduced to a point, we compactify the notation  $\tau(\mathbf{U}) =: \tau(x)$  and  $\tau^+(\mathbf{U}) =: \tau^+(x)$ .

The conductance function  $c^k$  associated with  $c$  and  $m$  induces a Markov chain  $S^\rho$  on  $\mathbf{V}^\rho$ . The RW  $S^\rho$  stopped at  $\rho$  is denoted by  $S^k$  and is called the *killed random walk* (KRW):

$$\forall i \in \mathbb{Z}_{\geq 0}, S_i^k := S_{i \wedge \tau(\rho)}^\rho$$

where we use the standard notation  $\wedge$  for the minimum. The cemetery vertex  $\rho$  is an absorbing state for the Markov chain  $S^k$ . The sub-Markovian transition kernel  $Q^k$  associated with  $S^k$  is a matrix with rows and columns indexed by  $\mathbf{V}$  related to the massive Laplacian:

$$Q^k = I - D(c^k)^{-1} \Delta^k. \quad (7)$$

We denote by  $\tau^k$ , resp.  $(\tau^k)^+$ , the hitting times of sets for this Markov chain. When  $m = 0$ ,  $S^k = S^\rho = S$  and  $\tau^k(\rho) = (\tau^k)^+(\rho) = \infty$  by convention, so from now on we only state the definitions and results for the KRW as they include the RW case  $m = 0$ .

*Remark 2.6.* The RW  $S$  associated with  $c$  and the KRW  $S^k$  associated with  $(c, m)$  can be coupled. Let  $(U_i)_{i \geq 0}$  be a sequence of independent uniform variables in  $[0, 1]$ , independent of  $S$ . Define  $X^k$  recursively by  $X_0^k = S_0$  and

$$\forall i \in \mathbb{Z}_{\geq 0}, X_{i+1}^k = \begin{cases} \rho & \text{if } (X_i^k = \rho) \text{ or } (X_i^k = S_i = x \in \mathbb{V} \text{ and } U_i > c(x)/(c(x) + m(x))) \\ S_{i+1} & \text{if } X_i^k = S_i = x \in \mathbb{V} \text{ and } U_i \leq c(x)/(c(x) + m(x)). \end{cases}$$

Then,  $X^k$  has the law of the KRW associated with the conductance and mass functions  $(c, m)$  on  $\mathbb{G}$  so we can denote it  $X^k = S^k$ , and

$$\forall i < \tau^k(\rho), S_i^k = S_i.$$

The KRW  $S^k$  is *transient* if for all  $x \in \mathbb{V}$ ,

$$\mathbb{P}_x((\tau^k)^+(x) < \infty) < 1,$$

meaning that with probability 1, the walk reaches the cemetery state in finite time or “escapes to infinity”. When  $m = 0$ , this is the usual definition of a transient RW.

*Remark 2.7.* When  $\mathbb{G}$  is finite and  $m = 0$ , the RW is never transient. When  $m \neq 0$ , the KRW is always transient: if  $S^k$  is coupled to  $S$  as in Remark 2.6, either  $S$  is transient, in which case  $S^k$  also is, or it is recurrent. In this case, given  $x \in \mathbb{V}$  such that  $m(x) > 0$ ,  $S$  returns almost surely infinitely many times to  $x$  and each time  $S^k$  has the same positive probability of dying, so  $S^k$  dies almost surely and hence it is also transient.

The *potential* of a transient KRW or transient RW is well-defined:

$$\forall x, y \in \mathbb{V}, V^k(x, y) = \mathbb{E}_x \left[ \sum_{n=0}^{\tau^k(\rho)-1} \mathbb{1}_{\{S_n^k=y\}} \right] = \mathbb{E}_x \left[ \sum_{n=0}^{\infty} \mathbb{1}_{\{S_n^k=y\}} \right] = \sum_{n=0}^{\infty} \mathbb{P}_x(S_n^k = y) < \infty. \quad (8)$$

This definition coincides with the potential of a transient RW when  $m = 0$  (recall that in this case,  $\tau^k(\rho) = \infty$  by convention). By the simple Markov property, the potential satisfies,

$$\forall x, y \in \mathbb{V}, V^k(x, y) = \mathbb{1}_{\{x=y\}} + \sum_{z \sim x} \frac{c(x, z)}{c^k(x)} V^k(z, y).$$

In other words, in matrix notation,

$$\Delta^k V^k = D(c^k). \quad (9)$$

If  $\mathbb{G}$  is finite and  $m \neq 0$ , taking inverses implies that the potential is related to the massive Green function:  $V^k = G^k D(c^k)$ , or in other words

$$\forall x, y \in \mathbb{V}, V^k(x, y) = c^k(y) G^k(x, y). \quad (10)$$

If  $\mathbb{G}$  is infinite,  $m = 0$  and the RW  $S$  is transient, the potential  $V = V^k$  is well-defined but the Green function is not since the Laplacian is not a priori invertible so Equation (10) does not make sense.

We will soon need to use results of [Cha13], who has a different definition of the potential. Let  $(X_t^k)_{t \in \mathbb{R}_+}$  be the continuous time pure jump Markovian process on  $\mathbb{V} \cup \{\rho\}$  with sub-Markovian generator  $\Delta^k$  (see Chapter 2.2 of [Cha13] for more details). The explosion time of this process is infinite since we always assume that the total conductances are bounded. The embedded Markov chain is the discrete time KRW: if  $N_0 = 0, N_1, \dots$  are the times of the jumps,  $X_{N_i}^k = S_i^k$  for all  $i \in \mathbb{N}$ . The associated *continuous potential* is

$$\forall x, y \in \mathbb{V}, V_c^k(x, y) = \mathbb{E}_x \left[ \int_0^\infty \mathbb{1}_{\{X_t^k=y\}} \right].$$

It is well defined by transience of the embedded Markov chain and the fact that the conductances (hence the jump rates) are bounded. Chang uses the notation  $V_y^x = V_c^k(x, y)$ . Since the continuous

time process  $X^k$  has a jump rate  $c^k(y)$  at  $y$ , it spends an average time  $c^k(y)^{-1}$  in  $y$  every time it hits  $y$ , and since the embedded Markov chain is the KRW:

$$V_c^k(x, y) = \frac{V^k(x, y)}{c^k(y)},$$

hence when  $G$  is finite and  $m \neq 0$ , by Equation (10) the continuous potential coincides with the Green function:  $V_c^k(x, y) = G^k(x, y)$  for all  $x, y \in V$ .

The *transfer current operator* is defined for all  $\vec{e} = (w, x), \vec{f} = (y, z) \in \vec{E}^\rho$  by

$$H_{\vec{e}, \vec{f}}^k = V_c^k(w, y) - V_c^k(x, y) = \frac{V^k(w, y)}{c^k(y)} - \frac{V^k(x, y)}{c^k(y)},$$

with the convention that for all  $x \in V$ ,  $V^k(x, \rho) = V^k(\rho, x) = 0$ . When  $m = 0$ , the transfer current operator is defined on  $\vec{E}$  so the conventions are not needed and it coincides with the usual transfer current operator associated with a transient RW. Observe that this transfer current operator is not symmetric and if  $y = \rho$  or if  $\vec{e}$  is a loop,  $H_{\vec{e}, \vec{f}}^k = 0$ .

**Induced subgraph and wired boundary conditions.** Consider an infinite graph  $G^\infty = (V^\infty, E^\infty)$ . Let  $V \subset V^\infty$  be a strict subset of  $V^\infty$ . The subgraph of  $G^\infty$  *induced* by  $V$  is the graph  $G = (V, E)$  with edge set  $E$  such that for all  $x, y \in V$ ,  $xy \in E$  if and only if  $xy \in E^\infty$ . When  $G^\infty$  is planar, the induced subgraph  $G$  is *simply connected* if all the vertices of  $V^\infty \setminus V$  belong to the outer face of  $G$ . The (inner) *boundary*  $\partial V$  of  $G$  in  $G^\infty$  is

$$\partial V = \{x \in V \mid \exists y \in V^\infty \setminus V, x \stackrel{G^\infty}{\sim} y\}.$$

Given a conductance and mass functions  $(c^\infty, m^\infty)$  on  $G^\infty$ , a natural conductance and mass functions with *wired boundary conditions* are defined on the induced subgraph. It corresponds to assigning conductances by restriction  $c := c^\infty|_E$  and adding a mass for each positive conductance between a vertex in  $V$  and a vertex in  $V^\infty \setminus V$ :

$$\forall x \in V, m(x) := m^\infty(x) + \sum_{y \stackrel{G^\infty}{\sim} x, y \in V^\infty \setminus V} c_{(x,y)}^\infty.$$

Note that it differs from  $m^\infty$  only on  $\partial V$ . Denote by  $\Delta^k$  and  $S^k$  the massive Laplacian and KRW on  $G^\infty$  associated with the conductance and mass functions  $(c^\infty, m^\infty)$ . The massive Laplacian  $G$  associated to  $(c, m)$  is simply the restriction  $\Delta_V^k$ : for all  $x, y \in V$  such that  $x \neq y$  and  $x \sim y$ ,  $\Delta_{x,y}^k = -c_{(x,y)}^\infty = -c_{(x,y)}$  and

$$\forall x \in V, \Delta_{x,x}^k = m^\infty(x) + \sum_{y \stackrel{G^\infty}{\sim} x} c_{(x,y)}^\infty = m(x) + \sum_{y \stackrel{G}{\sim} x} c_{(x,y)}.$$

Equivalently, the KRW on  $G$  with conductance and mass functions  $(c, m)$  is the KRW  $S^k$  on  $G^\infty$  with conductance and mass functions  $(c^\infty, m^\infty)$  killed when it exits  $V$ . More formally, if we define  $X^k$  by  $X_i^k = S_i^k$  for all  $i < \tau^k(V^\infty \setminus V)$  and  $X_i^k = \rho$  for all  $i \geq \tau^k(V^\infty \setminus V)$ , then  $X^k$  has the law of the KRW on  $G$  associated with the conductance and mass functions  $(c, m)$ .

Assume now that  $m^\infty = 0$ . The wired conductance and mass functions are  $c := c_V^\infty : \vec{E} \rightarrow \mathbb{R}_{>0}$  and

$$\forall x \in V, m(x) := \sum_{y \stackrel{G^\infty}{\sim} x, y \in V^\infty \setminus V} c_{(x,y)}^\infty. \quad (11)$$

In particular when  $V$  is a strict subset of  $V^\infty$ ,  $m \neq 0$  even though  $m^\infty = 0$ . The KRW on  $G$  associated with the conductance and mass functions  $(c, m)$  is simply the RW  $S$  on  $G^\infty$  with conductance function  $c^\infty$  killed when it exits  $V$ , and the massive Laplacian on  $V$  associated with

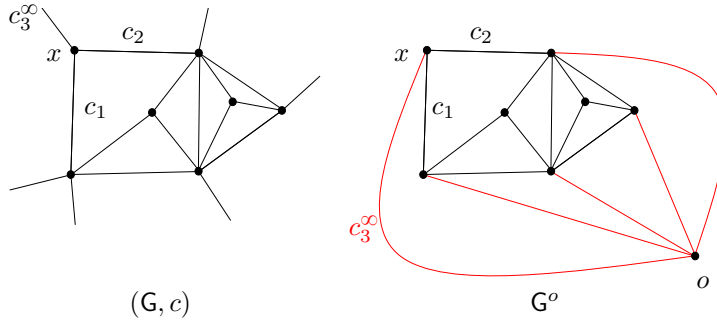


Figure 3: On the left: a simply connected induced subgraph  $G$  of a planar graph  $G^\infty$ . Edges leaving  $V$  are marked with no target vertex. On the right: the graph  $G^o$  is also planar. Edges are labelled by their conductances.

the conductance and mass functions  $(c, m)$  is simply the restriction  $\Delta_V$ . In this case, the forest-tree bijection is particularly easy to understand: denote by  $G^o$  the graph obtained from  $G^\infty$  by identifying all vertices of  $V^\infty \setminus V$  to a single vertex  $o$ . When  $G^\infty$  is planar and  $G$  is simply connected, the graph  $G^o$  is still planar: this will be useful later on. The forest-tree bijection states that RSF of  $G$  weighted by  $(c, m)$  are in weight-preserving bijection with RST of  $G^o$  rooted at  $o$  weighted by  $c$ , with the natural abuse of notation that the conductance of an edge  $(x, o)$  is the sum of conductances  $c_{(x,y)}^\infty$  of edges  $(x, y)$  with  $y \in V^\infty \setminus V$ . An example of the graph  $G^o$  is drawn on Figure 3. These RST of  $G^o$  rooted at  $o$  can be thought of as RST of  $G$  rooted at the outer boundary.

*Remark 2.8.* The *free boundary conditions* correspond to assigning conductances and masses by restriction  $c^f = c_{|\vec{E}}^\infty$  and  $m^f = m_{|V}^\infty$ . We will only mention it in this article. This is not compatible with the restriction of the Laplacian.

## 2.2 Wilson's algorithm and the transfer current theorem

The transfer current theorem (first proved by Burton and Pemantle [BP93]) expresses the probability that some fixed edges belong to a random uniform (undirected) spanning tree of a finite graph as determinants of submatrices of the transfer current operator. It shows in particular that the uniform spanning tree seen as a random point process on the set of edges is determinantal. This result was extended in many directions, but one which is particularly interesting to us is to the case of directed RSF proved by Chang in his PhD thesis [Cha13].

Recall that by the forest-tree bijection, the probability measure  $\mathbb{P}_{\text{RST}}^\rho$  on  $\mathcal{T}^\rho(G^\rho)$  associated with the conductance function  $c^k$  by Equation (3) coincides with the probability measure  $\mathbb{P}_{\text{RSF}}$  of  $G$  weighted by  $(c, m)$ . Theorem 5.2.3 of [Cha13] reads

**Theorem 2.9** (Finite Directed Transfer Current Theorem). *When  $G$  is finite and  $m \neq 0$ ,*

$$\forall \{\vec{e}_1, \dots, \vec{e}_k\} \subset \vec{E}^\rho, \mathbb{P}_{\text{RST}}^\rho(\vec{e}_1, \dots, \vec{e}_k) = \det \left( (H_{\vec{e}_i, \vec{e}_j}^k)_{i,j=1, \dots, k} \right) \prod_{i=1}^k c_{\vec{e}_i}^k.$$

In [Cha13] this result is stated for processes with finite state space that are absorbed by the cemetery state almost surely in finite time: it corresponds to  $G$  being finite and  $m \neq 0$ . The historical proof of [BP93] in the non-directed case relies on electrical networks. Other proofs, including that of [Cha13] which deals with the general directed case, rely on Wilson's algorithm. We do not repeat the proof but we introduce a slightly more general version of Wilson's algorithm in the setting of transient KRW on countable graphs. This algorithm is a combination of the algorithm of [Cha13] for a directed KRW that dies almost surely and Wilson's method rooted at infinity described in Section 10.1 of [LP16] (for reversible Markov chains with no killing). It actually unifies the two algorithms by identifying the cemetery state with infinity.

We first need to define the concept of loop-erasure. A path  $\gamma$  in  $\mathbf{G}$ , finite or infinite, is said to be *transient* if it visits no vertex infinitely many times. If  $\gamma = (x_0, x_1, \dots)$  is a transient path, its *loop-erasure* denoted by  $LE(\gamma)$  is also a path in  $\mathbf{G}$  built in the following way: start with  $\gamma_1 = (x_0, x_1)$ . Define recursively  $\gamma_{n+1}$  out of  $\gamma_n$  in the following way: let  $l_n = |\gamma_n|$ . If for all  $0 \leq i \leq l_n$ ,  $\gamma_n(i) \neq x_{n+1}$ , then  $\gamma_{n+1} := (\gamma_n(0), \dots, \gamma_n(l_n), x_{n+1})$ . Otherwise, let  $k = \inf\{0 \leq i \leq l_n, x_{n+1} = \gamma_n(i)\}$  and define  $\gamma_{n+1} = (\gamma_n(0), \dots, \gamma_n(k-1), x_{n+1})$ . This produces a new path in  $\mathbf{G}$  with no cycle. If  $\gamma$  is finite this algorithm stops after  $|\gamma|$  steps. If  $\gamma$  is infinite, a countable number of steps must be done. Nevertheless, the loop-erasure is well defined when the path is transient (see the proof of Proposition 10.1 of [LP16] for details).

**Definition 2.10** (Wilson's algorithm rooted at the cemetery state). Assume that  $S^k$  is transient. Choose an arbitrary order  $L = \{x_1, x_2, \dots\}$  on  $V$ . Let  $T_0 = \emptyset$ . Assume that for some  $i \geq 0$ , the random subgraph  $T_i \subset \vec{E}^\rho$  was constructed. Let  $x_{\phi(i)}$  be the first vertex of  $L$  which does not belong to an edge in  $T_i$ . Run a KRW  $S^k$  started at  $x_{\phi(i)}$ . Since  $S^k$  is transient, almost surely, one of the two following situations occur:

- $S^k$  hits  $\rho$  or one of the vertices in  $T_i$  in finite time. In this case, perform the chronological loop-erasure of the trajectory up to this time: it produces a finite path  $s_i$  with no cycle disjoint of  $T_i$  except for its endpoint. Add this path to  $T_i$  to form  $T_{i+1} = T_i \cup s_i$ .
- $S^k$  never hits  $T_i$  nor  $\rho$ . Perform the chronological loop-erasure of this infinite trajectory: this is almost surely well-defined because since  $S^k$  is transient, the path is almost surely transient. This produces an infinite path  $t_i$  with no cycle. Form  $T_{i+1}$  by adding this infinite path to  $T_i$ .

Run this algorithm until all the vertices of  $\mathbf{G}$  are spanned by  $T_i$  (it can take a countable number of steps if  $\mathbf{G}$  is infinite). This produces a random RST of  $\mathbf{G}^\rho$  (recall that with our definition, a RST can have infinite unrooted components, see Remark 2.1). By the forest-tree bijection, this RST corresponds to a RSF of  $\mathbf{G}$ .

**Definition 2.11.** We call the *Wilson's measure* (following [Con23]) the probability measure on RST of  $\mathbf{G}^\rho$  rooted at  $\rho$  obtained by applying this algorithm and denote it by  $\mathbb{P}_W$ .

When  $m = 0$ , this algorithm is Wilson's algorithm rooted at infinity described in Section 10.1 of [LP16] (for symmetric conductance functions). It produces an ESF of  $\mathbf{G}$  (see Remark 2.4). When  $\mathbf{G}$  is finite and  $m \neq 0$ , the Wilson's measure is supported on RST of  $\mathbf{G}^\rho$  and coincides with the probability measure defined in Equation (3) associated with the conductances  $c^k$ :

**Proposition 2.12.** *When  $\mathbf{G}$  is finite and  $m \neq 0$ ,  $\mathbb{P}_W$  coincides with the measure  $\mathbb{P}_{\text{RST}}^\rho$  defined in Equation (3) associated with the conductances  $c^k$ :  $\mathbb{P}_W = \mathbb{P}_{\text{RST}}^\rho$*

*Proof.* Proposition 5.2.6 of [Cha13] states that

$$\forall T \in \mathcal{T}^\rho(\mathbf{G}^\rho), \mathbb{P}_W(T) = \det(G^k) \prod_{\vec{e} \in T} c_e^k.$$

Since by the matrix-forest theorem (see Equation (5)) and the definition of the massive Green function,  $Z_{\text{RSF}}^\rho(\mathbf{G}^\rho, c^k) = \det(\Delta^k) = \det(G^k)^{-1}$ ,  $\mathbb{P}_W$  coincides indeed with  $\mathbb{P}_{\text{RST}}^\rho$ .  $\square$

When  $\mathbf{G}$  is infinite, another natural measure on RST of  $\mathbf{G}^\rho$  is the weak limit over any exhaustion of the measure (3). We now explain why this coincides with the Wilson's measure. An *exhaustion* of  $\mathbf{G}$  is a sequence of connected induced subgraphs  $\mathbf{G}_n = (V_n, E_n)$  such that  $V_n$  is increasing (for union) and  $\bigcup_{n \in \mathbb{N}} V_n = V$ . We denote by  $(c_n, m_n)$  the associated conductance and mass functions with wired boundary conditions. Recall that even when  $m = 0$ ,  $m_n$  takes at least one positive value since the finite set  $V_n$  is a strict subset of the infinite set  $V$ . Consider an exhaustion  $(\mathbf{G}_n)_{n \in \mathbb{N}}$  of  $\mathbf{G}$ , and denote by  $\mathbb{P}_W^n$  the Wilson's measures on RST of  $\mathbf{G}_n^\rho$  rooted at  $\rho$  associated with the conductance and mass functions  $(c_n, m_n)$ .

**Proposition 2.13.** *Assume that  $\mathsf{G}$  is infinite and one of the following holds:*

- (i)  $m = 0$  and  $S^k = S$  is transient.
- (ii)  $m \neq 0$  and for all  $x \in \mathsf{V}$ ,  $\mathbb{P}_x(\tau^k(\rho) < \infty) = 1$ :  $S^k$  dies almost surely in finite time.

The Wilson's measure  $\mathbb{P}_{\mathsf{W}}$  on RST of  $\mathsf{G}^\rho$  rooted at  $\rho$  associated with the conductance and mass functions  $(c, m)$  coincides with the weak limit of the  $\mathbb{P}_{\mathsf{W}}^n$ . More precisely,

$$\forall \{\vec{e}_1, \dots, \vec{e}_k\} \subset \vec{\mathsf{E}}^\rho, \mathbb{P}_{\mathsf{W}}^n(\vec{e}_1, \dots, \vec{e}_k) \xrightarrow{n \rightarrow \infty} \mathbb{P}_{\mathsf{W}}(\vec{e}_1, \dots, \vec{e}_k).$$

When  $m = 0$ , we simply have  $\vec{\mathsf{E}}^\rho = \vec{\mathsf{E}}$  and  $\mathbb{P}_{\mathsf{W}}$  is supported on EST of  $\mathsf{G} = \mathsf{G}^\rho$ .

Observe that the assumption in case (ii) is stronger than the transience assumption. Case (ii) is exactly Proposition 5.2.6 of [Cha13]. Case (i) is an immediate extension of Proposition 10.1 of [LP16] to the directed transient case. In their setting, the conductance function is symmetric, but the exact same proof works.

*Remark 2.14.* Proposition 2.13 holds more generally for transient KRW (which includes cases (i) and (ii): this corresponds to identifying  $\rho$  with the cemetery state) by simply combining the two proofs, but we will not need it. Case (ii) of Proposition 2.13 also holds for free boundary conditions (defined in Remark 2.8) instead of wired boundary conditions, but in general case (i) does not. Nevertheless the limits with free and wired boundary conditions coincide in case (i) for the simple random walk on  $\mathbb{Z}^d$  (see for example Corollary 10.9 of [LP16]).

This proposition combined with Proposition 2.12 shows that Wilson's measure that we introduced on RSF of infinite graphs corresponds to the weak limit along any exhaustion with wired boundary conditions of the measures defined in Equation (3) (for the conductances  $c^k$  on  $\mathsf{G}^\rho$ ). Taking weak limits for edge probabilities, one deduces that the transfer current theorem also holds for infinite directed graphs. Recall the definition of the transfer current operator  $H^k$  when  $S^k$  is transient. Recall that when  $m = 0$ ,  $H^k = H$  is the usual transfer current operator associated with a transient RW.

**Theorem 2.15** (Infinite Transfer Current Theorem). *Assume that  $\mathsf{G}$  is infinite and one of the following holds:*

- (i)  $m = 0$  and  $S^k = S$  is transient.
- (ii)  $m \neq 0$  and for all  $x \in \mathsf{V}$ ,  $\mathbb{P}_x(\tau^k(\rho) < \infty) = 1$ .

The Wilson's measure  $\mathbb{P}_{\mathsf{W}}$  on RST of  $\mathsf{G}^\rho$  satisfies

$$\forall \{\vec{e}_1, \dots, \vec{e}_k\} \subset \vec{\mathsf{E}}^\rho, \mathbb{P}_{\mathsf{W}}(\vec{e}_1, \dots, \vec{e}_k) = \det \left( (H_{\vec{e}_i, \vec{e}_j}^k)_{i,j=1, \dots, k} \right) \prod_{i=1}^k c_{\vec{e}_i}^k.$$

When  $m = 0$ ,  $\mathsf{G}^\rho = \mathsf{G}$ ,  $\vec{\mathsf{E}}^\rho = \vec{\mathsf{E}}$ ,  $H^k = H$  and  $\mathbb{P}_{\mathsf{W}}$  is supported on EST of  $\mathsf{G}$ .

*Proof.* Case (ii) is proved in the discussion after Remark 18 of [Cha13]. Let us prove case (i) in the same spirit. Assume that  $m = 0$  and  $S$  is transient. Let  $(\mathsf{G}_n)_{n \in \mathbb{N}}$  be an exhaustion of  $\mathsf{G}$ . Denote by  $\mathbb{P}_{\mathsf{W}}^n$  the Wilson's measure on RST of  $\mathsf{G}_n$  associated with  $(c_n, m_n)$ . Recall that  $m_n \neq 0$  since  $\mathsf{V}_n$  is a strict subset of  $\mathsf{V}$ . Let  $\vec{e}_1, \dots, \vec{e}_k \in \vec{\mathsf{E}}$ . Case (i) of Proposition 2.13 shows that

$$\mathbb{P}_{\mathsf{W}}(\vec{e}_1, \dots, \vec{e}_k) = \lim_{n \rightarrow \infty} \mathbb{P}_{\mathsf{W}}^n(\vec{e}_1, \dots, \vec{e}_k). \quad (12)$$

Since by Proposition 2.12, on finite graphs the Wilson's measure coincides with the natural measure defined in (3), the finite transfer current Theorem 2.9 applies with the transfer current operator  $H_n^k$  associated with the potential

$$\forall x, y \in \mathsf{V}_n, V_n^k(x, y) = \mathbb{E}_x \left[ \sum_{i=0}^{\tau(\mathsf{V} \setminus \mathsf{V}_n) - 1} \mathbb{1}_{\{S_i = y\}} \right],$$



by definition of the potential of a KRW and since the KRW on  $G_n$  associated with  $(c_n, m_n)$  is simply  $S$  killed when it exits  $V_n$ . For fixed  $x, y \in V$ , by transience of  $S$ :

$$V_n^k(x, y) \xrightarrow{n \rightarrow \infty} V(x, y) = \mathbb{E}_x \left[ \sum_{i=0}^{\infty} \mathbb{1}_{\{S_i=y\}} \right].$$

Together with (12), it proves the theorem with the associated transfer current operator

$$\forall \vec{e} = (wx), \vec{f} = (yz) \in \vec{E}, H_{\vec{e}, \vec{f}} = \frac{V(w, y)}{c(y)} - \frac{V(x, y)}{c(y)}. \quad \square$$

### 2.3 The Doob transform technique

Let  $(G^\infty, c^\infty, m^\infty)$  be a graph with conductance and mass functions. Let  $V \subset V^\infty$  and denote by  $G$  the induced subgraph and by  $(c, m)$  the conductance and mass function with wired boundary conditions. Recall that the restriction  $\Delta_V^k$  is the massive Laplacian on  $G$  with conductance and mass functions  $(c, m)$ . Consider a positive function  $\lambda : V^\infty \rightarrow \mathbb{R}_{>0}$ , and define a new conductance function on  $G^\infty$ :

$$\forall x \overset{G^\infty}{\sim} y \in V^\infty, \tilde{c}_{(x,y)}^\infty := \frac{\lambda(y)}{\lambda(x)} c_{(x,y)}^\infty.$$

Let  $\tilde{S}, \tilde{\Delta}, \tilde{Q}$  be respectively the RW, non-massive Laplacian and transition kernel on  $G^\infty$  associated with these conductances. The KRW on the induced subgraph  $G$  associated with the conductance and mass functions with wired boundary conditions  $(\tilde{c}, \tilde{m})$  induced on  $G$  by  $\tilde{c}^\infty$  is simply the RW  $\tilde{S}$  killed when it exits  $V$ . As it was already noted, the Laplacian associated with  $(G, \tilde{c}, \tilde{m})$  is the restriction  $\tilde{\Delta}_V$ . Recall that when  $V$  is a strict subset of  $V^\infty$ ,  $\tilde{m}$  takes at least one positive value even though  $\tilde{S}$  is a non-massive RW, so  $\tilde{\Delta}_V$  is a massive Laplacian. When  $V = V^\infty$ ,  $c = c^\infty$  and  $\tilde{\Delta}_V = \tilde{\Delta}$  is non-massive. This is the case that the reader should keep in mind as it is the more intuitive, the additional technicalities are only here to handle the general case.

**Definition 2.16.** [Definition 8-11 of [KGSK12]] The RW  $\tilde{S}$  is called the *Doob transform* (or sometimes *h-transform*) of  $S^k$  by  $\lambda$ .

Let  $\Lambda = D(\lambda)$  be the diagonal matrix with rows and columns indexed by  $V^\infty$  and diagonal entries  $\lambda$ . Recall that  $\lambda$  is said to be massive harmonic on  $V$  if for all  $x \in V$ ,  $(\Delta^k \lambda)(x) = 0$ . Recall that  $Q^k$  is the transition kernel of the KRW  $S^k$ . The following holds:

**Proposition 2.17.** *If  $\lambda$  is massive harmonic for  $\Delta^k$  on  $V$ , then*

$$\tilde{\Delta}_V = \Lambda_V^{-1} \Delta_V^k \Lambda_V.$$

*Equivalently,*

$$\forall x, y \in V, \tilde{Q}(x, y) = \frac{\lambda(y)}{\lambda(x)} Q^k(x, y). \quad (13)$$

The Laplacians  $\tilde{\Delta}_V$  and  $\Delta_V^k$  are said to be *gauge equivalent* with gauge  $\Lambda_V$ . When  $V = V^\infty$ , this is a gauge equivalence between a non-massive Laplacian  $\tilde{\Delta}$  and a massive Laplacian  $\Delta^k$ .

*Proof.* For all  $x \sim y \in V$ ,  $x \neq y$

$$(\Lambda_V^{-1} \Delta_V^k \Lambda_V)_{x,y} = \frac{\lambda(y)}{\lambda(x)} \Delta_{x,y}^k = -\frac{\lambda(y)}{\lambda(x)} c_{(x,y)} = -\tilde{c}_{(x,y)} = \tilde{\Delta}_{x,y}.$$

Moreover, for  $x \in V$ , by massive harmonicity of  $\lambda$  at  $x$  for  $\Delta^k$ :

$$c^k(x) = m(x) + c(x) = m^\infty(x) + c^\infty(x) = m^\infty(x) + \sum_{y \overset{G^\infty}{\sim} x} c_{(x,y)}^\infty = \sum_{y \overset{G^\infty}{\sim} x} \frac{\lambda(y)}{\lambda(x)} c_{(x,y)}^\infty = \tilde{m}(x) + \tilde{c}(x). \quad (14)$$

This implies, for  $x \in V$

$$(\Lambda_V^{-1} \Delta_V^k \Lambda_V)_{x,x} = \Delta_{x,x}^k = c^k(x) - c_{(x,x)} = \tilde{m}(x) + \tilde{c}(x) - \tilde{c}_{(x,x)} = \tilde{\Delta}_{x,x}, \quad (15)$$

which proves the first point of the Lemma. The second point of the Proposition is a direct consequence of Equation (14) and the definition of the transition kernels in Equation (7) (and Equation (6) when  $V = V^\infty$ ):

$$\tilde{Q}_V = I - D(\tilde{c} + \tilde{m})^{-1} \tilde{\Delta}_V = I - D(c^k) \Lambda_V^{-1} \Delta_V^k \Lambda_V = \Lambda_V^{-1} Q_V^k \Lambda_V.$$

□

A first property of the Doob transform is the following:

**Proposition 2.18.** *Assume that  $S^k$  is transient and  $V = V^\infty$ . If  $\lambda$  is massive harmonic for  $\Delta^k$  on  $V$ , the Doob transform  $\tilde{S}$  is also transient.*

This holds more generally for  $V \subset V^\infty$  with the same proof but we will not need it and we find it more clear to provide a proof in the intuitive case  $V = V^\infty$  where the notation is less cumbersome.

*Proof.* It is left as an exercise after Definition 8-11 of [KGSK12], we prove it to illustrate our definitions. For  $A \subset V$ , we denote respectively by  $(\tau^k)^+(A)$ ,  $(\tilde{\tau})^+(A)$  the first positive hitting time of  $A$  for  $S^k$  and  $\tilde{S}$ . Let  $x, y \in V$ ,  $n \in \mathbb{N}$ . By summing over all paths in  $V$  of length  $n$  from  $x$  to  $y$  that do not hit  $y$  except at their extremities,

$$\mathbb{P}_x((\tilde{\tau})^+(y) = n) \stackrel{(13)}{=} \frac{\lambda(y)}{\lambda(x)} \mathbb{P}_x((\tau^k)^+(y) = n).$$

Since  $S^k$  is transient, summing over all  $n$  and taking  $y = x$  gives

$$\mathbb{P}_x((\tilde{\tau})^+(x) < \infty) = \mathbb{P}_x((\tau^k)^+(x) < \infty) < 1. \quad \square$$

The *Doob transform technique* is often used to study general Markov chains conditioned by a certain event. We refer to Chapter 8 of [KGSK12] for a general introduction from the potential theory point of view and to Section 7 of [DHESC20] for a recent and clear introduction. It is a powerful tool that works in very general settings. The difficulty is to find a setting where these results can be applied and give quantitative estimate.

### 3 From rooted spanning forests to directed spanning trees

This section has two goals: the first one (developed in Section 3.1) is to show some applications of the Doob transform technique to the random RSF and RST models when a positive massive harmonic function is available. The idea is to show that a RSF model on a graph  $G$  can be related to a RST model *on the same graph* (infinite case) or on a modified but similar graph  $G^\circ$  (finite case). This is very different from the forest-tree bijection which relates RSF on  $G$  with RST on  $G^\rho$ : since our main applications will be to the case of planar graphs, it is very important that the modified graph  $G^\circ$  retains the properties of the initial graph, which is typically not the case of  $G^\rho$  (see Figure 2).

The second goal (developed in Section 3.2) is to justify that it is always possible to find positive massive harmonic functions on  $G$  (infinite case) or on  $G^\circ$  (finite case) and hence to apply Theorem 3.1. In the finite case it is impossible to find positive massive harmonic functions on  $G$ , see Paragraph 3.2.1 (which explains why the statements are heavier in the finite case).

#### 3.1 The Doob transform technique for rooted spanning forests

We start by introducing the notation in the intuitive case of an infinite graph, and then show how it has to be adapted in the finite case. The reader can in a first time skip the finite case and jump directly to the statement of Theorem 3.1.

### 3.1.1 The infinite case

Assume that  $\mathbf{G}$  is infinite and that  $S^k$  dies almost surely in finite time as in (ii) of Theorem 2.15. Let  $V^k, H^k, \mathbb{P}_W^k$  be respectively the associated potential, transfer current operator and Wilson's measure on RSF of  $\mathbf{G}$ .

Let  $\lambda$  be a positive massive harmonic function on  $\mathbf{V}$ . The Doob transform technique of Section 2.3 applies: denote by  $\tilde{S}$  the Doob transform of  $S^k$  by  $\lambda$  on  $\mathbf{V}$ : it is the RW associated with the conductance function  $\tilde{c}$  and the non-massive Laplacian  $\tilde{\Delta}$ . Denote by  $\tilde{V}$  the associated potential of the transient RW  $\tilde{S}$ :

$$\forall x, y \in \mathbf{V}, \quad \tilde{V}(x, y) = \mathbb{E}_x \left[ \sum_{i=0}^{\infty} \mathbb{1}_{\{\tilde{S}_i=y\}} \right]$$

and by  $\tilde{H}, \tilde{\mathbb{P}}_W$  the associated transfer current operator on  $\mathbf{V}$  and Wilson's measure on EST of  $\mathbf{G}$ . Proposition 2.17 shows that  $\Delta^k$  and  $\tilde{\Delta}$  are gauge equivalent:

$$\tilde{\Delta} = \Lambda^{-1} \Delta^k \Lambda.$$

The implications of this equation for the RSF and ERST models are the content of Theorem 3.1.

### 3.1.2 The finite case

Consider a connected graph  $\mathbf{G}^\infty = (\mathbf{V}^\infty, \mathbf{E}^\infty)$  with conductance and mass functions  $(c^\infty, m^\infty)$  and denote by  $\Delta^k$  and  $S^k$  the associated massive Laplacian and KRW. Let  $\mathbf{V} \subset \mathbf{V}^\infty$  be a strict *finite* subset such that the induced subgraph  $\mathbf{G}$  is connected, and denote by  $(c, m)$  the conductance and mass functions with wired boundary conditions induced by  $(c^\infty, m^\infty)$  on  $\mathbf{G}$ . Recall that the restriction  $\Delta_V^k$  is the massive Laplacian on  $\mathbf{G}$  associated with  $(c, m)$ . Let  $\lambda : \mathbf{V}^\infty \rightarrow \mathbb{R}_{>0}$  be massive harmonic for  $\Delta^k$  on  $\mathbf{V}$ . Denote by  $\tilde{S}$  and  $\tilde{\Delta}$  the Doob transform of  $S^k$  by  $\lambda$  on  $\mathbf{V}$ , and by  $(\tilde{c}, \tilde{m})$  the conductance and mass function with wired boundary conditions induced by  $\tilde{c}^\infty$  on  $\mathbf{G}$ :  $\tilde{\Delta}_V$  is the Laplacian associated with  $(\tilde{c}, \tilde{m})$ . Observe that unlike in Paragraph 3.1.1,  $\mathbf{V} \neq \mathbf{V}^\infty$  so  $\tilde{m} \neq 0$ . Proposition 2.17 holds and states that:

$$\tilde{\Delta}_V = \Lambda_V^{-1} \Delta_V^k \Lambda_V. \quad (16)$$

The Laplacians  $\Delta_V^k$  and  $\tilde{\Delta}_V$  are related to RSF models on  $\mathbf{G}$  with respective conductance and mass functions  $(c, m)$  and  $(\tilde{c}, \tilde{m})$ . Since  $\tilde{S}$  is a RW with no mass, we are in the setting described around Equation (11). Recall that  $\mathbf{G}^\circ$  is the graph obtained from  $\mathbf{G}$  by identifying all the vertices outside  $\mathbf{V}$  to an additional vertex  $o$ . By the forest-tree bijection, RSF of  $\mathbf{G}$  weighted by  $(\tilde{c}, \tilde{m})$  are in weight-preserving bijection with RST of  $\mathbf{G}^\circ$  weighted by  $\tilde{c}$ . *We use the notation  $o$  instead of  $\rho$  to insist on the fact that the graphs  $\mathbf{G}^\circ$  and  $\mathbf{G}^\rho$  are different (compare Figure 2 and 3), and this is the key point of our work.*

Let  $\tilde{V}, \tilde{H}$  be the potential and transfer current operator associated with the RW  $\tilde{S}$  killed when it exits  $\mathbf{V}$  (or more formally, with the KRW on  $\mathbf{G}$  associated with the conductance and mass functions  $(\tilde{c}, \tilde{m})$ ). Observe the difference with Paragraph 3.1.1 where they are the potential and transfer operators  $\tilde{V}$  and  $\tilde{H}$  associated with a RW *with no killing*. Let  $\mathbb{P}_W^k$  and  $\tilde{\mathbb{P}}_W$  be the Wilson's measure on RST of  $\mathbf{G}^\rho$  (rooted at  $\rho$ ) and RST of  $\mathbf{G}^\circ$  (rooted at  $o$ ) associated with  $S^k$  and  $\tilde{S}$ .

### 3.1.3 Statement and proof of the main result

Consider either the setting of Paragraph 3.1.1 ( $\mathbf{G}$  is infinite) or the setting of Paragraph 3.1.2 ( $\mathbf{G}$  is finite). The following holds:

**Theorem 3.1.** *The Wilson's measures  $\mathbb{P}_W^k$  and  $\tilde{\mathbb{P}}_W$  are naturally related: they both satisfy the transfer current Theorem 2.9 with respective transfer current operators*

$$\begin{cases} \forall \vec{e} = (w, x), \vec{f} = (y, z) \in \vec{\mathbf{E}}^\rho, & H_{\vec{e}, \vec{f}}^k = \frac{V^k(w, y)}{c^k(y)} - \frac{V^k(x, y)}{c^k(y)} \\ \forall \vec{e} = (w, x), \vec{f} = (y, z) \in \vec{\mathbf{E}}^o, & \tilde{H}_{\vec{e}, \vec{f}} = \frac{\lambda(y)}{\lambda(w)} \frac{V^k(w, y)}{c^k(y)} - \frac{\lambda(y)}{\lambda(x)} \frac{V^k(x, y)}{c^k(y)}. \end{cases} \quad (17)$$

When  $\mathbf{G}$  is infinite,  $\vec{\mathbb{E}}^o = \vec{\mathbb{E}}$ . When  $\mathbf{G}$  is finite, we use the conventions  $V^k(o, \cdot) = V^k(\cdot, o) = 0$ . Moreover, when  $\mathbf{G}$  is finite, the following equality of partition functions holds:

$$Z_{\text{RSF}}(\mathbf{G}, c, m) = Z_{\text{RST}}^\rho(\mathbf{G}^\rho, c^k) = Z_{\text{RST}}^o(\mathbf{G}^o, \tilde{c}). \quad (18)$$

When  $\mathbf{G}$  is infinite, this theorem relates the RSF model on  $\mathbf{G}$  with conductance and mass functions  $(c, m)$  with an EST model on  $\mathbf{G}$  with conductance function  $\tilde{c}$ . When  $\mathbf{G}$  is finite, it relates a RSF model on  $\mathbf{G}$  with another RSF model on  $\mathbf{G}$  with the mass “sent to the boundary”.

*Proof.* We first prove the equality of partition functions in the finite case. The first equality in (18) is the matrix-forest theorem stated in Equation (5). For the second equality, the matrix-tree theorem implies on the one hand

$$Z_{\text{RST}}^o(\mathbf{G}^o, \tilde{c}) = \det(\tilde{\Delta}_{\mathbf{V}})$$

and on the other hand

$$Z_{\text{RST}}^\rho(\mathbf{G}^\rho, c^k) = \det(\Delta_{\mathbf{V}}^k).$$

The gauge equivalence of Equation (16) and the fact that the determinant is invariant by gauge change imply the second equality of Equation (18).

In Paragraph 3.1.1, we assumed that  $S^k$  dies almost surely in finite time so the first statement of Equation (17) is case (ii) of Theorem 2.15. In this case, the RW  $\tilde{S}$  is transient by Proposition 2.18 so it satisfies the hypothesis of case (i) of Theorem 2.15 with the transfer current operator  $\tilde{H}$  on  $\mathbf{G}$ . In the setting of Paragraph 3.1.2, the first statement of Equation (17) is simply case (i) of Theorem 2.9. In this case, the RW  $\tilde{S}$  killed when exiting  $\mathbf{V}$  is a KRW on a finite graph and Theorem 2.9 holds with the potential and transfer current operators  $\tilde{V}, \tilde{H}$  on  $\mathbf{G}^o$ . Note that by convention,  $\tilde{V}(x, o) = \tilde{V}(o, x) = 0$  for all  $x \in \mathbf{V}$ .

We only have to show that in both cases, the transfer current operator  $\tilde{H}$  takes the form given in Equation (17). By Equation (13),

$$\forall i \in \mathbb{N}, \forall x, y \in \mathbf{V}, \mathbb{P}_x(\tilde{S}_i = y) = \frac{\lambda(y)}{\lambda(x)} \mathbb{P}_x(S_i^k = y),$$

which implies for all  $x, y \in \mathbf{V}$

$$\tilde{V}(x, y) = \sum_{i=0}^{\infty} \mathbb{P}_x(\tilde{S}_i = y) = \frac{\lambda(y)}{\lambda(x)} \sum_{i=0}^{\infty} \mathbb{P}_x(S_i^k = y) = \frac{\lambda(y)}{\lambda(x)} V^k(x, y).$$

Note that in the finite case, this also holds when  $x = o$  or  $y = o$  since both sides are 0. Moreover, Proposition 2.17 (and more precisely Equation (15)) implies that for all  $x \in \mathbf{V}$ ,  $c^k(x) = \tilde{c}(x)$ . Hence, for all  $\vec{e} = (w, x) \in \vec{\mathbb{E}}^o, \vec{f} = (y, z) \in \vec{\mathbb{E}}^o$  (this is simply  $\vec{\mathbb{E}}$  when  $\mathbf{G}$  is infinite), the definition of the transfer current operator  $\tilde{H}$  becomes

$$\tilde{H}_{\vec{e}, \vec{f}} = \frac{\tilde{V}(w, y)}{\tilde{c}(y)} - \frac{\tilde{V}(x, y)}{\tilde{c}(y)} = \frac{\lambda(y)}{\lambda(w)} \frac{V^k(w, y)}{c^k(y)} - \frac{\lambda(y)}{\lambda(x)} \frac{V^k(x, y)}{c^k(y)}. \quad \square$$

## 3.2 Positive massive harmonic functions

### 3.2.1 The finite case

Consider a finite graph  $\mathbf{G}$  with conductance and mass functions  $(c, m)$  and associated massive Laplacian  $\Delta^k$ .

**Proposition 3.2.** *If  $m \neq 0$ , there are no positive massive harmonic functions on  $\mathbf{V}$ .*

*Proof.* We proceed by contradiction and assume that there exists a positive massive harmonic function  $\lambda$ . We apply the maximum principle to show that  $\lambda$  is constant: by contradiction, if  $\lambda$  is not constant, since  $\mathsf{G}$  is connected we can pick a point  $x \in \mathsf{V}$  where  $\lambda$  reaches its maximum and a neighbouring point  $y \sim x$  with  $\lambda(y) < \lambda(x)$ . Massive harmonicity at  $x$  implies

$$\sum_{y \sim x} c_{(x,y)} \lambda(x) \leq \left( m(x) + \sum_{y \sim x} c_{(x,y)} \right) \lambda(x) = \sum_{y \sim x} c_{(x,y)} \lambda(y) < \sum_{y \sim x} c_{(x,y)} \lambda(x),$$

which is a contradiction. Hence  $\lambda$  is constant (and positive), and for all  $x \in \mathsf{V}$ , massive harmonicity at  $x$  implies  $c(x) = c^k(x) = c(x) + m(x)$  which contradicts the fact that  $m \neq 0$ .  $\square$

This justifies why when  $\mathsf{G}$  is finite we cannot use the same simple setting as in Paragraph 3.1.1. Consider now  $(\mathsf{G}^\infty, c^\infty, m^\infty)$  and a finite connected induced subgraph with wired boundary conditions  $(\mathsf{G}, c, m)$  as in Paragraph 3.1.2.

**Proposition 3.3.** *There exists a positive function  $\lambda$  on  $\mathsf{V}^\infty$  satisfying  $(\Delta^k \lambda)(x) = 0$  for all  $x \in \mathsf{V}$ .*

*Proof.* Recall that  $S^k$  is the KRW on  $\mathsf{G}^\infty$  associated with  $(c^\infty, m^\infty)$  and let  $V^{\infty,k}$  be the associated potential operator on  $\mathsf{V}^\infty$ . Be careful that it does not coincide with  $V^k$  even when restricting to  $\mathsf{V}$ . Fix any  $z \in \mathsf{V}^\infty \setminus \mathsf{V}$  and let  $\lambda(\cdot) = V^{\infty,k}(\cdot, z)$ . This defines a function which is positive on  $\mathsf{V}^\infty$  since  $\mathsf{G}^\infty$  is connected and massive harmonic on  $\mathsf{V}^\infty \setminus \{z\}$  by Equation (9), so in particular it is massive harmonic on  $\mathsf{V}$ .  $\square$

Proposition 3.3 justifies that the statement of Theorem 3.1 are non empty. A more explicit way of finding massive harmonic functions as in Paragraph 3.1.2 is to use an explicit massive harmonic function  $\lambda$  on an infinite graph  $\mathsf{G}^\infty$  with massive Laplacian  $\Delta^k$  (see Example 3.6 in the next paragraph) and apply the Doob transform technique on a finite induced subgraph  $\mathsf{G}$  of  $\mathsf{G}^\infty$ . This is the starting point of Section 6.

### 3.2.2 The infinite case: Martin boundary of a killed random walk

For some specific infinite graphs and Laplacians (for example isoradial graphs or  $\mathbb{Z}^d$ -periodic graphs), explicit positive massive harmonic functions are known. This will be detailed in Example 3.6 and in Section 6 and Section 5.2.

In this paragraph, we show that there always exist positive massive harmonic functions on an infinite graph, which justifies that the statements of Section 3.1 are non-empty. For non-massive RW, or KRW with a constant killing function  $m(x) = m > 0 \forall x \in \mathsf{V}$ , harmonic and superharmonic functions are described by the theory of the Martin boundary (see Section 24 of [Woe00] for a general introduction in the transient case). To our knowledge, the general theory has not been developed for KRW with non-constant killing functions, but some examples (which we will recall at the end of the paragraph) were considered in [BR22]. We give a sketch of what such a general theory would look like and prove a first result. It would be interesting to push this further.

Consider an infinite irreducible graph with a conductance function  $c$  and a mass function  $m \neq 0$ . Recall that the KRW is transient and its potential  $V^k$  is defined by Equation (8). Let  $x_0 \in \mathsf{V}$  be a reference point. For  $y \in \mathsf{V}$ , the *killed Martin kernel* is defined as the function  $\mathcal{K}(\cdot, y) : \mathsf{V} \rightarrow \mathbb{R}_{>0}$

$$\forall x \in \mathsf{V}, \mathcal{K}(x, y) = \frac{V^k(x, y)}{V^k(x_0, y)}.$$

Consider a sequence of vertices  $\zeta = (z_n)_{n \in \mathbb{N}} \in \mathsf{V}^\mathbb{N}$  going to  $\infty$  (in the sense that it visits each fixed vertex of  $\mathsf{V}$  for only a finite set of indices). We say that  $\zeta$  converges to the *killed Martin boundary* if for all  $x \in \mathsf{V}$ , the killed Martin kernel converges towards a limit in  $(0, \infty)$

$$\mathcal{K}(x, \zeta) := \lim_{n \rightarrow \infty} \frac{V^k(x, z_n)}{V^k(x_0, z_n)} \in (0, +\infty). \quad (19)$$

Two sequences  $\zeta_1, \zeta_2$  converging to the Martin boundary are *equivalent* and we write  $\zeta_1 \sim \zeta_2$  if the corresponding Martin kernel  $\mathcal{K}(\cdot, \zeta_1)$  and  $\mathcal{K}(\cdot, \zeta_2)$  coincide. The *killed Martin boundary*  $\partial^M(\mathbf{G})$  is defined as the set of these converging sequences quotiented by the equivalence relation  $\sim$ .

**Proposition 3.4.** *The killed Martin boundary of a transient KRW is non empty. More precisely, for all sequence  $(z_n)_{n \in \mathbb{N}} \in \mathbf{V}^{\mathbb{N}}$  such that  $z_n \xrightarrow{n \rightarrow \infty} \infty$ , there exists an extraction  $k_n \xrightarrow{n \rightarrow \infty} \infty$  such that the subsequence  $(z_{k_n})_{n \in \mathbb{N}}$  converges to the killed Martin boundary.*

We could not find this result under such general hypothesis so we provide a proof even though this will not seem new to the reader acquainted with the usual Martin boundary theory.

*Proof.* For all  $x \in \mathbf{V}$ , since  $\mathbf{G}$  is connected there exists  $n_0 \in \mathbb{N}$  such that  $\mathbb{P}_{x_0}(S_{n_0}^k = x) > 0$ . Hence by the simple Markov property,

$$\frac{V^k(x, z_n)}{V^k(x_0, z_n)} \geq \frac{\mathbb{P}_{x_0}(S_{n_0}^k = x, \tau^\rho(z_n) > n_0) V^k(x_0, z_n)}{V^k(x_0, z_n)} \xrightarrow{n \rightarrow \infty} \mathbb{P}_{x_0}(S_{n_0}^k = x) > 0$$

where the convergence is justified by the fact that  $z_n \rightarrow \infty$ . This shows that the ratio is bounded from below: there exists  $\varepsilon(x) > 0$  such that for all  $n$  large enough,

$$\frac{V^k(x, z_n)}{V^k(x_0, z_n)} \geq \varepsilon(x).$$

The reverse argument works the same way (by exchanging  $x_0$  and  $x$ ): there exists  $E(x) > 0$  such that for all  $n$  large enough,

$$\frac{V^k(x, z_n)}{V^k(x_0, z_n)} \leq E(x).$$

Hence, for fixed  $x$ , there exists a subsequence  $z_{k_n}$  with  $k_n \rightarrow \infty$  such that the ratio  $\frac{V^k(x, z_{k_n})}{V^k(x_0, z_{k_n})}$  converges. Since this holds for all  $x \in \mathbf{V}$ , by diagonal extraction we can find a common converging subsequence.  $\square$

In the usual Martin boundary theory (i.e. the KRW with constant mass function  $m \geq 0$ ), an integral representation theorem shows that all harmonic functions are obtained as integrals of Martin kernels. We do not know if such a result holds for KRW with general killing function, but we can still use that limits of killed Martin kernels provide us with massive harmonic functions on the full graph:

**Proposition 3.5.** *If  $\zeta \in \partial^M(\mathbf{G})$ ,  $\mathcal{K}(\cdot, \zeta)$  is a positive massive harmonic function on  $\mathbf{V}$ .*

Observe that this is in some sense the limit of the finite case since it corresponds to sending  $z \rightarrow \infty$  in Proposition 3.3.

*Proof.* Let  $\zeta = (z_n)_{n \in \mathbb{N}}$ . Since by Equation (9)  $V^k(\cdot, z_n)$  is a massive harmonic function on  $\mathbf{V} \setminus \{z_n\}$ , we have that for fixed  $x \in \mathbf{V}$  and  $n$  large enough such that  $x \neq z_n$ :

$$(\Delta^k V^k(\cdot, z_n))(x) = 0.$$

Dividing by  $V^k(x_0, z_n)$  and letting  $n \rightarrow \infty$  gives the massive harmonicity of the killed Martin kernel.  $\square$

We now give some examples where this killed Martin boundary was studied.

*Example 3.6.*

- The degenerate case of our model where the RW associated to the conductances  $c$  is recurrent and the mass is infinite at a point and 0 elsewhere is the setting of the recurrent Martin boundary theory, which is well developed and written out for example in Chapter 9 of [KGSK12]. In particular, for the simple random walk on  $\mathbb{Z}^2$  killed almost surely at the origin, the convergence in Equation (19) holds and the left hand side is the usual potential kernel for any sequence  $(z_n)_{n \in \mathbb{N}}$  going to infinity.

- If the graph  $G$  is isoradial and  $\Delta^k$  is the  $Z$ -invariant massive Laplacian of [BdTR17], the main result of [BR22] is the asymptotics of the killed Martin kernel  $\mathcal{K}(\cdot, \zeta)$  depending on how the sequence  $\zeta$  goes to infinity. This asymptotic is described explicitly: it is the discrete massive exponential evaluated at a certain point, so the killed Martin boundary is fully described in this case.
- If the graph  $G$  with conductance and mass functions  $(c, m)$  are  $\mathbb{Z}^d$ -periodic, the asymptotic of the killed Martin kernel is also discussed in [BR22] (building on earlier results). Theorem 5 of [BR22] establishes convergence of the killed Martin kernel for translated copies of the same vertex  $x \in V$ . More precisely, they show that for  $x \in V$ ,  $y = x + (k_1, \dots, k_d)$  and  $z_n = u + (i_1^n, \dots, i_d^n)$  such that  $z_n \rightarrow \infty$  and  $\frac{z_n}{|z_n|}$  converges to a limit  $r$ ,

$$\lim_{n \rightarrow \infty} \frac{V^k(x, z_n)}{V^k(y, z_n)} = \exp(\nu(r) \cdot (k_1, \dots, k_d))$$

where  $\nu$  is defined precisely in [BR22].

## 4 The planar case: from rooted spanning forests to dimers

The goal of this section is to explain how our results relating a RSF model on a planar graph  $G$  with a RST model on a related planar graph  $G^o$  can be used to relate the RSF model on  $G$  with a dimer model on a planar graph related to  $G^o$ .

### 4.1 Preliminary: Temperley's bijection

Temperley's bijection was first stated in [TF61] as a bijection between (undirected) spanning trees of a finite grid and dimer configurations of an associated grid. It was later extended by several authors. The version that we describe is a weight-preserving bijection between weighted RST of a planar graph  $G$  and weighted dimer configurations of its double graph. This extension can be found in Section 2 of [KPW00]. For the notation we align as much as possible with Section 2.3.1 of [dT20]. *Warning:* in the bijection that we present here, the primal and dual graph are exchanged compared to the literature.

**Dimers on a bipartite graph.** Let  $G = (V, E)$  be a bipartite graph (finite or infinite). Its vertex set can be partitioned into white and black vertices  $V = W \sqcup B$  such that there is no edge between two vertices of the same color. If  $G$  is finite, we assume that it has the same number of white and black vertices:  $|W| = |B|$ . A *dimer configuration* on  $G$  is a subset of edges  $M \subset E$  such that every vertex in  $V$  is incident to exactly one edge of this subset. The set of dimer configurations of  $G$  is denoted by  $\mathcal{M}(G)$ . A dimer model on  $G$  is specified by a weight function on the edges  $\nu : e \in E \rightarrow \nu_e \geq 0$ . The weight of a dimer configuration is

$$\forall M \in \mathcal{M}(G), \nu_{\dim}(M) = \prod_{e \in M} \nu_e. \quad (20)$$

The partition function of the dimer model is the weighted sum

$$Z_{\dim}(G, \nu) = \sum_{M \in \mathcal{M}(G)} \nu_{\dim}(M).$$

When  $G$  is finite and  $\mathcal{M}(G) \neq \emptyset$ ,  $0 < Z_{\dim}(G, \nu) < \infty$  so this weight function induces a probability measure on dimer configurations:

$$\forall M \in \mathcal{M}(G), \mathbb{P}_{\dim}(M) = \frac{\nu_{\dim}(M)}{Z_{\dim}(G, \nu)}. \quad (21)$$

For all  $\{e_1, \dots, e_k\} \subset \mathbf{E}$ , we denote by  $\mathbb{P}_{\dim}(e_1, \dots, e_k)$  the measure of the set of all dimer configurations containing  $\{e_1, \dots, e_k\}$ .

A *set of phases*  $\zeta$  is the attribution of a complex number of modulus one  $\zeta_{wb}$  to each edge  $wb \in \mathbf{E}$ . It satisfies the *Kasteleyn property* if, for all inner face of  $\mathbf{G}$  bounded by the  $2p$  vertices  $w_1, b_1, \dots, w_p, b_p$  in clockwise order,

$$\frac{\zeta_{w_1 b_1} \cdots \zeta_{w_p b_p}}{\zeta_{w_2 b_1} \cdots \zeta_{w_1 b_p}} = (-1)^{p-1}.$$

Given a set of phases  $\zeta$  on  $\mathbf{G}$  satisfying the Kasteleyn property, the *Kasteleyn matrix* of the dimer model associated with these phases is the matrix with rows indexed by white vertices and columns by black vertices

$$\forall w \in \mathbf{W}, b \in \mathbf{B}, K_{w,b} = \begin{cases} \zeta_{wb} \nu_{wb} & \text{if } w \sim b \\ 0 & \text{otherwise.} \end{cases}$$

It has been known since [TF61] (see [Kup98] for the version with phases that we use here) that the determinant of the Kasteleyn matrix counts the weighted sum of dimer configurations:

$$|\det K| = Z_{\dim}(\mathbf{G}, \nu). \quad (22)$$

When  $\mathcal{M}(\mathbf{G}) \neq \emptyset$ , this implies that the Kasteleyn matrix is invertible and the probability measure  $\mathbb{P}_{\dim}$  is related to the Kasteleyn matrix  $K$  by the *local statistics formula* of [Ken97]:

$$\forall \{w_1 b_1, \dots, w_k b_k\} \subset \mathbf{E}, \mathbb{P}_{\dim}(w_1 b_1, \dots, w_k b_k) = \left| \prod_{i=1}^k K_{w_i, b_j} \det (K_{b_i, w_j}^{-1})_{1 \leq i, j \leq k} \right|. \quad (23)$$

The dimer models associated with two weight functions  $\nu^1, \nu^2$  are *gauge equivalent* with gauge functions  $\phi : \mathbf{W} \rightarrow \mathbb{R}_{>0}, \psi : \mathbf{B} \rightarrow \mathbb{R}_{>0}$  if

$$\forall wb \in \mathbf{E}, \nu_{wb}^2 = \phi(w) \psi(b) \nu_{wb}^1.$$

The dimer models are gauge equivalent with gauge functions  $\phi : \mathbf{W} \rightarrow \mathbb{R}_{>0}, \psi : \mathbf{B} \rightarrow \mathbb{R}_{>0}$  if and only if their respective Kasteleyn matrix  $K^1, K^2$  (with respect to the same fixed set of phases) satisfy

$$K^2 = \Phi K^1 \Psi,$$

where  $\Phi = D(\phi)$  and  $\Psi = D(\psi)$ . The Kasteleyn matrices are also said to be *gauge equivalent* with gauge  $(\Phi, \Psi)$ . If  $\mathbf{G}$  is finite, gauge equivalence preserves the measure since for all  $\mathbf{M} \in \mathcal{M}(\mathbf{G})$ ,

$$\nu^2(\mathbf{M}) = \prod_{e \in \mathbf{M}} \nu_e^2 = \prod_{w \in \mathbf{W}} \phi(w) \prod_{b \in \mathbf{B}} \psi(b) \prod_{e \in \mathbf{M}} \nu_e^1 = \left( \prod_{w \in \mathbf{W}} \phi(w) \prod_{b \in \mathbf{B}} \psi(b) \right) \nu^1(\mathbf{M}). \quad (24)$$

where  $\prod_{w \in \mathbf{W}} \phi(w) \prod_{b \in \mathbf{B}} \psi(b)$  is a constant which does not depend on  $\mathbf{M}$ . In particular, gauge equivalence preserves the local statistics formula Equation 23.

**Dual and double graph of a planar graph.** For the rest of the section, let  $(\mathbf{G}, c, m)$  be an infinite connected planar graph with mass and conductance functions as in Paragraph 3.1.1 or a finite simply connected induced subgraph with wired boundary conditions of a planar graph with conductance and mass functions  $(\mathbf{G}^\infty, c^\infty, m^\infty)$  as in Paragraph 3.1.2.

Recall the definition of the boundary  $\partial\mathbf{V}$  of  $\mathbf{G}$  in  $\mathbf{G}^\infty$ . Note that when  $\mathbf{G}$  is infinite,  $\partial\mathbf{V} = \emptyset$ . Recall from Paragraph 3.1.2 the definition of  $\mathbf{G}^\circ$  obtained by identifying all vertices in  $\mathbf{V}^\infty \setminus \mathbf{V}$  to a unique vertex  $o$ , and recall that we abuse notation and also write  $c$  for the conductance function on  $\mathbf{G}^\circ$ . We already observed that since  $\mathbf{G}$  is simply connected,  $\mathbf{G}^\circ$  is planar. If  $\mathbf{G}$  is infinite, we use the convention  $\mathbf{G}^\circ = \mathbf{G}$ .

Let  $\mathbf{G}^* = (\mathbf{V}^*, \mathbf{E}^*)$  be the restricted dual graph of  $\mathbf{G}^\circ$  (in black on Figure 4). It is the dual graph of  $\mathbf{G}^\circ$  from which the dual vertex corresponding to the outer face and all the edges incident



to it are removed. If  $G$  is infinite, it is simply the dual graph of  $G = G^o$ . Observe that  $G$  is the restricted dual of  $G^*$  so we are in the same situation as [KPW00] with primal and dual exchanged. If  $G$  is finite, given  $r \in G^*$ , every RST of  $G^o$  rooted at  $o$ ,  $T \in \mathcal{T}^o(G^o)$ , has a *dual directed spanning tree* rooted at  $r$  which we denote by  $T^*$ . It is the unique RST of  $G^*$  rooted at  $r$  which does not intersect  $T$ .

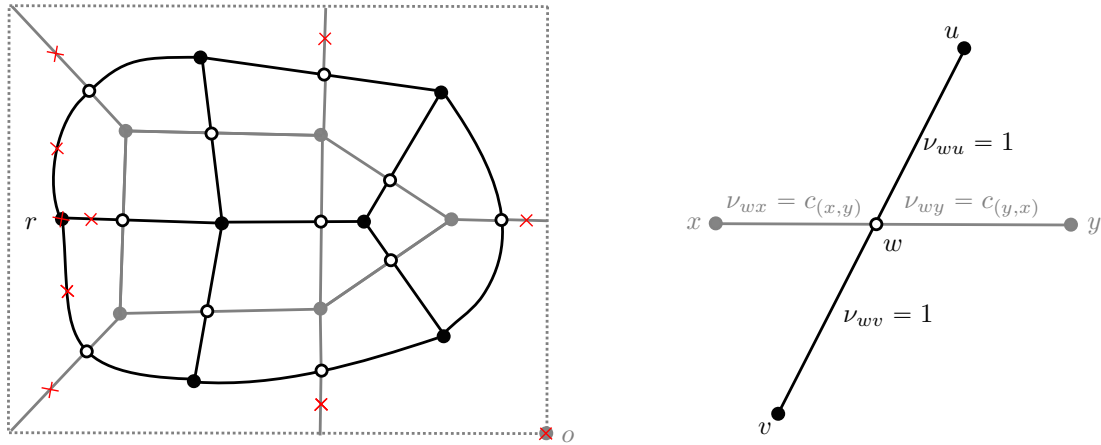


Figure 4: On the left:  $G$  (in gray) is a finite simply connected induced subgraph of a planar graph. The additional vertex  $o$  in the outer face is represented in a spread-out way by a dotted gray line. The restricted dual  $G^*$  is drawn in black. The white vertices of the double graph  $G^D$  are drawn as white balls. The graph  $\underline{G}^D$  is obtained from  $G^D$  by removing the vertices  $o$  and  $r$  and the edges incident to them (crossed in red). On the right: the weight function  $\nu$  on the edges of  $\underline{G}^D$ .

Embed the graphs  $G^o$  and  $G^*$  in the plane so that the primal and dual edges intersect at a single point. The *double graph*  $G^D = (V^D, E^D)$  is constructed by superimposing the graphs  $G^o$  and  $G^*$  and adding an extra vertex at each crossing of a primal and dual edge. It has two types of vertices: the original vertices are called *black vertices* (drawn as gray and black bullets on Figure 4) and denoted by  $B = V^o \sqcup V^*$  while the extra vertices are called *white vertices* and denoted by  $W$  (drawn as white bullets on Figure 4). The graph  $G^D$  is bipartite: its edges correspond to half-edges of  $G^o$  or  $G^*$  linking a white vertex with a black vertex.

Every (undirected) edge of the double graph  $G^D$  corresponds to a directed edge of the primal graph  $G^o$  or dual graph  $G^*$  (and conversely): if an edge  $e = xw \in E^D$  links a primal vertex  $x \in V^o$  to a white vertex  $w \in W$ , since this white vertex corresponds to the intersection between a primal edge  $xy \in E^o$  and a dual edge  $uv \in E^*$ , we say that the (undirected) half-edge  $xw \in E^D$  corresponds to the (directed) edge  $(x, y) \in E^o$  of the primal graph. The same correspondence holds between edges  $uw \in E^D$  with  $u \in V^*$ ,  $w \in W$  and directed edges of the dual graph  $G^*$ .

For the rest of the section, when  $G$  is finite,  $r \in \partial G^*$  denotes a vertex on the boundary of the dual graph (which is also planar). Denote by  $\underline{G}^D = (\underline{V}^D, \underline{E}^D)$  the graph obtained from  $G^D$  by removing  $o$ ,  $r$  and all the edges incident to them (on Figure 4 it corresponds to removing everything crossed in red). Using our notation for induced subgraphs,  $\underline{G}^D = G_{V^D \setminus \{o, r\}}^D$ . When  $G$  is infinite, we use the convention  $\underline{G}^D = G^D$ . The graph  $\underline{G}^D$  is also bipartite with white vertices  $\underline{W} = W$  and black vertices  $\underline{B} := V \sqcup V^*$  with  $V^* = V^* \setminus \{r\}$ . Note that if  $G$  is infinite,  $V = V^o$  and  $\underline{V}^* = V^*$ . When  $G$  is finite,  $\underline{G}^D$  has the same number of black and white vertices:  $|\underline{W}| = |\underline{B}|$  (by Euler's formula applied to the planar graph  $G^*$ , which has  $|V^o|$  faces,  $|V^*|$  vertices and  $|\underline{W}|$  edges).

Even if  $G$  is not bipartite, its double graph  $G^D$  is and there is a convenient way to choose phases on  $G^D$ . It goes as follows: around each white vertex, there are four vertices of  $G^D$  alternating between vertices of  $G$  and  $G^*$ , say  $x, u, y, v$  in the clockwise order. We attribute phases to edges

$$\zeta_{wx} = 1, \zeta_{wu} = i, \zeta_{wy} = -1, \zeta_{wv} = -i$$

(the choice of starting with 1 on  $x$  or  $y$  is arbitrary and can be made independently for each  $w \in W$ ). Such a choice of phases is represented on Figure 5. These phases satisfy the Kasteleyn

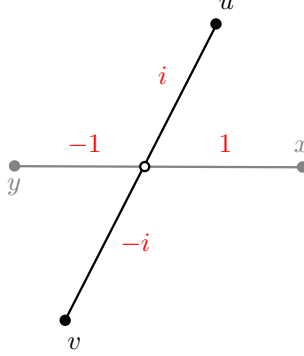


Figure 5: A choice of phases  $\zeta$  on the double graph  $\underline{G}^D$  satisfying the Kasteleyn property (in red).

property: the alternating product around the faces is  $-1$  (and all the inner faces of  $G^D$  are bounded by four vertices). The restriction of these phases to  $\underline{G}^D$  also satisfies the Kasteleyn property since inner faces of  $\underline{G}^D$  are also inner faces of  $G^D$ . For the rest of this section, such a choice of phases is fixed.

**Temperley's bijection.** We define weights on the edges of  $\underline{G}^D$  (see Figure 4).

$$\forall wb \in \underline{E}^D, \nu_{wb} = \begin{cases} 1 & \text{if } b = u \in \underline{V}^* \\ c_{(x,y)} & \text{if } b = x \in \underline{V} \text{ and } xw \in \underline{E}^D \text{ corresponds to } (x, y) \in \underline{E}^o \end{cases} \quad (25)$$

*Temperley's bijection* is a weight-preserving correspondence between dimer configurations and RST of a finite planar graph. Denote by  $\nu_{\text{RST}}^o$  the weight function on  $\mathcal{T}^o(G^o)$  associated to  $c$  by Equation (2) and by  $\nu_{\text{dim}}$  the weight function on  $\mathcal{M}(\underline{G}^D)$  associated to  $\nu$  by (20).

**Theorem 4.1** (Temperley's bijection, Theorem 1 of [KPW00]). *Assume that  $G$  is finite. Dimer configurations of  $\underline{G}^D$  are in weight-preserving bijection with RST of  $G^o$  rooted at  $o$ . More precisely, for every  $T \in \mathcal{T}^o(G^o)$ , the corresponding dimer configuration  $M \in \mathcal{M}(\underline{G}^D)$  is obtained by replacing each directed edge of  $T$  and of its dual directed spanning tree  $T^*$  by the corresponding (undirected) edge of  $\underline{G}^D$ , and satisfies*

$$\nu_{\text{RST}}^o(T) = \nu_{\text{dim}}(M).$$

Pictures of Temperley's bijection in this context can be found in Figure 1 of [KPW00] or Figure 2 of [dT20]. Recall the notation  $\mathbb{P}_{\text{dim}}$  for the probability measure on  $\mathcal{M}(\underline{G}^D)$  associated with  $\nu$  by Equation (21) and denote by  $\mathbb{P}_{\text{RST}}^o$  the probability measure on RST of  $G^o$  associated with the conductance function  $c$  by Equation (3).

**Corollary 4.2.** *Assume that  $G$  is finite. The partition function of the dimer and tree models are equal:*

$$Z_{\text{RST}}^o(G^o, c) = Z_{\text{dim}}(\underline{G}^D, \nu).$$

Moreover, for all  $\vec{e}_1, \dots, \vec{e}_k \in \underline{E}^o$ , if  $f_1, \dots, f_k$  are the corresponding undirected edges of  $\underline{G}^D$ :

$$\mathbb{P}_{\text{RST}}^o(\vec{e}_1, \dots, \vec{e}_k) = \mathbb{P}_{\text{dim}}(f_1, \dots, f_k).$$

## 4.2 The drifted and killed dimer models.

The goal of this Section is to combine Theorem 3.1 with Temperley's bijection to relate the RSF model on  $G$  with a dimer model. We introduce two families of dimer models and show how they

are related to the RSF model. For the rest of this section, we assume that  $c$  is a *symmetric* conductance function (the symmetry is necessary here unlike in the rest of the paper).

Let  $\lambda : \mathbf{V}^\circ \rightarrow \mathbb{R}_{>0}$ , not necessarily massive harmonic for the moment. Recall that when  $\mathbf{G}$  is finite, it is an induced subgraph of a larger graph  $\mathbf{G}^\circ$ . Recall the definition of the conductance function associated with the Doob transform  $\forall (x, y) \in \mathbf{E}$ ,  $\tilde{c}_{(x,y)} = \frac{\lambda(y)}{\lambda(x)} c_{xy}$ . The conductance function  $\tilde{c}$  is not in general symmetric, even though  $c$  is. Recall that when  $\mathbf{G}$  is finite, we abuse notation and also denote by  $\tilde{c}$  the conductance function on  $\mathbf{G}^\circ$  obtained from  $\tilde{c}^\circ$  by identifying vertices of  $\mathbf{V}^\circ \setminus \mathbf{V}$  to  $o$ .

We introduce two dimer models on  $\underline{\mathbf{G}}^{\mathbf{D}}$  (local weights are pictured in Figure 1): they are a generalization on any graph with any symmetric conductance function of the “drifted” and “killed” dimer models of [Chh12] (which were defined on  $\mathbb{Z}^2$  for specific conductance functions) and of the “killed” dimer model defined in Section 3.1 of [dT20] underlying the  $Z^u$ -Dirac operator (which was defined on an isoradial graph, with specific elliptic weights). The local weights associated with these dimer models are pictured on Figure 1.

**Definition 4.3** (The drifted dimer model). Denote by  $\nu^{\mathbf{d}}$  the weight function on the edges of  $\underline{\mathbf{G}}^{\mathbf{D}}$  associated to  $\tilde{c}$  by Equation (25):

$$\forall wb \in \underline{\mathbf{E}}^{\mathbf{D}}, \nu_{wb}^{\mathbf{d}} = \begin{cases} 1 & \text{if } b = u \in \underline{\mathbf{V}}^\star \\ \tilde{c}_{(x,y)} & \text{if } b = x \in \mathbf{V} \text{ and } xw \in \underline{\mathbf{E}}^{\mathbf{D}} \text{ corresponds to } (x, y) \in \vec{\mathbf{E}}^\circ. \end{cases}$$

(see Figure 1). When  $\mathbf{G}$  is infinite,  $\underline{\mathbf{G}}^{\mathbf{D}} = \mathbf{G}^{\mathbf{D}}$ ,  $\vec{\mathbf{E}}^\circ = \vec{\mathbf{E}}$  and  $\forall (x, y) \in \mathbf{E}$ ,  $\tilde{c}_{(x,y)} = \frac{\lambda(y)}{\lambda(x)} c_{xy}$ . The *drifted dimer model* associated with  $\lambda$  is the dimer model on  $\underline{\mathbf{G}}^{\mathbf{D}}$  with weights  $\nu^{\mathbf{d}}$ .

Let  $\lambda^\star : \mathbf{V}^\star \rightarrow \mathbb{R}_{>0}$  be an arbitrary positive function on the dual.

**Definition 4.4** (The killed dimer model). We define a weight function  $\nu^{\mathbf{k}}$  associated to  $\lambda, \lambda^\star$  on the edges of  $\underline{\mathbf{G}}^{\mathbf{D}}$  (see Figure 1). If  $w \in \underline{\mathbf{W}}$  corresponds to the intersection of the primal and dual edges  $xy \in \mathbf{E}^\circ$  (with  $x \neq o$ ) and  $uv \in \mathbf{E}^\star$  (with  $u \neq r$ ), let

$$\nu_{wb}^{\mathbf{k}} = \begin{cases} \lambda^\star(u)^{-1} (c_{xy} \lambda(x) \lambda(y))^{-\frac{1}{2}} & \text{if } b = u \\ (c_{xy} \lambda(y))^{\frac{1}{2}} \lambda(x)^{-\frac{1}{2}} & \text{if } b = x \end{cases} \quad (26)$$

(see Figure 1). Recall that when  $\mathbf{G}$  is infinite,  $\underline{\mathbf{G}}^{\mathbf{D}} = \mathbf{G}^{\mathbf{D}}$  and  $\mathbf{E}^\circ = \mathbf{E}$ . The dimer model on  $\underline{\mathbf{G}}^{\mathbf{D}}$  with weight function  $\nu^{\mathbf{k}}$  is called the *killed dimer model*.

We denote by  $K^{\mathbf{d}}$  and  $K^{\mathbf{k}}$  the Kasteleyn matrices of the drifted and killed dimer models, with rows indexed by  $\underline{\mathbf{W}}$  and columns by  $\underline{\mathbf{B}}$  (associated with the fixed phases  $\zeta$ ). If  $\mathbf{G}$  is finite we denote by  $\mathbb{P}_{\text{dim}}^{\mathbf{d}}$  and  $\mathbb{P}_{\text{dim}}^{\mathbf{k}}$  the associated probability measure on  $\mathcal{M}(\underline{\mathbf{G}}^{\mathbf{D}})$  (see Equation (21)).

**Proposition 4.5.** *The drifted and killed dimer models are gauge equivalent.*

*Proof.* Define a gauge function  $\psi$  on  $\underline{\mathbf{B}}$  coinciding with  $\lambda$  on  $\mathbf{V}$  and  $(\lambda^\star)^{-1}$  on  $\underline{\mathbf{V}}^\star$ . We also define a function  $\phi$  on the white vertices. For  $w$  corresponding to the primal edge  $xy$  (we can assume that  $x \neq o$  upon exchanging  $x$  and  $y$ ) let  $\phi(w) = (c_{xy} \lambda(x) \lambda(y))^{-\frac{1}{2}}$ . It is left to the reader to check that for all  $w \in \underline{\mathbf{W}}$ ,  $b \in \underline{\mathbf{B}}$

$$(K^{\mathbf{k}})_{w,b} = \phi(w) \psi(b) K_{w,b}^{\mathbf{d}}. \quad (27)$$

□

We first show that when  $\lambda$  is massive harmonic on  $\mathbf{V}$ , a consequence of the Doob transform technique is that the partition function of the drifted dimer model is equal to the partition function of the RSF model. It is a direct corollary of Temperley’s bijection (Corollary 4.2) and of Theorem 3.1:

**Corollary 4.6** (Temperley’s bijection at the level of partition functions). *Assume that  $\mathbf{G}$  is finite and  $\lambda$  is massive harmonic for  $\Delta^k$  on  $\mathbf{V}$ . The RST model on  $\mathbf{G}^\circ$  with conductance function  $\tilde{c}$  and the drifted dimer model are in weight-preserving bijection. In particular:*

$$Z_{\text{RSF}}(\mathbf{G}, c, m) = Z_{\text{RST}}(\mathbf{G}^\circ, \tilde{c}) = Z_{\text{dim}}(\underline{\mathbf{G}}^{\text{D}}, \nu^{\text{d}})$$

*Proof.* Temperley’s bijection holds by definition of the drifted dimer model. The first equality is stated in Theorem 3.1. The last one is Corollary 4.2.  $\square$

This result can be seen as a partial extension of Temperley’s bijection since it relates RSF of  $\mathbf{G}$  with dimer configurations on the double graph  $\underline{\mathbf{G}}^{\text{D}}$ . Our equality only holds for partition function, unlike Temperley’s bijection which is a one-to-one bijection. This result has no direct extension for an infinite planar graph  $\mathbf{G}$  since the partition function is not well-defined, but an analogous result for infinite  $\mathbb{Z}^2$ -periodic graphs is given in Section 5.3, relating the free energy (which can be thought of as the infinite equivalent of the partition function) of a  $\mathbb{Z}^2$ -periodic forest model with the free energy of a  $\mathbb{Z}^2$ -periodic dimer model.

When  $\lambda$  is massive harmonic, the killed dimer model is also related to the RSF model via the massive Laplacian  $\Delta^k$ . To state this result, we need a bit of notation. The *hermitian conjugate*  $M^\dagger$  of a matrix  $M$  is the conjugate of the transposed matrix. If a square matrix  $M$  has rows and columns indexed by  $\underline{\mathbf{B}}$ , it is *block diagonal* if all its coefficients outside of the blocks  $\mathbf{V} \times \mathbf{V}$  and  $\underline{\mathbf{V}}^* \times \underline{\mathbf{V}}^*$  are 0. In which case, we write

$$M = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}.$$

A natural *dual conductance function* and *modified dual conductance function*  $c^*$  and  $\tilde{c}^*$  are defined on  $\mathbf{E}^*$ . Each edge  $uv$  of  $\mathbf{G}^*$  intersects a unique edge  $xy$  of  $\mathbf{G}^\circ$ . Upon exchanging  $x$  and  $y$ , we can assume that  $x \in \mathbf{V}$ . The dual conductance function is then  $c_{uv}^* = (c_{xy})^{-1}$  and the dual modified conductance function is then  $\tilde{c}_{uv}^* = (c_{xy}\lambda(x)\lambda(y))^{-1}$ . They are both symmetric. The Laplacian operator on  $\mathbf{G}^*$  associated with the conductance function  $c^*$  is denoted by  $\Delta^*$ . We also define a matrix  $\tilde{\Delta}^*$  with rows and columns indexed by  $\underline{\mathbf{V}}^*$  and coefficients

$$\forall u, v \in \underline{\mathbf{V}}^*, \tilde{\Delta}_{u,v}^* = \begin{cases} -\tilde{c}_{uv}^* \lambda^*(u)^{-1} \lambda^*(v)^{-1} & \text{if } u \stackrel{\mathbf{G}^*}{\sim} v, u \neq v \\ \sum_{v \stackrel{\mathbf{G}^*}{\sim} u, v \neq u} \tilde{c}_{uv}^* \lambda^*(u)^{-2} & \text{if } u = v \\ 0 & \text{otherwise} \end{cases}. \quad (28)$$

We will comment on this definition after the statement of Proposition 4.7.

**Proposition 4.7.** *The matrix  $(K^k)^\dagger K^k$  is block diagonal. Moreover, if  $\lambda$  is massive harmonic for  $\Delta^k$  on  $\mathbf{V}$ ,*

$$(K^k)^\dagger K^k = \begin{pmatrix} \Delta_{\mathbf{V}}^k & 0 \\ 0 & \tilde{\Delta}^* \end{pmatrix}. \quad (29)$$

*When  $\mathbf{G}$  is infinite, reciprocally, if  $\nu : \underline{\mathbf{E}}^{\text{D}} \rightarrow \mathbb{R}_{>0}$  is a weight function and  $K$  denotes the associated Kasteleyn matrix (with the fixed set of phases  $\zeta$ ), if*

$$K^\dagger K = \begin{pmatrix} \Delta_{\mathbf{V}}^k & 0 \\ 0 & * \end{pmatrix}, \quad (30)$$

*there exists  $\lambda : \mathbf{V} \rightarrow \mathbb{R}_{>0}$  massive harmonic for  $\Delta^k$  and  $\lambda^* : \mathbf{V}^* \rightarrow \mathbb{R}_{>0}$  such that  $\nu$  is the weight function of the killed dimer model associated with  $\lambda, \lambda^*$ .*

We first make a few observations on the definitions and statements, then state some consequences, and finally provide a proof of this proposition at the end of the section.

- Recall that when  $\mathbf{G}$  is infinite,  $\Delta_{\mathbf{V}}^k = \Delta^k$ .

- When  $\mathsf{G}$  is finite,  $r \in \mathsf{V}^*$  acts as a cemetery state in the definition of  $\tilde{\Delta}^*$  since for  $u \stackrel{\circ}{\sim} r$ ,  $r$  contributes to the diagonal coefficient  $\tilde{\Delta}_{u,u}^*$  even though there is no extra diagonal coefficient  $\tilde{\Delta}_{u,r}^*$  since the matrix is indexed by  $\underline{\mathsf{V}}^* = \mathsf{V}^* \setminus \{r\}$ .
- The value of  $\lambda^*$  at  $u$  is not used in the construction of the killed dimer model, but it will be useful that  $\lambda^*$  is defined on all  $\mathsf{V}^*$  when we discuss duality.
- The matrix  $\tilde{\Delta}^*$  is not in general the (massive) Laplacian operator associated with a conductance (and mass) function. In the infinite case though, for the choice  $\lambda^* = 1$ ,  $\tilde{\Delta}^*$  is the (non massive) Laplacian operator on  $\mathsf{G}^*$  associated with the symmetric conductance function  $\tilde{c}^*$ . In the finite case, for the choice  $\lambda^* = 1$ ,  $\tilde{\Delta}^*$  is the massive Laplacian associated with the conductance function  $\tilde{c}^*$  restricted to  $\underline{\mathsf{V}}^* = \mathsf{V}^* \setminus \{r\}$ . It is associated with a RST model on  $\mathsf{G}^*$  rooted at  $r$ .

We state some consequences of Proposition 4.7. When  $\mathsf{G}$  is finite this Proposition gives an alternative proof of our Theorem 3.1 for symmetric conductance functions. We explain why.

*Alternative proof of Theorem 3.1 when  $c$  is symmetric.* Denote by  $\mathbb{P}_{\text{RST}}^{\text{d}}$  the probability measure on RST of  $\mathsf{G}^o$  rooted at  $o$  associated with the conductance function  $\tilde{c}$ . By Temperley's bijection Theorem 4.1 and the local statistics formula (23), for all  $\vec{e}_1 = (x_1, y_1), \dots, \vec{e}_k = (x_k, y_k) \in \tilde{\mathsf{E}}^o$  with  $x_1, \dots, x_k \in \mathsf{V}$ , if  $f_1 = x_1 w_1, \dots, f_k = x_k w_k \in \underline{\mathsf{E}}^{\text{D}}$  are the corresponding edges of the double graph,

$$\begin{aligned} \mathbb{P}_{\text{RST}}^{\text{d}}(\vec{e}_1, \dots, \vec{e}_k) &= \mathbb{P}_{\text{dim}}^{\text{d}}(f_1, \dots, f_k) = \left| \prod_{i=1}^k K_{w_i, x_i}^{\text{d}} \det \left( ((K^{\text{d}})^{-1})_{x_i, w_j} \right)_{1 \leq i, j \leq k} \right| \\ &= \left| \prod_{i=1}^k K_{w_i, x_i}^{\text{k}} \det \left( ((K^{\text{k}})^{-1})_{x_i, w_j} \right)_{1 \leq i, j \leq k} \right| \end{aligned} \quad (31)$$

because gauge equivalence preserves the local statistics formula (see Equation (24)) and by Proposition 4.7 the drifted and killed dimer models are gauge equivalent. The inverse Kasteleyn matrix  $(K^{\text{k}})^{-1}$  can be computed using Equation (29): for every  $1 \leq i, j \leq k$ :

$$\begin{aligned} ((K^{\text{k}})^{-1})_{x_i, w_j} &\stackrel{(29)}{=} ((\Delta^{\text{k}})^{-1} (K^{\text{k}})^{\dagger})_{x_i, w_j} \stackrel{(4)}{=} (G^{\text{k}} (K^{\text{k}})^{\dagger})_{x_i, w_j} \\ &= G_{x_i, x_j}^{\text{k}} \overline{K_{w_j, x_j}^{\text{k}}} + G_{x_i, y_j}^{\text{k}} \overline{K_{w_j, y_j}^{\text{k}}} \\ &= \overline{\zeta_{w_j x_j}} \left( \nu_{w_j x_j}^{\text{k}} G_{x_i, x_j}^{\text{k}} - \nu_{w_j y_j}^{\text{k}} G_{x_i, y_j}^{\text{k}} \right) \\ &\stackrel{(26)}{=} \frac{\overline{\zeta_{w_j x_j}} (c_{x_j y_j} \lambda(x_j) \lambda(y_j))^{\frac{1}{2}}}{\lambda(x_i)} \left( \frac{\lambda(x_i) G_{x_i, x_j}^{\text{k}}}{\lambda(x_j)} - \frac{\lambda(x_i) G_{x_i, y_j}^{\text{k}}}{\lambda(y_j)} \right). \end{aligned}$$

Observe that since the conductance function  $c$  is symmetric,  $(\Delta^{\text{k}})^T = \Delta^{\text{k}}$  and hence  $(G^{\text{k}})^T = G^{\text{k}}$ . This also holds for the extension of  $G^{\text{k}}$  to  $\mathsf{V}^o$  since we defined by convention  $G_{x,o}^{\text{k}} = G_{o,y}^{\text{k}} = 0$  for any  $x, y \in \mathsf{V}^o$ . Using this symmetry, we can write

$$\begin{aligned} ((K^{\text{k}})^{-1})_{x_i, w_j} &= \frac{\overline{\zeta_{w_j x_j}} (c_{x_j y_j} \lambda(x_j) \lambda(y_j))^{\frac{1}{2}}}{\lambda(x_i)} \left( \frac{\lambda(x_i) G_{x_i, x_i}^{\text{k}}}{\lambda(x_j)} - \frac{\lambda(x_i) G_{x_i, y_j}^{\text{k}}}{\lambda(y_j)} \right) \\ &= \frac{\overline{\zeta_{w_j x_j}} (c_{x_j y_j} \lambda(x_j) \lambda(y_j))^{\frac{1}{2}}}{\lambda(x_i)} \tilde{H}_{\vec{e}_j, \vec{e}_i}. \end{aligned} \quad (32)$$

Inserting Equation (32) in Equation (31) and using the multilinearity of the determinant gives

$$\begin{aligned} \mathbb{P}_{\text{RST}}^{\text{d}}(\vec{e}_1, \dots, \vec{e}_k) &= \prod_{i=1}^k \nu_{w_i x_i}^{\text{k}} \prod_{j=1}^k (c_{x_j y_j} \lambda(x_j) \lambda(y_j))^{\frac{1}{2}} \prod_{i=1}^k \lambda(x_i)^{-1} \det(\tilde{H}_{\vec{e}_j, \vec{e}_i})_{1 \leq i, j \leq k} \\ &= \prod_{i=1}^k \tilde{c}_{x_i, x_j} \det(\tilde{H}_{\vec{e}_j, \vec{e}_i})_{1 \leq i, j \leq k}. \end{aligned}$$

which is precisely the statement of Theorem 3.1.  $\square$

When  $\mathsf{G}$  is finite, Proposition 4.7 implies the equality of determinants  $(\det K^{\mathsf{k}})^2 = \det \Delta^{\mathsf{k}} \det \tilde{\Delta}^{\star}$ . We can actually prove something stronger:

**Proposition 4.8.** *Assume that  $\mathsf{G}$  is finite and  $\lambda$  is massive harmonic for  $\Delta^{\mathsf{k}}$  on  $\mathsf{V}$ . There exists an explicit constant  $C(\lambda, \lambda^{\star})$  such that*

$$\det K^{\mathsf{k}} = C(\lambda, \lambda^{\star}) \det \Delta_{\mathsf{V}}^{\mathsf{k}} = C(\lambda, \lambda^{\star})^{-1} \det \tilde{\Delta}^{\star}.$$

Observe that by the Kasteleyn formula Equation (22) and the matrix-forest theorem, this proposition gives an equality up to an explicit constant  $C(\lambda, \lambda^{\star})$  between the partition function of the killed dimer model and the partition function of the RSF model associated with  $\Delta_{\mathsf{V}}^{\mathsf{k}}$ . This result is an extension of Theorem 21 of [dT20] in the case of a general finite graph (instead of an isoradial graph), and the proof is the same.

*Proof.* For  $w \in \underline{\mathsf{W}}$ , we denote by  $x_w y_w \in \mathsf{E}^o$  with  $x_w \neq o$  the associated primal edge. Using successively the gauge equivalence between the drifted and killed Kasteleyn matrices obtained in Equation (27), the multilinearity of the determinant, Temperley's bijection between directed trees and drifted dimers, the matrix-forest theorem and finally the gauge equivalence between  $\Delta_{\mathsf{V}}^{\mathsf{k}}$  and  $\tilde{\Delta}_{\mathsf{V}}$ , we obtain that

$$\begin{aligned} \det K^{\mathsf{k}} &\stackrel{(27)}{=} \left\{ \prod_{w \in \underline{\mathsf{W}}} (c_{x_w y_w} \lambda(x_w) \lambda(y_w))^{-\frac{1}{2}} \prod_{x \in \mathsf{V}} \lambda(x) \prod_{u \in \underline{\mathsf{V}}^{\star}} \lambda^{\star}(u) \right\} \det K^{\mathsf{d}} \\ &\stackrel{(22)}{=} C(\lambda, \lambda^{\star}) Z_{\dim}(\underline{\mathsf{G}}^{\mathsf{D}}, \nu^{\mathsf{d}}) \\ &\stackrel{(4.2)}{=} C(\lambda, \lambda^{\star}) Z_{\text{RST}}(\mathsf{G}^o, \tilde{c}) \stackrel{(5)}{=} C(\lambda, \lambda^{\star}) \det \tilde{\Delta}_{\mathsf{V}} \stackrel{(16)}{=} C(\lambda, \lambda^{\star}) \det \Delta_{\mathsf{V}}^{\mathsf{k}}. \quad \square \end{aligned}$$

We now prove Proposition 4.7

*Proof.* We first check that  $(K^{\mathsf{k}})^{\dagger} K^{\mathsf{k}}$  is block diagonal. Let  $x \in \mathsf{V}, u \in \underline{\mathsf{V}}^{\star}$ . If  $u$  is not adjacent to  $x$  the coefficient  $((K^{\mathsf{k}})^{\dagger} K^{\mathsf{k}})_{x,u}$  is zero because  $K^{\mathsf{k}}$  is non zero only on edges of  $\underline{\mathsf{G}}^{\mathsf{D}}$  so for all  $w \in \underline{\mathsf{W}}$  either  $(K^{\mathsf{k}})^{\dagger}_{x,w} = 0$  or  $K^{\mathsf{k}}_{w,x} = 0$ . If  $x$  is a corner of the face  $u$  in  $\underline{\mathsf{G}}^{\mathsf{D}}$ ,  $x$  and  $u$  are the opposite vertices of a face  $xw_1uw_2$  (with vertices in the clockwise order) of  $\underline{\mathsf{G}}^{\mathsf{D}}$  and

$$((K^{\mathsf{k}})^{\dagger} K^{\mathsf{k}})_{x,u} = (K^{\mathsf{k}})^{\dagger}_{x,w_1} K^{\mathsf{k}}_{w_1,u} + (K^{\mathsf{k}})^{\dagger}_{x,w_2} K^{\mathsf{k}}_{w_2,u} = \sum_{i=1,2} \nu_{w_i x} \nu_{w_i u} \overline{\zeta_{w_i x}} \zeta_{w_i u}.$$

On the one hand, by definition of  $\nu^{\mathsf{k}}$ ,

$$\nu_{xw_1}^{\mathsf{k}} \nu_{uw_1}^{\mathsf{k}} = \nu_{xw_2}^{\mathsf{k}} \nu_{uw_2}^{\mathsf{k}} = (\lambda^{\star}(u) \lambda(x))^{-1}.$$

On the other hand, the Kasteleyn property of the phases around the face  $xw_1uw_2$  implies that

$$\overline{\zeta_{w_1 x} \zeta_{w_1 u}} = -\overline{\zeta_{w_2 x} \zeta_{w_2 u}}$$

and hence

$$((K^{\mathsf{k}})^{\dagger} K^{\mathsf{k}})_{x,u} = 0.$$

The same holds for  $((K^{\mathsf{k}})^{\dagger} K^{\mathsf{k}})_{u,x}$  since

$$((K^{\mathsf{k}})^{\dagger} K^{\mathsf{k}})_{u,x} = \overline{((K^{\mathsf{k}})^{\dagger} K^{\mathsf{k}})_{x,u}^{\dagger}} = \overline{((K^{\mathsf{k}})^{\dagger} K^{\mathsf{k}})_{x,u}} = 0.$$

Assume further that  $\lambda$  is massive harmonic for  $\Delta^{\mathsf{k}}$  on  $\mathsf{V}$ . We compute the  $\underline{\mathsf{V}}^{\star} \times \underline{\mathsf{V}}^{\star}$  block of  $(K^{\mathsf{k}})^{\dagger} K^{\mathsf{k}}$ . For  $u, v \in \underline{\mathsf{V}}^{\star}$ ,  $((K^{\mathsf{k}})^{\dagger} K^{\mathsf{k}})_{u,v}$  can be non zero only if  $u \stackrel{\mathsf{G}^{\star}}{\sim} v$  or  $u = v$  (since otherwise

for all  $w \in \underline{W}$  either  $K_{w,u}^k = 0$  or  $K_{w,v}^k = 0$ ). For  $u \stackrel{\star}{\sim} v$  with  $u \neq v$ , if  $w$  is the white vertex of  $\underline{G}^D$  corresponding to the edge  $uv \in \mathbf{E}^\star$

$$((K^k)^\dagger K^k)_{u,v} = (K^k)^\dagger_{u,w} K_{w,v}^k = -\tilde{c}_{uv}^\star (\lambda^\star(u)\lambda^\star(v))^{-1} = \tilde{\Delta}_{u,v}^\star.$$

For  $u \in \underline{V}^\star$ , if we denote by  $u_1, \dots, u_k$  the neighbours of  $u$  in  $\mathbf{G}^\star$  (including  $r$ ) and let  $w_1, \dots, w_k \in \underline{W}$  such that for all  $1 \leq i \leq k$  the directed edge  $(u, u_k)$  of  $\mathbf{G}^\star$  corresponds to the undirected edge  $uw_k$  of  $\underline{G}^D$ ,

$$((K^k)^\dagger K^k)_{u,u} = \sum_{i=1}^k \tilde{c}_{uu_k}^\star \lambda^\star(u)^{-2} = \tilde{\Delta}_{u,u}^\star.$$

We identify the  $\mathbf{V} \times \mathbf{V}$  block of  $(K^k)^\dagger K^k$ . For  $x, y \in \mathbf{V}$ ,  $((K^k)^\dagger K^k)_{x,y}$  can be non zero only if  $x \sim y$  or  $x = y$  (since otherwise for all  $w \in \underline{W}$  either  $K_{w,x}^k = 0$  or  $K_{w,y}^k = 0$ ). For  $x \sim y$  with  $y \neq x$ , if  $w \in \underline{W}$  is the white vertex corresponding to the edge  $xy \in \mathbf{E}$

$$((K^k)^\dagger K^k)_{x,y} = (K^k)^\dagger_{x,w} K_{w,y}^k = -c_{xy} = \Delta_{x,y}^k.$$

For  $x \in \mathbf{V}$ , if we denote by  $y_1, \dots, y_k$  the neighbours of  $x$  in  $\mathbf{G}^\circ$  and let  $w_1, \dots, w_k \in \underline{W}$  such that for all  $1 \leq i \leq k$  the directed edge  $(x, y_k)$  of  $\mathbf{G}^\circ$  corresponds to the undirected edge  $xw_k$  of  $\underline{G}^D$ ,

$$((K^k)^\dagger K^k)_{x,x} = \sum_{i=1}^k c_{xy_i} \frac{\lambda(y_i)}{\lambda(x)}. \quad (33)$$

This is equal to  $\Delta_{x,x}^k$  if and only if  $\lambda$  is massive for  $\Delta^k$  at  $x$ , which concludes the first part of the proof.

We now assume that  $\mathbf{G}$  is infinite so the notation simplify, in particular  $\mathbf{B} = \underline{\mathbf{B}}$ ,  $\mathbf{W} = \underline{\mathbf{W}}$  and  $\underline{\mathbf{V}}^\star = \mathbf{V}^\star$ . We prove the reciprocal statement. Let  $\nu : \bar{\mathbf{E}} \rightarrow \mathbb{R}_{>0}$  be any weight function such that the associated Kasteleyn matrix  $K$  satisfies Equation (30). In particular, for any  $x \in \mathbf{V}$ ,  $u \in \mathbf{V}^\star$ ,  $(K^\dagger K)_{x,u} = 0$ . Let  $xw_1uw_2$  (with vertices in the clockwise order) be any face of  $\mathbf{G}^D$ . Since  $K^\dagger K$  vanishes outside the diagonal blocks,

$$0 = (K^\dagger K)_{x,u} = \overline{K_{w_1,x}} K_{w_1,u} + \overline{K_{w_2,x}} K_{w_2,u}. \quad (34)$$

We define the auxiliary matrix  $L$  with rows and columns indexed by  $\mathbf{W}$  and  $\mathbf{B}$ :

$$\forall w \in \mathbf{W}, b \in \mathbf{B}, L_{w,b} = \begin{cases} \overline{K_{w,x}} & \text{if } b = x \in \mathbf{V} \\ (K_{w,u})^{-1} & \text{if } b = u \in \mathbf{V}^\star. \end{cases} \quad (35)$$

Equation (34) implies that  $K$  has the alternating product property: around each face  $xw_1uw_2$  of  $\mathbf{G}_\delta$  (with vertices in the clockwise order),

$$\frac{L_{w_1,x} L_{w_2,u}}{L_{w_1,u} L_{w_2,x}} = -1. \quad (36)$$

Let  $K^{k,1}$  and  $\nu^{k,1}$  be the Kasteleyn matrix and weight function of the killed dimer model with the particular choice  $\lambda(x) = 1$  for all  $x \in \mathbf{V}$  (which is not massive harmonic) and  $\lambda^\star(u) = 1$  for all  $x \in \mathbf{V}^\star$  and  $L^{k,1}$  the auxiliary matrix associated to  $K^{k,1}$  by Equation (35). For  $w \in \mathbf{W}$  the intersection of the primal and dual edges  $xy$  and  $uw$ ,  $K_{w,x}^{k,1} = \zeta_{w,x} c_{xy}^{1/2}$  and  $K_{w,u}^{k,1} = \zeta_{w,u} c_{xy}^{-1/2}$ . By the first statement of Proposition 4.7 which we already proved,  $K^{k,1}$  vanishes outside the diagonal blocks so  $L^{k,1}$  satisfies the alternating product property Equation (36). It is classical (see for example Lemma 72 in the Appendix of [dT20]) that for two matrices with non-zero coefficients on the edges of a bipartite planar graph, equality of the alternating products around every inner face implies gauge equivalence. Hence  $L$  and  $L^{k,1}$  are gauge equivalent and there exist gauge functions  $\phi : \mathbf{W} \rightarrow \mathbb{C} \setminus \{0\}$  and  $\psi : \mathbf{B} \rightarrow \mathbb{C} \setminus \{0\}$  such that

$$\forall w \in \mathbf{W}, \forall b \in \mathbf{B}, L_{w,b} = \phi(w)\psi(b)L_{w,b}^{k,1}.$$

The gauge functions  $\phi, \psi$  are real and positive since they are written in [dT20] as product and quotient of ratios of the form  $\frac{L_{w,b}}{L_{w,b}^{k,1}} = \left( \frac{\nu_{w,b}}{\nu_{w,b}^{k,1}} \right)^{\pm 1} \in \mathbb{R}_{>0}$ . Let  $\lambda = \psi|_{\mathbb{V}}$ ,  $\lambda^* = \psi|_{\mathbb{V}^*}$  and because the Kasteleyn phases are the same. We can write for all  $w \in \mathbb{W}$ ,  $b \in \mathbb{B}$ ,

$$K_{w,b} = \begin{cases} \overline{L_{w,x}} & = \overline{\lambda(x)\phi(w)L_{w,x}^{k,1}} & = \lambda(x)\phi(w)K_{w,x}^{k,1} & \text{if } b = x \in \mathbb{V} \\ (L_{w,u})^{-1} & = (\lambda^*(u)\phi(w)L_{wu}^{k,1})^{-1} & = (\lambda^*(u)\phi(w))^{-1} K_{wu}^{k,1} & \text{if } b = u \in \mathbb{V}^*. \end{cases} \quad (37)$$

We can now identify the gauge function  $\phi$ . Since Equation (30) holds, for all  $w \in \mathbb{W}$  corresponding to the primal edge  $xy \in \mathbb{E}$ :

$$\Delta_{x,y}^k = -c_{xy} = (K^\dagger K)_{x,y} = \overline{K_{w,x}} K_{w,y} = \lambda(x)\lambda(y)\phi(w)^2 \overline{K_{w,x}^{k,1}} K_{w,y}^{k,1} = -\lambda(x)\lambda(y)\phi(w)^2 c_{xy}.$$

and hence

$$\phi(w)^2 = (\lambda(x)\lambda(y))^{-1}.$$

Combined with Equation (37), it implies that the weight function  $\nu$  coincides with the weight function of the killed dimer model associated with the positive functions  $\lambda, \lambda^*$ . It remains to prove that the function  $\lambda$  is massive harmonic: for  $x \in \mathbb{V}$ , Equation (30) applied at  $x$  implies massive harmonicity of  $\lambda$  at  $x$  by (33).  $\square$

### 4.3 Massive holomorphy and self-duality.

For the clarity of exposition, we assume in this paragraph that  $\mathbb{G}$  is infinite. Similar statements could be given in the finite case, but the self-duality would be less clear since  $\mathbb{G}^o$  with distinguished vertex  $o$  and  $\mathbb{G}^*$  with distinguished vertex  $r$  do not play perfectly dual roles.

Proposition 4.7 can be seen as a generalization of the theory of discrete massive holomorphy on isoradial graphs developed in [Ken02] for the non-massive case and in [MS09], [dT20], [Par22] (among others) for the massive case. Equation (29) implies that the Kasteleyn matrix  $K^k$  of the killed dimer model is a discrete massive Dirac operator associated with the massive Laplacian  $\Delta^k$ . In other words, a function  $f : \mathbb{B} \rightarrow \mathbb{C}$  satisfying  $K^k f = 0$  can be decomposed into a *real part*  $f_0 : \mathbb{V} \rightarrow \mathbb{C}$  harmonic for  $\Delta^k$  and an *imaginary part*  $f_1 : \mathbb{V}^* \rightarrow \mathbb{C}$  harmonic for  $\tilde{\Delta}^*$ .

When  $K^k$  is invertible,  $\tilde{\Delta}^*$  also is and we can define the *dual Green function*  $\tilde{G}^* = (\tilde{\Delta}^*)^{-1}$ . Similarly to the papers in the isoradial setting, the inverse Dirac operator  $(K^k)^{-1}$  can be expressed in terms of “discrete derivatives” of the (massive) Green function and the dual Green function: Equation (29) implies that

$$(K^k)^{-1} = \begin{pmatrix} G^k & 0 \\ 0 & \tilde{G}^* \end{pmatrix} (K^k)^\dagger.$$

This means that for all  $b \in \mathbb{B}$ ,  $w \in \mathbb{W}$ , if we denote by  $x_w, y_w, u_w, v_w$  the neighbours of  $w$  in  $\mathbb{G}^D$  in clockwise order,

$$((K^k)^{-1})_{b,w} = \overline{\zeta_{wx_w}} \begin{cases} \nu_{wx_w}^k G_{x_w, x_w}^k - \nu_{wy_w}^k G_{x_w, y_w}^k & \text{if } b = x \in \mathbb{V} \\ i(\nu_{wu_w}^k \tilde{G}_{u_w, u_w}^* - \nu_{wv_w}^k \tilde{G}_{u_w, v_w}^*) & \text{if } b = u \in \mathbb{V}^* \end{cases}$$

In both papers [Ken02] and [dT20], the (massive) Laplacian and (massive) Dirac operator are “self-dual”: the dual operator  $\tilde{\Delta}^*$  is the (massive) Laplacian on  $\mathbb{G}^*$  associated with the dual conductance function  $c^*$  that we already introduced (and a natural dual mass function  $m^*$  in [dT20]). This leads to the following question: is there a choice of function  $\lambda^* : \mathbb{V} \rightarrow \mathbb{R}_{>0}$  that makes Equation (29) “self-dual” ? Or in other words, does there exist a function  $\lambda^* : \mathbb{V}^* \rightarrow \mathbb{R}_{>0}$  such that  $\tilde{\Delta}^*$  is the massive Laplacian operator on  $\mathbb{G}^*$  associated with the dual conductance function  $c^*$  and a dual mass function  $m^*$  ? More formally, let  $m^* : \mathbb{V} \rightarrow \mathbb{R}_{\geq 0}$  be an arbitrary mass function. Recall that a dual conductance function  $c^*$  is naturally associated to  $c$ : if  $xy \in \mathbb{E}$  is the unique primal edge intersecting the dual edge  $uv \in \mathbb{E}^*$ ,  $c_{uv}^* = c_{xy}^{-1}$ . Denote by  $(\Delta^*)^k$  the Laplacian on  $\mathbb{G}^*$  associated with the conductance and mass functions  $(c^*, m^*)$ .



**Definition 4.9.** The killed dimer model associated with the positive functions  $\lambda, \lambda^*$  is *self-dual* for the conductance function  $c$  and mass functions  $(m, m^*)$  if and only if

$$(K^k)^\dagger K^k = \begin{pmatrix} \Delta^k & 0 \\ 0 & (\Delta^*)^k \end{pmatrix}.$$

**Proposition 4.10.** *If  $\lambda$  is massive harmonic for  $\Delta^k$  on  $\mathbb{V}$ ,  $\lambda^*$  is massive harmonic for  $(\Delta^*)^k$  on  $\mathbb{V}^*$  and for all intersecting primal and dual edges  $xy \in \mathbb{E}$ ,  $uv \in \mathbb{E}^*$ ,*

$$\lambda(x)\lambda(y)\lambda^*(u)\lambda^*(v) = 1. \quad (38)$$

*the killed dimer model associated with the positive functions  $\lambda, \lambda^*$  is self dual for the conductance function  $c$  and mass functions  $(m, m^*)$*

*Remark 4.11.* The hypothesis of this proposition are satisfied in the isoradial settings of [Ken02] and [dT20] where  $\lambda$  and  $\lambda^*$  are the restriction to the primal and dual graph of the discrete (massive) exponential function.

*Proof.* Comparing the extra-diagonal coefficients of the Laplacian  $\tilde{\Delta}^*$  (see Equation (28)) with the extra diagonal coefficients of  $(\Delta^*)^k$ , shows that when Equation (38) holds, for all  $u \neq v \in \mathbb{V}^*$  such that  $u \stackrel{\mathbb{E}^*}{\sim} v$ , if  $xy \in \mathbb{E}$  is the unique primal edge intersecting  $uv$

$$\tilde{\Delta}_{u,v}^* = \tilde{c}^*(\lambda^*(u)\lambda^*(v))^{-1} = (c_{xy}\lambda(x)\lambda(y)\lambda^*(u)\lambda^*(v))^{-1} = (\Delta^*)_{u,v}^k.$$

Moreover, for all  $u \in \mathbb{V}^*$ , by massive harmonicity of  $\lambda^*$  at  $u$  for  $(\Delta^*)^k$ .

$$\tilde{\Delta}_{u,u}^* = \sum_{v \stackrel{\mathbb{E}^*}{\sim} u, v \neq u} \tilde{c}_{(u,v)}^* \lambda^*(u)^{-2} \stackrel{(38)}{=} \sum_{v \stackrel{\mathbb{E}^*}{\sim} u, v \neq u} c_{uv}^* \frac{\lambda^*(v)}{\lambda^*(u)} = m^*(u) + \sum_{v \stackrel{\mathbb{E}^*}{\sim} u, v \in \mathbb{V}^*} c_{uv}^* = (\Delta^*)_{u,u}^k. \quad \square$$

## 5 Application: rooted spanning forests of $\mathbb{Z}^2$ -periodic graphs

In this paragraph we study random RSF (and more generally, cycle rooted spanning forests) of  $\mathbb{Z}^2$ -periodic graphs. This study was initiated independently in [Ken19] (for the massive undirected case) and [Sun16] (for the non-massive directed case). We first give some preliminaries and definitions, then explain how to find massive harmonic functions on such graphs, and finally we give applications of the Doob transform technique.

### 5.1 Definitions and first results.

We recall the setting and some results of [Ken19]. In this section, we always assume that  $\mathbb{G}$  is infinite.

**$\mathbb{Z}^d$ -periodic graphs and Laplacians.** The graph  $\mathbb{G}$  is  $\mathbb{Z}^d$ -periodic if it can be embedded in  $\mathbb{R}^d$  in such a way that it is invariant under translations by the canonical basis  $(e_1, \dots, e_d)$  of  $\mathbb{R}^d$  with finite quotient  $\mathbb{G}_0 = (\mathbb{V}_0, \mathbb{E}_0) = \mathbb{G}/\mathbb{Z}^d$ . The graph  $\mathbb{G}_0$  is naturally embedded on the  $d$ -dimensional torus. We choose an embedding, and define the *fundamental domain*  $\mathbb{V}_0 = \mathbb{V} \cap [0, 1]^d$ : it is identified with the vertices of  $\mathbb{G}_0$ . The vertices of  $\mathbb{G}$  can be written uniquely under the form  $(x_0, i)$  with  $x_0 \in \mathbb{V}_0$  and  $i = (i_1, \dots, i_d) \in \mathbb{Z}^d$ , and for all  $x, y \in \mathbb{V}$ ,  $x \sim y$  if and only if  $\forall i \in \mathbb{Z}^d$ ,  $x + i \sim y + i$ . The conductance and mass functions  $(c, m)$  are  $\mathbb{Z}^d$ -periodic if for all  $x, y \in \mathbb{V}$  and  $i \in \mathbb{Z}^d$ ,  $c_{(x+i, y+i)} = c_{(x, y)}$  and  $m(x+i) = m(x)$ . The most simple example  $\mathbb{G} = \mathbb{Z}^d$ ,  $c$  and  $m$  are constant corresponds to the simple random walk in  $\mathbb{Z}^d$  killed with constant rate  $m$ : in this case,  $\mathbb{V}_0$  is reduced to a single point. In the rest of this section, we always assume that  $\mathbb{G}$ ,  $c$  and  $m$  are  $\mathbb{Z}^d$ -periodic. For  $z = (z_1, \dots, z_d) \in (\mathbb{C} \setminus \{0\})^d$  and  $i \in \mathbb{Z}^d$ , we write

$$z^i = \prod_{k=1}^d z_k^{i_k}.$$

As in Section 4 of [Ken19], for any  $z \in (\mathbb{C} \setminus \{0\})^d$ , a function  $f : \mathbb{V} \rightarrow \mathbb{C}$  is  $z$ -periodic if it satisfies

$$\forall x \in \mathbb{V}, \forall i \in \mathbb{Z}^d, f(x+i) = f(x)z^i.$$

In this case we write  $f \in \Omega_0(z)$ . We denote by  $\Delta^k(z) : \Omega_0(z) \rightarrow \Omega_0(z)$  the restriction of the massive Laplacian operator  $\Delta^k$  to  $\Omega_0(z)$ : this is well-defined since  $c$  and  $m$  are  $\mathbb{Z}^d$ -periodic. The canonical base of  $\Omega_0(z)$  is the set of functions  $(f_{x_0})_{x_0 \in \mathbb{V}_0}$  where for all  $x_0 \in \mathbb{V}_0$ ,  $f_{x_0} : \mathbb{V} \rightarrow \mathbb{C}$  is defined by

$$\forall y_0 \in \mathbb{V}_0, \forall i \in \mathbb{Z}^d, f_{x_0}(y_0+i) = \mathbb{1}_{\{x_0=y_0\}}z^i.$$

In this base,  $\Delta^k(z)$  is represented by a square matrix  $\Delta^k(z)$  with rows and columns indexed by  $\mathbb{V}_0$ : for all  $x_0, y_0 \in \mathbb{V}_0$ ,

$$\Delta^k(z)_{x_0, y_0} = \begin{cases} -\sum_{(y_0, j) \sim (x_0, 0)} c_{((x_0, 0), (y_0, j))} z^j & \text{if } x_0 y_0 \in \mathbf{E}_0, x_0 \neq y_0, \\ m(x_0) + \sum_{(w_0, k) \sim (x_0, 0)} c_{((x_0, 0), (w_0, k))} - \sum_{(x_0, j) \sim (x_0, 0)} c_{((x_0, 0), (x_0, j))} z^j & \text{if } x_0 = y_0, \\ 0 & \text{otherwise.} \end{cases}$$

(The underlying notion is that of Laplacian on a line bundle, as in [Ken19], but we will not need to introduce it in full generality here so we restrain to elementary definitions). The *characteristic polynomial* of  $\Delta^k$  is the (Laurent) polynomial of  $d$  complex variables

$$P^k(z) = \det \Delta^k(z).$$

We denote by  $N^k$  its *Newton polygon*: it is the convex hull in  $\mathbb{R}^d$  of the set of points  $i \in \mathbb{Z}^d$  such that the monomial  $z^i$  has a non-zero coefficient in  $P^k$ .

## 5.2 Massive harmonic functions on $\mathbb{Z}^d$ -periodic graphs

Consider a  $\mathbb{Z}^d$ -periodic graph  $\mathbf{G}$  with  $\mathbb{Z}^d$ -periodic mass and conductance functions  $(c, m)$ .

**Proposition 5.1.** *There exists  $z \in (\mathbb{R}_{>0})^d$  and a  $z$ -periodic positive function  $\lambda : \mathbb{V} \rightarrow \mathbb{R}_{>0}$  which is massive harmonic, i.e.  $(\Delta^k \lambda)(x) = 0$  for all  $x \in \mathbb{V}$ . It can be found explicitly by solving a finite system of linear equations.*

*Proof.* This argument was shown to us by Cédric Boutillier when we realized that we were working on related problems (see a forthcoming paper of Ballu, Boutillier, Mkrtchyan and Raschel). The first step is to observe that finding a periodic massive harmonic function is not a hard problem in principle: for fixed  $z \in (\mathbb{R}_{>0})^d$ , finding a  $z$ -periodic positive massive harmonic function is equivalent to finding a vector with positive coordinates in the kernel of the finite matrix  $\Delta^k(z)$ , i.e. solving a finite linear system.

If  $m = 0$  any positive constant is a  $(1, \dots, 1)$ -periodic function so we can assume that  $m \neq 0$ . Recall that  $D(c^k)$  denote the diagonal matrix on  $\mathbb{V}_0$  with coefficients  $c^k(x_0)$ . If we define, similarly to Equation (7),  $Q^k(z) = I - D(c^k)^{-1} \Delta^k(z)$ ,  $Q(z)$  is the operator on  $\Omega_0(z)$  satisfying for all  $f \in \Omega_0(z)$ , for all  $x \in \mathbb{V}$ ,

$$(Q^k(z)f)(x) = \sum_{y \sim x} \frac{c(x, y)}{c^k(x)} f(y).$$

It is also represented by a square matrix with rows and columns indexed by the canonical base of  $\Omega_0(z)$ :

$$\forall x_0, y_0 \in \mathbb{V}_0, Q^k(z)_{x_0, y_0} = \frac{1}{c^k(x_0, 0)} \sum_{(y_0, j) \overset{c}{\sim} (x_0, 0)} c_{((x_0, 0), (y_0, j))} z^j.$$

Since  $D(c^k)$  is diagonal with positive coefficients,

$$(\mathbb{R}_{>0})^{\mathbb{V}_0} \cap \ker(\Delta^k(z)) = (\mathbb{R}_{>0})^{\mathbb{V}_0} \cap \ker(I - Q^k(z)).$$

Since  $\mathbf{G}$  is connected, the matrix  $Q^k(z)$  is irreducible for all  $z \in (\mathbb{R}_{>0})^d$  (i.e. for all  $x_0, y_0 \in \mathbf{V}_0$ , there exists  $n > 0$  such that  $(Q^k)_{x_0, y_0}^n > 0$ ). Observe that  $Q^k := Q^k(1, \dots, 1)$  is the transition kernel of the KRW on the  $d$ -dimensional torus  $\mathbf{G}_0$  associated to  $\Delta^k$  since the space of  $(1, \dots, 1)$ -periodic function on  $\mathbf{V}$  is exactly the space of functions on the torus. It is a sub-Markovian matrix:

$$\forall x_0 \in \mathbf{V}_0, \quad \sum_{y_0 \stackrel{\mathbf{G}_0}{\sim} x_0} Q^k(x_0, y_0) = \sum_{y \stackrel{\mathbf{G}}{\sim} (x_0, 0)} \frac{c_{((x_0, 0), y)}}{c^k(x_0)} = 1 - \frac{m(x_0)}{c^k(x_0)} \leq 1. \quad (39)$$

Coefficients of  $Q^k$  are non-negative and since  $m \neq 0$ , one of the sums (39) is strictly smaller than 1 so by the Perron-Frobenius theorem the maximal eigenvalue  $\beta$  of  $Q^k$  satisfies  $\beta < 1$ , which implies  $\ker(I - Q^k) = \{0\}$ . If (for example) we fix  $z_2, \dots, z_d = 1$  and let  $z_1 \in \mathbb{R}^+ \rightarrow \infty$ ,  $Q^k(z_1, 1, \dots, 1)$  remains irreducible with non-negative coefficients, and the Perron-Frobenius theorem still applies and yields a maximal eigenvalue  $\beta(z_1, 1, \dots, 1)$  and an associated eigenvector  $\lambda(z_1, 1, \dots, 1)$  with positive coefficients. Assume that

$$\rho(z_1, 1, \dots, 1) \xrightarrow{z_1 \rightarrow \infty} \infty \quad (40)$$

(this will be proved just after). By continuity of the Perron-Frobenius eigenvalue there exists  $z_1 > 1$  such that  $\beta(z_1, 1, \dots, 1) = 1$  and hence  $\lambda(z_1, 1, \dots, 1) \in \ker(I - Q^k(z_1, 1, \dots, 1)) \cap (\mathbb{R}_{>0})^{\mathbf{V}_0}$  is a positive  $(z_1, 1, \dots, 1)$ -periodic massive harmonic function.

We now prove Equation (40), which concludes the proof. Let  $x_0 \in \mathbf{V}_0$ . Since  $\mathbf{G}$  is connected we can find a path  $(x(0), x(1), \dots, x(n))$  with  $x(0) = (x_0, (0, \dots, 0))$  and  $x(n) = (x_0, (1, 0, \dots, 0)) = x(0) + (1, 0, \dots, 0)$  such that  $c_{(x(i), x(i+1))} > 0$  for all  $0 \leq i \leq n-1$ . Hence

$$(Q^k(z_1, 1, \dots, 1))_{x_0, x_0}^n \geq z_1 \prod_{i=0}^{n-1} \frac{c_{(x(i), x(i+1))}}{c^k(x(i))}.$$

Hence for any  $N \in \mathbb{N}$ ,

$$(Q^k(z_1, 1, \dots, 1))_{x_0, x_0}^{Nn} \geq \left( z_1 \prod_{i=0}^{n-1} \frac{c_{(x(i), x(i+1))}}{c^k(x(i))} \right)^N,$$

so the Perron-Frobenius eigenvalue of  $Q^k(z_1, 1, \dots, 1)$  is bounded from below by  $\left( z_1 \prod_{i=0}^{n-1} \frac{c_{(x(i), x(i+1))}}{c^k(x(i))} \right)^{\frac{1}{n}}$ . Since this quantity goes to  $\infty$  when  $z_1 \rightarrow \infty$ , it proves Equation (40).  $\square$

### 5.3 Temperley's bijection for the free energy of $\mathbb{Z}^2$ -periodic models

In this section, we assume that  $d = 2$  and  $c$  is symmetric. The massive Laplacian  $\Delta^k$  that we introduced is related to a RSF model on  $\mathbf{G}$ . This model is obtained by taking the weak limit of Boltzmann measures on *cycle-rooted spanning forests* (CRSF) along the toroidal exhaustion  $\mathbf{G}_n = \mathbf{G}/n\mathbb{Z}^2$  (see [Ken19] for details). The characteristic polynomial  $P^k$  is an important tool to study the RSF model on the graph  $\mathbf{G}$ : it is linked to the Ronkin function and the surface tension of the RSF model. We show that the characteristic polynomial of the massive Laplacian  $\Delta^k$  can be related to the characteristic polynomial of a non-massive directed Laplacian.

Let  $z_0 \in (\mathbb{R}_{>0})^2$  and  $\lambda$  a  $z_0$ -periodic positive massive harmonic function as in Proposition 5.2. Let  $\tilde{S}$  be the Doob transform of  $S^k$  by  $\lambda$  on  $\mathbf{V}$ ,  $\tilde{\Delta}$  the associated Laplacian with conductance function  $\tilde{c}$ : recall that we simply have for all  $x \sim y \in \mathbf{V}$ ,  $\tilde{c}_{(x, y)} = \frac{\lambda(y)}{\lambda(x)} c_{xy}$ . These conductances are also periodic:

$$\forall x \sim y \in \mathbf{V}, \quad \forall i \in \mathbb{Z}^d, \quad \tilde{c}_{(x+i, y+i)} = \frac{\lambda(y+i)}{\lambda(x+i)} c_{(x+i)(y+i)} = \frac{\lambda(y)z_0^i}{\lambda(x)z_0^i} c_{xy} = \tilde{c}_{(x, y)}.$$

Denote by  $P^k$  and  $\tilde{P}$  the characteristic polynomials of  $\Delta^k$  and  $\tilde{\Delta}$ . The Doob transform technique implies

**Proposition 5.2.**

$$\forall z \in \mathbb{C}^2, \tilde{P}\left(\frac{z}{z_0}\right) = P^k(z).$$

*Proof.* Denote by  $\Lambda$  the diagonal matrix with rows and columns indexed by  $\mathbf{V}$  with entries  $\lambda$  on the diagonal. Observe that for all  $z \in (\mathbb{C} \setminus \{0\})^2$ , if  $f \in \Omega_0(z)$ ,  $\Lambda f \in \Omega_0(z z_0)$ :

$$\forall x \in \mathbf{V}, \forall i \in \mathbb{Z}^2, (\Lambda f)(x+i) = \lambda(x+i)f(x+i) = (z z_0)^i (\Lambda f)(x).$$

Hence  $\Lambda : \Omega_0(z) \rightarrow \Omega_0(z z_0)$ . By Proposition 2.17,  $\tilde{\Delta} = \Lambda^{-1} \Delta^k \Lambda$ , so when restricting to  $\Omega_0(z/z_0)$ , we obtain

$$\tilde{\Delta}\left(\frac{z}{z_0}\right) = \Lambda^{-1} \Delta^k(z) \Lambda.$$

Taking determinants concludes the proof.  $\square$

The non-massive Laplacian  $\tilde{\Delta}$  can be related to the drifted dimer model associated with the positive function  $\lambda$  that we introduced in the preceding section (see Definition 4.3) by using Temperley's bijection: this is a result of [Sun16]. Denote by  $P_{\text{dim}}^{\text{d}}$  the characteristic polynomial of the drifted dimer model (see [Sun16] for the definition):

**Proposition 5.3** (Proposition 3.1 of [Sun16]). *For all  $(z, w) \in \mathbb{C}^2$ ,  $\tilde{P}(z, w) = \det(\tilde{\Delta}(z, w)) = P_{\text{dim}}^{\text{d}}(z, w)$ .*

Combining this proposition with our Proposition 5.2 gives:

**Proposition 5.4.** *For all  $(z, w) \in \mathbb{C}^2$ ,*

$$P^k(z, w) = P_{\text{dim}}^{\text{d}}\left(\frac{z}{z_0}, \frac{w}{w_0}\right).$$

This result can be seen as an extension of Temperley's bijection for infinite periodic RSF models, as it relates the characteristic polynomial of a RSF model with the characteristic polynomial of a dimer model. Observe that since the free energy of the models can be computed from the characteristic polynomial, this result can be seen as an infinite analog of Temperley's bijection at the level of partition functions. Proposition 5.4 also sheds a new light on a result of Kenyon: Theorem 1.4. of [Ken19] states among other things that

**Theorem 5.5.** *The spectral curve  $\{(z, w) \in \mathbb{C}^2 \mid P^k(z, w) = 0\}$  associated with the RSF model is a simple Harnack curve.*

This could come as a surprise since the RSF model is not known to be in bijection with a dimer model while it is known by [KO06] that every Harnack curve arises as the spectral curve of a dimer model (and conversely by [KOS03] the spectral curve of a periodic dimer model is always a Harnack curve). Proposition 5.4 together with the result of [KOS03] give an alternative proof of Theorem 5.5 which fills a gap by introducing a dimer model corresponding to the Harnack curve (actually: a full family of dimer models, corresponding to all the positive massive harmonic  $z$ -periodic functions  $\lambda$ ).

## 6 Application to near-critical dimers

In this section, we study a near-critical dimer and RST model naturally related to a near-critical KRW by the Doob transform technique and show convergence of the associated Temperleyan tree, extending the results of [HSB22] and answering in particular (iv) of their open questions section 1.7. We introduce the model that generalizes their model to the case of isoradial graphs. We follow closely their proof to obtain analogous results and adapt the arguments when needed.

## 6.1 General setting: near-critical drifted trees and dimers on isoradial graphs

For the conventions, we follow as much as possible [BdTR17] and [CS11] for isoradial graphs and the massive Laplacian, and [HSB22] for dimers and trees.

### 6.1.1 Isoradial graphs, $Z$ -invariant Laplacian and discrete exponential functions.

**Isoradial lattice.** For each  $\delta > 0$ , consider an infinite *isoradial grid*  $G_\delta^\infty = (V_\delta^\infty, E_\delta^\infty)$  embedded in  $\mathbb{C}$  with *mesh*  $\delta$ , that is an infinite connected planar graph such that all faces are circumscribed by circles of the same radius  $\delta$  in such a way that the center of the circles are in the interior of the faces. We use the same local parameters as in [BdTR17], see Figure 6: for all  $x \sim y \in V_\delta^\infty$ ,  $x$  and  $y$  are the opposite vertices of a lozenge of side length  $\delta$ , and the two (vector) sides of the lozenge starting from  $x$  are denoted by  $\delta e^{i\bar{\alpha}_{xy}}$  and  $\delta e^{i\bar{\beta}_{xy}}$  with  $\bar{\alpha}_{xy}, \bar{\beta}_{xy} \in (-\pi, \pi]$  and  $\bar{\beta}_{xy} - \bar{\alpha}_{xy} \in [0, \pi)$ . We denote by  $\bar{\theta}_j$  the *half angle*  $\bar{\theta}_{xy} = (\bar{\beta}_{xy} - \bar{\alpha}_{xy})/2$ . Note that  $y - x = \delta e^{i\bar{\alpha}_{xy}} + \delta e^{i\bar{\beta}_{xy}}$ .

We always make the so-called *bounded angle assumption*: there exists an absolute constant  $\varepsilon \in (0, \pi/2)$  independent of  $\delta$  such that all half-angles  $\bar{\theta}_{xy}$  belong to  $[\varepsilon, \pi/2 - \varepsilon]$ .

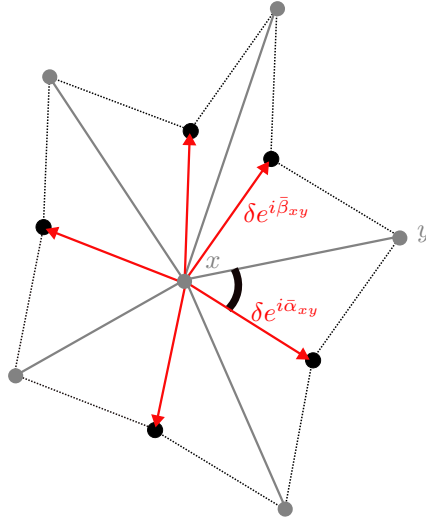


Figure 6: A point  $x \in V_\delta^\infty$  with its neighbours in grey. All red arrows and dotted lines have length  $\delta$ . The vertices of the dual graph are represented in black: they are the centers of the circumscribing circles (not drawn here).

**Critical Laplacian.** The critical Laplacian  $\Delta_\delta$  on  $G_\delta^\infty$ , first introduced in [Ken02], is defined by

$$\forall f : V_\delta^\infty \rightarrow \mathbb{C}, \forall x \in V_\delta^\infty, (\Delta_\delta f)(x) = \sum_{y \sim x} \tan(\bar{\theta}_{xy})(f(x) - f(y)). \quad (41)$$

Note that this is the opposite of the Laplacian of [CS11]. Define

$$\forall x \in V_\delta^\infty, \forall \delta > 0, T_\delta(x) := \frac{\sum_{y \sim x} \sin(2\bar{\theta}_{xy})}{\sum_{y \sim x} \tan(\bar{\theta}_{xy})} ; \quad T := \inf_{\delta > 0, x \in V_\delta^\infty} T_\delta(x) > 0.$$

The latter is positive by the bounded angle assumption. Unlike [CS11], we choose not to scale by  $\delta^2$ , which is more natural in our setting. We denote by  $X_\delta$  the RW associated with  $\Delta_\delta$ , and

by  $\xi_\delta(x)$  the law of its increments: conditionally on  $(X_\delta)_n = x$ , it is the law of  $(X_\delta)_{n+1} - (X_\delta)_n$ . Equation (1.2) of [CS11] writes, for all  $x \in \mathbf{V}_\delta^\infty$ ,

$$\mathbb{E}(\xi_\delta(x)) = 0 \quad ; \quad \mathbb{E}(\Re(\xi_\delta(x))^2) = \mathbb{E}(\Im(\xi_\delta(x))^2) = \delta^2 T_\delta(x) \quad ; \quad \mathbb{E}(\Re(\xi_\delta(x))\Im(\xi_\delta(x))) = 0. \quad (42)$$

As mentioned in [CS11], this implies that a certain time-reparametrization of  $X_\delta$  converges towards brownian motion.

**Z-invariant Laplacian.** Let  $k \in [0, 1[$  be a parameter called the *elliptic modulus*,  $k' = \sqrt{1 - k^2}$  be the *complementary elliptic modulus*,  $K = K(k) = \int_0^{\pi/2} (1 - k^2 \sin^2(t))^{-1/2} dt$  and  $E = E(k) = \int_0^{\pi/2} (1 - k^2 \sin^2(t))^{1/2} dt$  be the *complete elliptic integral of the first and second kind*. Denote by  $K' = K(k')$  and  $E' = E(k')$  their *complementary*. For fixed  $k \in [0, 1[$  (and any angle  $\bar{\theta} \in \mathbb{R}$ ), we denote by  $\theta = 2K\bar{\theta}/\pi$  the associated *abstract angle*. To an elliptic modulus  $k$  is associated a massive Laplacian  $\Delta_\delta^k$  (and hence a KRW  $S_\delta^k$ ) by [BdTR17]:

$$\forall f : \mathbf{V}_\delta^\infty \rightarrow \mathbb{C}, \quad \forall x \in \mathbf{V}_\delta^\infty, \quad \Delta_\delta^k f(x) = m^2(x|k)f(x) + \sum_{y \sim x} \text{sc}(\theta_{xy}|k)(f(x) - f(y))$$

where the conductance function  $c_{xy}^\infty := \text{sc}(\theta_{xy}|k)$  is the Jacobian elliptic function defined in (22.2.9) of [DLMF] and the mass function is

$$m^\infty(x) := m^2(x|k) = \sum_{y \sim x} \left[ \frac{1}{k'} \left( \int_0^{\theta_{xy}} \text{dc}^2(v|k) dv + \frac{E - K}{K} \theta_{xy} \right) - \text{sc}(\theta_{xy}|k) \right],$$

with  $\text{dc}$  the Jacobian elliptic function defined in (22.2.8) of [DLMF]. We will not need this explicit expression of the mass and provide it only for completeness. The positivity of the mass function  $m^2(x|k) > 0$  for all  $x \in \mathbf{V}_\delta^\infty$  is established in Proposition 6 of [BdTR17].

At  $k = 0$ , the massive Laplacian  $\Delta_\delta^k$  coincides with the critical Laplacian  $\Delta_\delta$  of Equation (41):  $m(\cdot|0) = 0$  and  $\text{sc}(\cdot|0) = \tan(\cdot)$ .

*Remark 6.1.* The notation might be a bit confusing since the  $k$  of the elliptic modulus is not the same as the superscript  $k$  of  $\Delta^k$  which means killing. Since  $k = 0$  corresponds to  $\Delta^k$  being a non-massive Laplacian, we find it rather convenient.

**The discrete massive exponential functions.** A very useful property of the massive Laplacian is the existence of an explicit family of multiplicative local massive harmonic functions: we recall its definition and summarize some of its properties from [BdTR17].

**Proposition 6.2.** *Let  $k \in [0, 1[$ ,  $\bar{u} \in \mathbb{C}$ . The discrete massive exponential function with parameter  $\bar{u}$  is defined for all  $x, y \in \mathbf{V}_\delta^\infty$  and any path  $x = x_0, \dots, x_n = y$  in  $\mathbf{G}_\delta^\infty$  from  $x$  to  $y$  by*

$$\mathbf{e}_{(x_j, x_{j+1})}(u|k) := -k' \text{sc}\left(\frac{u - \alpha_{x_j x_{j+1}}}{2} | k\right) \text{sc}\left(\frac{u - \beta_{x_j x_{j+1}}}{2} | k\right), \quad \mathbf{e}_{(x, y)}(u|k) := \prod_{j=0}^{n-1} \mathbf{e}_{(x_j, x_{j+1})}(u|k) \quad (43)$$

*It is massive harmonic in both variables: for any fixed  $x \in \mathbf{V}_\delta^\infty$ ,*

$$\Delta_\delta^k \mathbf{e}_{(\cdot, x)}(u|k) = \Delta_\delta^k \mathbf{e}_{(x, \cdot)}(u|k) = 0.$$

*Moreover, when  $\bar{u} \in \mathbb{R}$ , the discrete massive exponential function evaluated at  $u - 2K - 2iK'$  is positive:*

$$\forall \bar{u} \in \mathbb{R}, \quad \forall x, y \in \mathbf{V}_\delta^\infty, \quad \mathbf{e}_{(x, y)}(u - 2K - 2iK' | k) \in \mathbb{R}_{>0}.$$

Equation (43) is the definition (see Definition 3.3 of [BdTR17]): the fact that the definition does not depend of the path is their Lemma 9. Observe that they define  $\mathbf{e}$  and prove that it is multiplicative on the full diamond graph (i.e. the union of  $\mathbf{G}_\delta^\infty$  and its dual), but we will not need it here so we restrict the definition to  $\mathbf{V}_\delta^\infty$ . The massive harmonicity is their Proposition 11. The positivity is known but we could not find a reference so we provide a short proof:

*Proof.* Fix  $\bar{u} \in \mathbb{R}$ . It is enough to prove that for all  $x \sim y \in \mathbb{V}_\delta^\infty$ ,  $e_{(x,y)}(u - 2K - 2iK'|k) \in \mathbb{R}_{>0}$ . By the change of variable formula for elliptic functions Equations (2.2.11)-(2.2.13) and (2.2.17)-(2.2.19) of [Law89] we can write

$$ik' \operatorname{sc}\left(\frac{u - 2K - 2iK' - \alpha_{xy}}{2} \middle| k\right) = \operatorname{dn}\left(\frac{u - \alpha_{xy}}{2} \middle| k\right) \quad (44)$$

where  $\operatorname{dn}$  is the Jacobian elliptic function defined in (22.2.6) of [DLMF]. Since the same holds for  $\beta_{xy}$ ,

$$e_{(x,y)}(u - 2K - 2iK'|k) = \frac{1}{k'} \operatorname{dn}\left(\frac{u - \alpha_{xy}}{2} \middle| k\right) \operatorname{dn}\left(\frac{u - \beta_{xy}}{2} \middle| k\right).$$

This concludes the proof since  $\operatorname{dn}$  is positive on  $\mathbb{R}$ . Indeed, on the explicit expression (22.2.6) of  $\operatorname{dn}$  in [DLMF], we see that  $\operatorname{dn}(0) = 1$  and that  $\operatorname{dn}$  is the quotient of two real continuous functions which do not vanish on the real line (by (20.2.(iv)) of [DLMF]).  $\square$

### 6.1.2 Isoradial approximation of an open set.

For the rest of the article, we fix a simply connected open set  $\Omega$  with smooth boundary. We say that a sequence of subgraphs  $\mathbb{G}_\delta$  of  $\mathbb{G}_\delta^\infty$  induced by finite subsets of vertices  $\mathbb{V}_\delta \subset \mathbb{V}_\delta^\infty$  approximates  $\Omega$  if the domains  $\hat{\mathbb{G}}_\delta$  obtained by gluing together all lozenges with two opposite vertices in  $\mathbb{V}_\delta$  converge towards  $\Omega$  in the Caratheodory topology. In other words, every  $x \in \Omega$  belongs to  $\hat{\mathbb{G}}_\delta$  for  $\delta$  small enough, and every boundary point  $a \in \partial\Omega$  can be approximated by a sequence  $a_\delta \in \partial\mathbb{V}_\delta$  (see [HSB22], Section 3.1).

We can construct an approximating sequence  $\mathbb{G}_\delta$  by keeping the largest connected component of the subgraph induced by  $\mathbb{V}_\delta^\infty \cap \Omega$  (see also Section 1.1 of [Par22]). For the rest of the article, we consider a fixed approximating sequence  $(\mathbb{G}_\delta)_{\delta>0}$  of  $\Omega$ . Recall that the restriction  $(\Delta_\delta^k)_{\mathbb{V}_\delta}$  is the

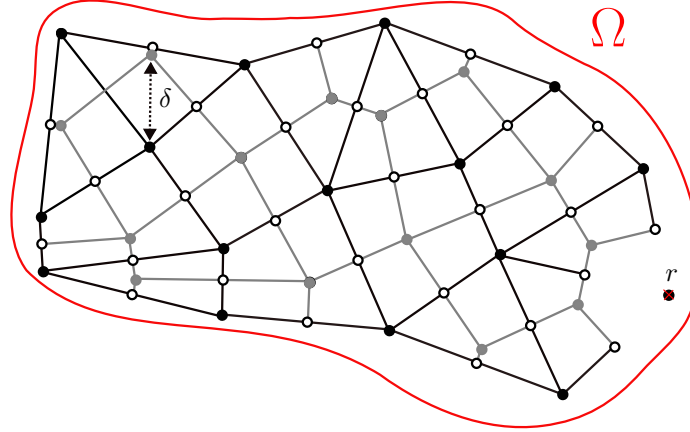


Figure 7: An isoradial approximation  $\mathbb{G}_\delta$  of  $\Omega$  in grey, and the associated Temperleyan graph. The dual  $\mathbb{G}_\delta^*$  with a vertex  $r$  removed is drawn in black. The additional white vertices of  $\mathbb{G}_\delta^D$  are in white.

Laplacian on  $\mathbb{G}_\delta$  associated with the conductance and mass functions  $(c, m)$  obtained by restricting  $(c^\infty, m^\infty)$  to  $\mathbb{G}_\delta$  with wired boundary conditions. In other words, for all  $xy \in \mathbb{E}_\delta$ ,  $c_{xy} = \operatorname{sc}(\theta_{xy}|k)$  and the mass function

$$\forall x \in \mathbb{V}_\delta, m(x) = m^2(x|k) + \sum_{\substack{y \stackrel{\mathbb{G}_\delta^\infty}{\sim} x, \\ y \notin \mathbb{V}_\delta}} \operatorname{sc}(\theta_{xy}|k)$$

differs from  $m^2(x|k)$  only on the boundary.

### 6.1.3 Application of the Doob transform technique and introduction of the model.

**Doob transform.** For the rest of the article, fix  $\bar{u} \in \mathbb{R}$  which we call the *drift* parameter (for reasons that will become clear later). Recall that the complete integral of the first kind  $K$  and its complementary  $K'$  are real constants depending only on  $k$ . For all  $\delta > 0$ , fix  $x_0 \in \mathbb{V}_\delta^\infty$  arbitrary and define

$$\forall x \in \mathbb{V}_\delta^\infty, \lambda^u(x) = e_{(x_0, x)}(u - 2K - 2iK'|k).$$

By Proposition 6.2,  $\lambda^u$  is a positive massive harmonic function for  $\Delta_\delta^k$  on  $\mathbb{V}_\delta^\infty$ , so in particular it is massive harmonic on  $\mathbb{V}_\delta$  and we can apply the Doob transform technique (this is exactly the setting mentioned in the comments after Proposition 3.3). Recall that the Doob transform  $\tilde{S}_\delta$  of the KRW  $S_\delta^k$  is the RW on  $\mathbb{G}_\delta^\infty$  with conductances  $\tilde{c}_{(x,y)} = \frac{\lambda^u(y)}{\lambda^u(x)} \text{sc}(\theta_{xy}|k)$ , and denote by  $\tilde{\Delta}_\delta$  the non-massive Laplacian associated with these conductances. Denote by  $\Lambda$  the diagonal matrix with rows and columns indexed by  $\mathbb{V}_\delta^\infty$  and entries  $\lambda^u$  on the diagonal. Proposition 2.17 implies

$$(\tilde{\Delta}_\delta)_{\mathbb{V}_\delta} = \Lambda_{\mathbb{V}_\delta}^{-1} (\Delta_\delta^k)_{\mathbb{V}_\delta} \Lambda_{\mathbb{V}_\delta}.$$

**RSF and dimers.** To the massive Laplacian  $(\Delta_\delta^k)_{\mathbb{V}_\delta}$  is associated a RSF model on  $\mathbb{G}_\delta$  with conductance and mass functions  $(c, m)$ . As was already mentioned, for  $k > 0$ , the mass function  $m$  is positive at all  $x \in \mathbb{V}_\delta$ : this is a typical example where the graph  $\mathbb{G}_\delta^o$  of the forest-tree bijection is highly non-planar. To  $(\tilde{\Delta}_\delta)_{\mathbb{V}_\delta}$  is associated a RST model on  $\mathbb{G}_\delta^o$  (obtained by identifying all vertices in  $\mathbb{V}_\delta^\infty \setminus \mathbb{V}_\delta$  to  $o$ , see Figure 3) rooted at  $o$  weighted by  $\tilde{c}$  (see around Equation (11) for details).

As in Section 4, fix an arbitrary vertex  $r \in \partial\mathbb{V}_\delta^*$  on the boundary of  $\mathbb{G}_\delta^*$ , the restricted dual of  $\mathbb{G}_\delta^o$ . Recall from Section 4.1 the definition of the double graph with  $o$  and  $r$  removed  $\underline{\mathbb{G}}_\delta^D$  associated with  $\mathbb{G}_\delta$  (see Figure 7). By Temperley's bijection and our Definition 4.3 of the drifted dimer model, RST of  $\mathbb{G}_\delta^o$  weighted by  $\tilde{c}$  are in weight-preserving bijection with dimer configurations of the drifted dimer model on  $\underline{\mathbb{G}}_\delta^D$ . The local weights for the RST model and the drifted dimer model away from the boundary are depicted in Figure 8.

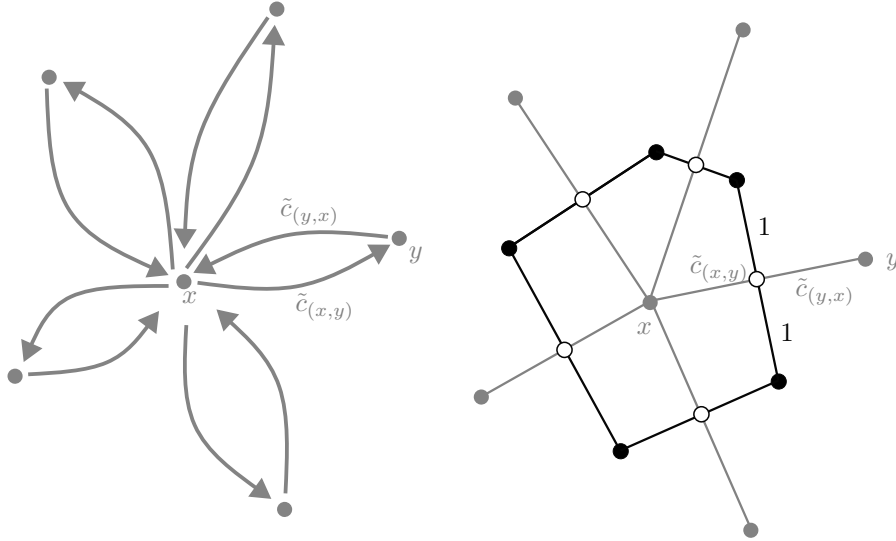


Figure 8: On the left: the conductances  $\tilde{c}$  around a vertex  $x \in \mathbb{V}_\delta$  for the RST model on  $\mathbb{G}_\delta^o$  rooted at  $o$ . On the right: the weights for the drifted dimer model on  $\underline{\mathbb{G}}_\delta^D$ . The asymptotics of  $\tilde{c}$  in the near-critical regime are given in Equation (47) in terms of the local parameters.



#### 6.1.4 Near-critical regime.

We will study these models in the near-critical regime where the elliptic modulus  $k$  goes to 0 appropriately with  $\delta$ . Intuitively, it corresponds to the mass being of order  $\delta^2$  at every point (see the forthcoming Lemma 6.4): it is an interesting regime where the probability that  $S_\delta^k$  leaves  $V_\delta$  before dying is bounded away from 0 and 1, so the shape of  $\Omega$  has a non-trivial effect on the model.

More precisely, let  $q = q(k) = \exp(-\pi K'/K)$  be the *nome* associated with the elliptic parameter  $k$ . The *near-critical regime* corresponds to  $\delta \rightarrow 0$  and  $q = \frac{1}{2}M\delta$  where  $M \in \mathbb{R}_{>0}$  is a real positive parameter called the *mass parameter* fixed in the rest of the article (as in [Par22]). In what follows, we use only one scale parameter  $\delta$ , and write implicitly  $q = q(\delta)$ ,  $k = k(\delta)$ . In this regime, we can compute the asymptotics of several of the quantities defined before.

**Lemma 6.3.** *In the near-critical regime  $q = \frac{1}{2}M\delta$  with  $\delta \rightarrow 0$ ,*

$$\forall \bar{\theta} \in \mathbb{R}, \quad \begin{cases} k^2 & = 8M\delta - 32M^2\delta^2 + O(\delta^3) \\ \theta & = (1 + 2M\delta + M^2\delta^2 + O(\delta^3))\bar{\theta} \\ \text{sc}(\theta|k) & = (1 + 2M\delta + O(\delta^2)) \tan(\bar{\theta}). \end{cases}$$

Moreover, for all  $\bar{u} \in \mathbb{R}$ ,

$$\forall x, y \in V_\delta^\infty, \quad \mathbf{e}_{(x,y)}(u - 2K - 2iK'|k) = \exp(2M\langle e^{i\bar{u}}, y - x \rangle) + O(\delta^2|y - x|)$$

where  $|y - x|$  is the euclidean distance on  $\mathbb{R}^2$  and  $\exp : \mathbb{R} \rightarrow \mathbb{R}$  is the usual exponential function. These asymptotics hold uniformly in the isoradial grid  $\mathbf{G}_\delta^\infty$  (given the bounded angle assumption), the angle  $\bar{\theta} \in \mathbb{R}$ , the parameter  $\bar{u} \in \mathbb{R}$  and the points  $x, y \in V_\delta^\infty$ .

**Lemma 6.4.** *In the near-critical regime  $q = \frac{1}{2}M\delta$  with  $\delta \rightarrow 0$ , uniformly in  $\delta$  and  $x$ ,*

$$\forall x \in V_\delta^\infty, \quad m^2(x|k) = 2M^2\delta^2 \sum_{y \sim x} \sin(2\bar{\theta}_{xy}) + O(\delta^3).$$

*Remark 6.5.* In the bounded angle assumptions, the  $\sin(\bar{\theta}_j)$  are positive and bounded away from 0, so the mass is of order  $\delta^2$  (uniformly in  $x, \delta$ ).

We will prove Lemma 6.4 in Appendix A.1 using only the massive harmonicity of the discrete exponential and the approximation property of the Laplacian (which will be introduced only in the next section). We now prove Lemma 6.3.

*Proof.* Using the expression of the elliptic modulus in terms of the nome and the theta functions (see (22.2.2) of [DLMF]):

$$k^2 = \left( \frac{2q^{1/4}(1 + O(q^2))}{1 + 2q + O(q^2)} \right)^4 = 16q(1 + O(q^2))(1 - 8q + O(q^2)) = 16q - 128q^2 + O(q^3).$$

Moreover, by (22.2.2) of [DLMF] and our definition of the abstract angle,

$$\forall \bar{\theta} \in \mathbb{R}, \quad \frac{\theta}{\bar{\theta}} = \frac{2K}{\pi} = (1 + 2q + O(q^3))^2 = 1 + 2M\delta + M^2\delta^2 + O(\delta^3).$$

The asymptotic of the conductances can be computed using (22.2.9) of [DLMF]. Observe that for  $\bar{\theta} \in \mathbb{R}$ , their notation  $\zeta(\bar{\theta}) = \pi\bar{\theta}/(2K)$  corresponds to our notation for the abstract angle:  $\theta = \zeta(\bar{\theta})$ . Hence,

$$\forall \bar{\theta} \in \mathbb{R}, \quad \text{sc}(\theta|k) = (1 + 4q + O(q^2)) \frac{\sin(\bar{\theta}) + O(q^2)}{\cos(\bar{\theta}) + O(q^2)} = (1 + 2M\delta + O(\delta^2)) \tan(\bar{\theta}), \quad (45)$$

where the  $O$  is uniform (using the bounded angle assumption for the last equality).

We now prove the asymptotic expression of the discrete massive exponential function. A proof of this result could probably be extracted from Section 5 of [CIM21] among much more technical

computations, but for the sake of clarity we write down the computations in our (simpler) setting. Fix  $\bar{u} \in \mathbb{R}$ ,  $x, y \in \mathbb{V}_\delta^\circ$ . Since the massive exponential is multiplicative, it can be computed along any path from  $x$  to  $y$ . It is a classical consequence of the bounded angle assumption (see for example Assumption 1 of [SSW23]) that we can choose a path  $\gamma_\delta : x \rightarrow y$  of length  $\leq \frac{C|y-x|}{\delta}$  where  $|\cdot|$  is the euclidean distance and where the constant  $C$  depends only on the lower bound in the bounded angle assumption. Denoting  $\gamma_\delta = (x_0, \dots, x_n)$ ,

$$\mathbf{e}_{(x,y)}(u - 2K - 2iK'|k) = \prod_{j=0}^{n-1} \mathbf{e}_{(x_j, x_{j+1})}(u - 2K - 2iK'|k).$$

We now compute the asymptotics of each term of the product. Let  $x_j \sim x_{j+1} \in \mathbb{V}_\delta$ ,  $x_{j+1} - x_j = \delta e^{i\bar{\alpha}_j} + \delta e^{i\bar{\beta}_j}$  and let  $\bar{\theta}_j = (\bar{\beta}_j - \bar{\alpha}_j)/2$  be the half angle associated to the edge  $x_j x_{j+1}$ . Using the change of variable formula as in Equation (44),

$$\mathbf{e}_{(x_j, x_{j+1})}(u - 2K - 2iK'|k) = \frac{1}{k'} \operatorname{dn}\left(\frac{u - \alpha_j}{2} \middle| k\right) \operatorname{dn}\left(\frac{u - \beta_j}{2} \middle| k\right). \quad (46)$$

By definition of  $\operatorname{dn}$  Equation (22.2.6) of [DLMF]:

$$\operatorname{dn}\left(\frac{u - \alpha_j}{2}\right) = \sqrt{k'} \frac{1 + 2q \cos(\bar{u} - \bar{\alpha}_j) + O(q^3)}{1 - 2q \cos(\bar{u} - \bar{\alpha}_j) + O(q^3)} = \sqrt{k'} \exp(4q \cos(\bar{u} - \bar{\alpha}_j)) + O(q^3)$$

because it is easily checked that  $x \rightarrow \frac{1+x}{1-x}$  and  $x \rightarrow \exp(2x)$  coincide up to  $O(x^3)$ . The  $O$  is uniform (the constant depends only on the mass parameter  $M \in \mathbb{R}_{>0}$ ). Using the fact that  $\bar{u} \in \mathbb{R}$  and the scaling  $q = \frac{1}{2}M\delta$  yields

$$\frac{1}{\sqrt{k'}} \operatorname{dn}\left(\frac{u - \alpha_j}{2}\right) = \exp(2M\delta \langle e^{i\bar{u}}, e^{i\bar{\alpha}_j} \rangle) + O(\delta^3).$$

Since we can write the same thing with  $\beta_j$  instead of  $\alpha_j$ , Equation (46) gives

$$\begin{aligned} \mathbf{e}_{(x_j, x_{j+1})}(u - 2K - 2iK'|k) &= \exp(2M\delta \langle e^{i\bar{u}}, e^{i\bar{\alpha}_j} + e^{i\bar{\beta}_j} \rangle) + O(\delta^3) \\ &= \exp(2M\delta \langle e^{i\bar{u}}, x_{j+1} - x_j \rangle) + O(\delta^3). \end{aligned}$$

By multiplying along the path  $\gamma_\delta$  of length  $\leq \frac{C|y-x|}{\delta}$ , we get the desired result.  $\square$

As an application of Lemma 6.3, we can compute the asymptotics of the weights of the RST and drifted dimer models in the near-critical regime. Fix  $\bar{u} \in \mathbb{R}$ . Let  $x \sim y \in \mathbb{V}_\delta$ . Using the asymptotics of  $\operatorname{sc}$  and  $\mathbf{e}$  of Lemma 6.3, we obtain that the conductance  $\tilde{c}$  attributed to the edge  $(x, y)$  by the RST model on  $\mathbb{G}_\delta^o$  associated to the Laplacian  $\tilde{\Delta}_\delta$  is asymptotically

$$\begin{aligned} \tilde{c}_{(x,y)} &= \operatorname{sc}(\theta_{xy}|k) \frac{\lambda^u(y)}{\lambda^u(x)} = (1 + 2M\delta + O(\delta^2)) \tan(\bar{\theta}_{xy}) \left( \exp\left(2M\delta \langle e^{i\bar{u}}, e^{i\bar{\alpha}_{xy}} + e^{i\bar{\beta}_{xy}} \rangle\right) + O(\delta^3) \right) \\ &= (1 + 2M\delta) \tan(\bar{\theta}_{xy}) \left( 1 + 4M\delta \cos(\bar{\theta}_{xy}) \cos\left(\bar{u} - \frac{\bar{\alpha}_{xy} + \bar{\beta}_{xy}}{2}\right) \right) + O(\delta^2). \end{aligned} \quad (47)$$

*Example 6.6.* If the isoradial grid  $\mathbb{G}_\delta^\circ$  is the square lattice  $\delta\mathbb{Z}^2$ , we check that our model of drifted dimers coincides with the near-critical dimer model of [Chh12] and [HSB22] with drift parameter  $\bar{u}$ : any  $x \in \mathbb{G}_\delta$  has four neighbours  $x \pm \delta$  and  $x \pm i\delta$ , with respective edge conductances  $1 \pm 2M\delta\sqrt{2} \cos(\bar{u})$  and  $1 \pm 2M\delta\sqrt{2} \sin(\bar{u})$  (up to an error  $O(\delta^2)$  and a global factor  $(1 + 2M\delta)$  which does not change the transition probabilities). This is exactly Equation (1.6) of [HSB22] with drift vector  $\sqrt{2}M e^{i\bar{u}}$ .

## 6.2 From the killed random walk to the drifted random walk

In this section, we follow closely [HSB22]: we provide discrete Girsanov identities between the KRW  $S_\delta^k$  and its Doob transform  $\tilde{S}_\delta$ , and between their loop-erasures.

Recall that the Doob transform  $\tilde{S}_\delta$  and associated Laplacian  $\tilde{\Delta}_\delta$  depend on the drift parameter  $\bar{u}$  via  $\lambda^u$ . We adapt to our setting Corollary 2.5 of [HSB22]. Denote by  $\tau_\delta = \tau_\delta(\rho) \wedge \tau_\delta(\mathbf{V}_\delta^\infty \setminus \mathbf{V}_\delta)$  and  $\tilde{\tau}_\delta = \tilde{\tau}_\delta(\mathbf{V}_\delta^\infty \setminus \mathbf{V}_\delta)$  the escape times of  $\mathbf{V}_\delta$  for  $S_\delta^k$  and  $\tilde{S}_\delta$  respectively.

**Lemma 6.7.** *Let  $\gamma_\delta$  be a sequence of simple paths in  $\mathbf{G}_\delta$  from  $x_\delta \in \mathbf{V}_\delta$  to  $y_\delta \in \partial\mathbf{V}_\delta$ , with  $x_\delta \xrightarrow{\delta \rightarrow 0} x \in \Omega$  and  $y_\delta \xrightarrow{\delta \rightarrow 0} y \in \Omega$ . Then:*

$$\frac{\mathbb{P}_{x_\delta}(\forall 0 \leq i < \tilde{\tau}_\delta, (\tilde{S}_\delta)_i = \gamma_\delta(i))}{\mathbb{P}_{x_\delta}(\forall 0 \leq i < \tau_\delta, (S_\delta^k)_i = \gamma_\delta(i))} \xrightarrow{\delta \rightarrow 0} \exp(2M\langle e^{i\bar{u}}, y - x \rangle)$$

uniformly in the isoradial grid, the drift parameter  $\bar{u}$ , and the path.

Observe that this is precisely the same statement as Corollary 2.5 of [HSB22], justifying the name “drift parameter” for  $\bar{u}$ :  $\tilde{S}_\delta$  is a RW with a drift in the direction  $e^{i\bar{u}}$  of intensity  $M$ .

*Proof.* With the notation of the lemma, by definition of the Doob transform

$$\frac{\mathbb{P}_{x_\delta}(\forall 0 \leq i < \tilde{\tau}_\delta(\rho), (\tilde{S}_\delta)_i = \gamma_\delta(i))}{\mathbb{P}_{x_\delta}(\forall 0 \leq i < \tau_\delta(\rho), (S_\delta^k)_i = \gamma_\delta(i))} = \mathbf{e}_{(x_\delta, y_\delta)}(u - 2K - 2iK'|k) \frac{\mathbb{P}_{y_\delta}((\tilde{S}_\delta)_1 \in \mathbf{V}_\delta^\infty \setminus \mathbf{V}_\delta)}{\mathbb{P}_{y_\delta}((S_\delta^k)_1 \in \{\rho\} \cup \mathbf{V}_\delta^\infty \setminus \mathbf{V}_\delta)}.$$

On the one hand, by the asymptotic expression of  $\mathbf{e}$  computed in Lemma 6.3,

$$\mathbf{e}_{(x_\delta, y_\delta)}(u - 2K - 2iK'|k) = \exp(2M\langle e^{i\bar{u}}, y_\delta - x_\delta \rangle) + O(\delta^2|y_\delta - x_\delta|).$$

On the other hand, by definition of the Doob transform, and since  $m^2(x|k) = O(\delta^2)$  by Lemma 6.4 and for all  $z \sim y_\delta$ ,  $\mathbf{e}_{(y_\delta, z)}(u - 2K - 2iK'|k) = 1 + O(\delta)$  by Lemma 6.3,

$$\begin{aligned} \frac{\mathbb{P}_{y_\delta}((\tilde{S}_\delta)_1 \in \mathbf{V}_\delta^\infty \setminus \mathbf{V}_\delta)}{\mathbb{P}_{y_\delta}((S_\delta^k)_1 \in \{\rho\} \cup \mathbf{V}_\delta^\infty \setminus \mathbf{V}_\delta)} &= \frac{\sum_{z \sim y_\delta, z \notin \mathbf{V}_\delta} \mathbf{e}_{(y_\delta, z)}(u - 2K - 2iK'|k) \text{sc}(\theta_{y_\delta z}|k)}{m^2(y_\delta|k) + \sum_{z \sim y_\delta, z \notin \mathbf{V}_\delta} \text{sc}(\theta_{y_\delta z}|k)} \\ &= \frac{(1 + O(\delta)) \sum_{z \sim y_\delta, z \notin \mathbf{V}_\delta} \text{sc}(\theta_{y_\delta z}|k)}{O(\delta^2) + \sum_{z \sim y_\delta, z \notin \mathbf{V}_\delta} \text{sc}(\theta_{y_\delta z}|k)} = 1 + O(\delta). \end{aligned}$$

It is also important that since  $y \in \partial\mathbf{V}_\delta$ ,  $\sum_{z \sim y_\delta, z \notin \mathbf{V}_\delta} \text{sc}(\theta_{y_\delta z}|k)$  is bounded away from 0 since there is at least one term in the sum (and by the bounded angle assumption).  $\square$

Recall from Section 2.2 that  $LE(\gamma)$  denotes the loop-erasure of the path  $\gamma$ . For a transient KRW  $X^k$  (it is the case of both  $S_\delta^k$  and  $\tilde{S}_\delta$  killed when exiting  $\mathbf{V}_\delta$ : they die almost surely since  $\mathbf{G}_\delta$  is finite),  $LE(X^k)$  denotes the loop-erasure of the trajectory up to the killing time. Lemma 6.7 implies, as in [HSB22],

**Lemma 6.8.** *Let  $\gamma_\delta$  be a sequence of simple paths in  $\mathbf{G}_\delta$  from  $x_\delta \in \mathbf{V}_\delta$  to  $y_\delta \in \partial\mathbf{V}_\delta$ , with  $x_\delta \xrightarrow{\delta \rightarrow 0} x \in \Omega$  and  $y_\delta \xrightarrow{\delta \rightarrow 0} y \in \partial\Omega$ . Then,*

$$\frac{\mathbb{P}_{x_\delta}(LE(\tilde{S}_\delta) = \gamma_\delta)}{\mathbb{P}_{x_\delta}(LE(S_\delta^k) = \gamma_\delta)} \xrightarrow{\delta \rightarrow 0} \exp(2M\langle e^{i\bar{u}}, y - x \rangle).$$

uniformly in the isoradial grid, the drift parameter  $\bar{u} \in \mathbb{R}$  and the path.

*Proof.* It follows from Lemma 6.7 by summing over all ways to get  $\gamma_\delta$  as a loop-erasure, and observing that for each of these paths the ratio of the probability is exactly the right-hand side by the preceding lemma.  $\square$

### 6.3 Convergence of the loop-erased killed random walk to massive $SLE_2$

The next step towards convergence of the near-critical RST and dimer models is to prove convergence of the massive LERW conditioned to survive. We will verify the hypothesis of a general result for convergence of the killed LERW, Theorem 4.2 of [HSB22], which we now recall.

**Notation for continuous processes.** We first introduce some definitions related to continuous processes. The *brownian motion* (BM) is the diffusion process  $B$  on  $\mathbb{R}^2$  with unit covariance matrix and 0 drift vector. Recall from Equation (4.4) of [HSB22] that the *killed brownian motion* (KBM) with killing rate  $M(\cdot)$  (a continuous non-negative bounded function on  $\Omega$ ) is the process  $B^k$  which behaves like  $B$  until a stopping time  $\tau$  given by  $\mathbb{P}(\tau > t \mid \mathcal{F}_t) = \exp\left(-\int_0^t M^2(B_s) ds\right)$ , where  $(\mathcal{F}_t, t \geq 0)$  is the canonical filtration of the BM, after which it goes to an absorbing cemetery. We refer to the original article for all definitions involving Loewner evolutions, in particular  $SLE_2$  and its massive version.

Given a sequence of discrete RW  $(Y_\delta)_{\delta>0}$  on planar graphs  $\Omega_\delta$  embedded in  $\mathbb{C}$ , we also denote by  $Y_\delta$  the affine interpolation (that is for all  $n \in \mathbb{N}$ ,  $t \in [n, n+1)$ ,  $(Y_\delta)_t = (n+1-t)(Y_\delta)_n + (t-n)(Y_\delta)_{n+1}$ ). When we say that such a sequence converges towards a process  $X$  with continuous trajectories, we mean the uniform convergence on bounded time intervals.

**A general convergence result.** Let  $\Omega_\delta$  be a sequence of planar graphs embedded in the complex plane with conductance function  $c^\infty$ , and suppose that there is no accumulation point in the sense that for every compact set  $K$ , the number of vertices of  $\Omega_\delta$  inside  $K$  is finite. Let  $M(\cdot) : \Omega \rightarrow [0, \infty)$  be a bounded nonnegative continuous function. Denote by  $Y_\delta$  the RW with conductances  $c^\infty$  and  $Y_\delta^k$  the massive random walk with conductances  $c^\infty$  and mass function  $m^\infty = M^2(\cdot)\delta^2$ . Recall that we can couple  $Y_\delta^k$  and  $Y_\delta$  such that  $Y_\delta^k$  is killed in position  $x$  with probability proportional to  $M^2(\cdot)\delta^2$  and otherwise moves like  $Y_\delta$ .

Let  $\mathcal{R}$  be the horizontal rectangle  $[0, 3] \times [0, 1]$  and  $\mathcal{R}'$  be the vertical rectangle  $[0, 1] \times [0, 3]$ . Let  $B^s = (B^s)' = B((1/2, 1/2), 1/4)$  be the starting ball and  $B^t = B((5/2, 1/2), 1/4)$ , resp.  $(B^t)' = B((1/2, 5/2), 1/4)$ , be the target ball. Let  $\mathcal{D} = \mathcal{R} \setminus B^t$ . For  $r > 0$ ,  $z \in \mathbb{C}$ , let  $\mathcal{R}(r, z) = r\mathcal{R} + z$ , resp.  $\mathcal{R}(r, z)'$ , and define similarly  $B^s(r, z)$ ,  $B^t(r, z)$ ,  $\mathcal{D}(r, z)$ .

For all  $\delta > 0$ , let  $\text{Cross}_\delta^k(r, z)$ , resp.  $\text{Cross}_\delta^k(r, z)'$ , denote the event that the KRW  $Y_\delta^k$  started at a point of the starting ball  $x_\delta \in B^s(r, z) \cap V_\delta^\infty$  enters the target ball  $B^t(r, z)$  before leaving the rectangle  $\mathcal{R}(r, z)$  or dying:

$$\text{Cross}_\delta^k(r, z) = \left( \tau_\delta(B^t(r, z)) < \tau_\delta(\rho) \wedge \tau_\delta(V_\delta^\infty \setminus \mathcal{R}(r, z)) \right).$$

The KRW  $Y_\delta^k$  satisfies the *uniform crossing estimate* if for all  $R > 0$ , there is a constant  $\eta = \eta(R) > 0$  such that uniformly in  $0 < r < R$ ,  $z \in \mathbb{C}$ ,  $\delta > 0$  and  $x_\delta \in B^s(r, z) \cap V_\delta^\infty$ ,

$$\mathbb{P}_{x_\delta}(\text{Cross}_\delta^k(r, z)) \geq \eta, \quad \mathbb{P}_x(\text{Cross}_\delta^k(r, z)') \geq \eta. \quad (48)$$

**Theorem 6.9** (Theorem 4.2 of [HSB22]). *Assume that  $Y_\delta^k$  satisfies the uniform crossing estimate and that  $Y_\delta^k$  starts at  $x_\delta \xrightarrow{\delta \rightarrow 0} x \in \Omega$ . Suppose that  $((Y_\delta)_{\delta-2t}, t \geq 0)$  converges towards the BM, or equivalently that  $((Y_\delta^k)_{\delta-2t}, t \geq 0)$  converges towards the KBM  $B^k$  killed with rate  $M(\cdot)$ . Then the loop erasure  $LE(X_\delta^k)$  of  $Y_\delta^k$ , conditioned to leave  $\Omega$  (at the stopping time  $\tau_\delta$ ) before dying, and conditioned on  $(Y_\delta^k)_{\tau_\delta} = a_\delta$  with  $a_\delta \xrightarrow{\delta \rightarrow 0} a \in \partial\Omega$ , converges to a radial Loewner evolution with driving function  $\zeta_t = e^{i\zeta_t}$  satisfying Equation (4.5) of [HSB22] called massive  $SLE_2$  with mass parameter  $M(\cdot)$ .*

To apply this theorem, we will need the following result which is proved in Appendix A.2, using the weak Beurling estimate of [CS11] on the critical Laplacian.

**Proposition 6.10.** *The near-critical KRW  $S_\delta^k$  satisfies the uniform crossing estimate Equation (48).*

Using Theorem 6.9 and Proposition 6.10, we are able to prove the following.

**Theorem 6.11.** *If  $x_\delta \xrightarrow{\delta \rightarrow 0} x \in \Omega$ , the loop erasure  $LE(S_\delta^k)$  of the KRW  $S_\delta^k$  started at  $x_\delta \in \mathbf{V}_\delta$  conditioned to leave  $\Omega$  (or equivalently  $\mathbf{V}_\delta$ , at the stopping time  $\tau_\delta$ ) before dying, and conditioned on  $(S_\delta^k)_{\tau_\delta} = a_\delta \in \partial\mathbf{V}_\delta$  with  $a_\delta \rightarrow a \in \partial\Omega$  converges to a radial Loewner evolution with driving function  $\zeta_t = e^{i\zeta_t}$  satisfying Equation (4.5) of [HSB22] with mass parameter  $\sqrt{2}M$ .*

Note that we cannot apply Theorem 6.9 directly to  $S_\delta^k$  which is not killed with probability  $M(x)\delta^2$  for a continuous function  $M$  (see Lemma 6.4), and moreover the RW  $S_\delta$  with conductances  $c_{xy}^\infty = \text{sc}(\theta_{xy}|k)$  converges towards the BM only after a time re-parametrization. These two matters are related, and we will solve them by considering an associated lazy RW  $S_\delta^l$ .

*Proof.* Denote by  $S_\delta$  the RW with conductances  $c_{xy}^\infty = \text{sc}(\theta_{xy}|k)$  and by  $\zeta_\delta(x)$  the law of its increments: conditionally on  $(S_\delta)_n = x$ , it is the law of  $(S_\delta)_{n+1} - (S_\delta)_n$ . Using Equation (42), the asymptotics of  $\text{sc}$  of Lemma 6.3 and the fact that two neighbours in  $\mathbf{G}_\delta^\infty$  are at distance  $\leq 2\delta$ , we obtain that

$$\mathbb{E}(\zeta_\delta(x)) = O(\delta^3); \mathbb{E}(\Re(\zeta_\delta(x))^2) = \mathbb{E}(\Im(\zeta_\delta(x))^2) = \delta^2 T_\delta(x) + O(\delta^4); \mathbb{E}(\Re(\zeta_\delta(x))\Im(\zeta_\delta(x))) = O(\delta^4). \quad (49)$$

Hence, a certain lazy version of  $S_\delta$  converges towards the BM. To make it formal, we define the *laziness* of a vertex

$$\forall x \in \mathbf{V}_\delta^\infty, l_\delta(x) := \frac{T_\delta(x) - T}{T} \sum_{y \sim x} \text{sc}(\theta_{xy}|k).$$

Let  $S_\delta^l$  be the *lazy RW*, or more precisely the RW on the graph  $\mathbf{G}_\delta^\infty$  with an additional loop edge  $(x, x)$  with conductance  $l_\delta(x)$  at each vertex  $x \in \mathbf{V}_\delta^\infty$ : in other words, it is the RW which stays in  $x$  with probability proportional to  $l_\delta(x)$  and otherwise moves towards  $y \sim x$  with probability proportional to  $c_{xy}^\infty = \text{sc}(\theta_{xy}|k)$ . The time spent by  $S_\delta^l$  at  $x$  follows a geometric law with parameter  $\pi_\delta(x) = l_\delta(x) / (l_\delta(x) + \sum_{y \sim x} \text{sc}(\theta_{xy}|k))$ . If we denote by  $\zeta_\delta^l$  the law of the increments of  $S_\delta^l$ , we have for all  $x \in \mathbf{V}_\delta^\infty$  and all  $\phi : \mathbb{C} \rightarrow \mathbb{C}$  with  $\phi(0) = 0$ ,

$$\mathbb{E}(\phi(\zeta_\delta^l(x))) = (1 - \pi_\delta(x))\mathbb{E}(\phi(\zeta_\delta(x))).$$

This applies in particular with  $\phi(x) = x, \Re(x)^2, \Im(x)^2, \Re(x)\Im(x)$  so Equation (49) gives

$$\mathbb{E}(\zeta_\delta^l(x)) = O(\delta^3); \mathbb{E}(\Re(\zeta_\delta^l(x))^2) = \mathbb{E}(\Im(\zeta_\delta^l(x))^2) = T\delta^2 + O(\delta^4); \mathbb{E}(\Re(\zeta_\delta^l(x))\Im(\zeta_\delta^l(x))) = O(\delta^4).$$

This implies that  $((S_\delta^l)_{(T\delta^2)^{-1}t}, t \geq 0)$  converges towards BM, see Theorem 11.2.3 of [SV79] with  $h = T\delta^2$  (it can also be seen as a consequence of the functional CLT for martingales after centering).

We are ready to deal with the KRW  $S_\delta^k$ . Let  $S_\delta^{l,k}$  be the *lazy KRW*, or more precisely the KRW on the graph  $\mathbf{G}_\delta^\infty$  with additional loop edges  $(x, x)$  with conductance  $l_\delta(x)$  at each vertex and mass function  $m^\infty(x) = m^2(x|k)$ . Define stopping times  $t_0 = 0$  and  $t_{i+1} = \inf\{n \geq t_i, (S_\delta^{l,k})_n \neq (S_\delta^{l,k})_{t_i}\}$ : they are the times at which the lazy KRW changes position. The law of the trajectories of  $S_\delta^{l,k}$  and  $S_\delta^k$  are the same, that is

$$((S_\delta^{l,k})_{t_i}, i \geq 0) \stackrel{(law)}{=} ((S_\delta^k)_i, i \geq 0).$$

This implies in particular that the loop-erasures of  $S_\delta^{l,k}$  and  $S_\delta^k$  have the same law. Using the asymptotics of  $\text{sc}$  and  $m^2(\cdot|k)$  from Lemmas 6.3 and 6.4, we get that the lazy KRW  $S_\delta^{l,k}$  is killed with probability

$$\begin{aligned} \mathbb{P}\left((S_\delta^{l,k})_{i+1} = \rho \mid (S_\delta^{l,k})_i = x\right) &= \frac{m^2(x|k)}{m^2(x|k) + l(x) + \sum_{y \sim x} \text{sc}(\bar{\theta}_{xy}|k)} \\ &= \frac{Tm^2(x|k)}{Tm^2(x|k) + \sum_{y \sim x} \sin(2\bar{\theta}_{xy}) + O(\delta)} = 2M^2T\delta^2 + O(\delta^2), \end{aligned}$$

so  $((S_\delta^{1,k})_{(T\delta^2)^{-1}t}, t \geq 0)$  converges towards the KBM with constant killing rate  $\sqrt{2}M$ .

We now apply Theorem 6.9 to the KRW  $Y_\delta^k := S_{T^{-1/2}\delta}^{1,k}$  on the graph  $\Omega_\delta$  which is  $\mathbb{G}_{T^{-1/2}\delta}$  with additional loop edges at each point (this graph is still planar). Proposition 6.10 implies that  $S_\delta^{1,k}$  and hence  $Y_\delta^k$  satisfy the uniform crossing assumption. Moreover, we just obtained that  $((Y_\delta^k)_{\delta^{-2}t}, t \geq 0)$  converges towards the KBM with constant killing rate  $\sqrt{2}M$ . Hence, conditionally on the endpoint,  $LE(Y_\delta^k)$  converges towards massive  $SLE_2$  with mass parameter  $\sqrt{2}M$ . In other words (replacing  $\delta$  by  $T^{-1/2}\delta$ ),  $LE(S_\delta^{1,k})$  converges towards massive  $SLE_2$  with mass parameter  $\sqrt{2}M$ . Since the loop-erasures of  $S_\delta^{1,k}$  and  $S_\delta^k$  have the same laws, this concludes the proof of Theorem 6.11.  $\square$

## 6.4 Convergence of the near-critical rooted spanning tree and drifted dimer models

We refer the reader to [HSB22] for more detailed explanations and proofs. We denote by  $\mathcal{T}_\delta^{M,\bar{u}}$  the RST of  $\mathbb{G}_\delta^0$  associated to the Laplacian  $\tilde{\Delta}_\delta$  by Wilson's algorithm: recall that it can be seen as a RST of  $\mathbb{G}_\delta$  rooted outside  $V_\delta$ . The law of a branch of this RST from a point  $x_\delta \in V_\delta$  to the boundary  $\partial V_\delta$  is the loop-erasure of  $\tilde{S}_\delta$  started at  $x_\delta$ . By Lemma 6.8, conditionally on the endpoint  $a_\delta \in \partial V_\delta$ , this law is the same as the loop-erasure of the KRW  $S_\delta^k$  conditioned on escaping  $V_\delta$  via  $a_\delta$  before dying, which by Theorem 6.11 converges towards massive  $SLE_2$  with mass parameter  $\sqrt{2}M$ . Moreover, if we denote by  $\sigma^k(y)dy$  the exit measure on  $\partial\Omega$  of the KBM with constant killing rate  $\sqrt{2}M$  (note that this is a sub-probability measure), Lemma 6.7 together with the convergence of the KRW towards the KBM imply that asymptotically, the law of the endpoint of a branch starting from  $x_\delta \xrightarrow{\delta \rightarrow 0} x \in \Omega$  is

$$\sigma^d(y)dy := \exp(2M\langle e^{i\bar{u}}, y \rangle)\sigma^k(y)dy$$

which is the exit law of a *drifted brownian motion* started at  $x$  with drift vector  $\sqrt{2}Me^{i\bar{u}}$ , that is a diffusion process with unit covariance matrix and drift vector  $\sqrt{2}Me^{i\bar{u}}$ . This gives the following theorem, extending Theorem 1.2 of [HSB22] to the isoradial case:

**Theorem 6.12.** *In the near-critical regime described above ( $q = \frac{1}{2}M\delta$ ), the tree  $\mathcal{T}_\delta^{M,\bar{u}}$  converges in the Schramm sense to a continuum limit tree  $\mathcal{T}^{M,\bar{u}}$  on  $\Omega$ . Conditionally on the endpoint  $y \in \partial\Omega$ , a branch of the tree from a point  $x \in \Omega$  has the law of massive radial  $SLE_2$  in  $\Omega$  with mass parameter  $\sqrt{2}M$  from  $x$  to  $y$ . The law of the endpoint  $y \in \partial\Omega$  is  $\sigma^d(y)dy$ .*

Note that this theorem contains the square case of Theorem 1.2 of [HSB22] as a particular case by Example 6.6, but not the hexagonal case as they consider a non-reversible random walk.

By Temperley's bijection (see our Section 4), the tree model is in weight-preserving bijection with the drifted dimer model on the double isoradial graph  $\mathbb{G}_\delta^D$ . As in [HSB22], convergence of the Temperleyan tree implies convergence of the height function of the drifted dimer model:

**Theorem 6.13.** *In the near-critical regime described above ( $q = \frac{1}{2}M\delta$ ), the centered height function  $h_\delta^{M,\bar{u}} - \mathbb{E}[h_\delta^{M,\bar{u}}]$  of the drifted dimer model with mass and drift parameters  $\sqrt{2}M > 0, \bar{u} \in \mathbb{R}$  converges in law to a limit when  $\delta \rightarrow 0$ .*

The limit is not explicit with these techniques, but [Mas23] seems to indicate that it is related to the Sine-Gordon field, see (i) of the open question Section 1.7. of [HSB22] for more details on this conjecture.

## A Critical and near-critical isoradial Laplacians

In this appendix, we use some tools developed in [CS11] for the analysis of the critical Laplacian on isoradial graphs and adapt them to the near-critical regime to prove Lemma 6.4 and Proposition 6.10. We use the notation of Section 6 for isoradial graphs and discrete  $Z$ -invariant and critical Laplacians.

## A.1 Proof of the asymptotic of the mass in the near-critical regime

**The approximation property.** Many of the properties of the critical Laplacian  $\Delta_\delta$  of Equation (41) follow from the fundamental *approximation property* Lemma 2.2 of [CS11] which we now recall. Let  $\Delta = \partial_{xx} + \partial_{yy}$  and  $D$  denote respectively the usual continuous Laplacian and derivative operators on  $\mathbb{R}^2$ . Note that the definition of  $\Delta_\delta^k$  and in particular  $\Delta_\delta$  are *local*: for  $x \in \mathbf{V}_\delta^\circ$ ,  $(\Delta_\delta f)(x)$  (and also  $(\Delta_\delta^k f)(x)$ ) is defined as long as  $f$  is defined on the neighbours of  $x$ . In particular, it makes sense if  $f$  is defined only on  $B(x, 2\delta)$  (the euclidean ball centered at  $x$  of radius  $r$ ) which contains all the neighbours of  $x$ .

**Lemma A.1** (Lemma 2.2 of [CS11]). *There exists an absolute constant  $C$  such that for all  $x \in \mathbf{V}_\delta^\circ$ , for all smooth function  $f \in \mathcal{C}^3(B(x, 2\delta), \mathbb{R})$ ,*

$$\left| (\Delta_\delta f)(x) + \frac{\delta^2}{2} \left( \sum_{y \sim x} \sin(2\bar{\theta}_{xy}) \right) (\Delta f)(x) \right| \leq C \delta^3 \sup_{B(x, 2\delta)} |D^3 f|.$$

From this approximation property follows the *approximation property for the massive Laplacian*:

**Lemma A.2.** *There exists an absolute constant  $C$  such that for all  $x \in \mathbf{V}_\delta^\circ$ , for all smooth function  $f \in \mathcal{C}^3(B(x, 2\delta), \mathbb{R})$  defined in a neighborhood of  $x$ ,*

$$\left| (\Delta_\delta^k f)(x) + \frac{\delta^2}{2} \left( \sum_{y \sim x} \sin(2\bar{\theta}_{xy}) \right) (\Delta f)(x) - m^2(x|k)f(x) \right| \leq C \delta^3 \sup_{B(x, 2\delta)} (|Df| + |\Delta f| + |D^3 f|)$$

*Proof.* Recall that in the near-critical regime, by Equation (45), the conductances of the massive Laplacian  $\Delta_\delta^k$  are  $c_{xy}^\circ = \text{sc}(\theta_{xy}|k) = (1 + 2M\delta + O(\delta^2)) \tan(\bar{\theta}_{xy})$ . Moreover, for all  $x \in \mathbf{V}_\delta^\circ$ ,  $f \in \mathcal{C}^3(B(x, 2\delta), \mathbb{R})$ ,  $y \sim x$ , we have  $|f(x) - f(y)| \leq \delta \sup_{B(x, 2\delta)} |Df|$  so

$$\begin{aligned} & \left| (\Delta_\delta^k f)(x) - m^2(x|k)f(x) - (1 + 2M\delta)(\Delta_\delta f)(x) \right| \\ &= \left| \sum_{y \sim x} (\text{sc}(\theta_{xy}|k) - (1 + 2M\delta) \tan(\bar{\theta}_{xy})) (f(x) - f(y)) \right| \leq C \delta^3 \sup_{B(x, 2\delta)} |Df|. \end{aligned}$$

For some universal constant  $C$ . The approximation property for the massive Laplacian follows by triangular inequality and Lemma A.1.  $\square$

**The asymptotic of the near-critical mass.** We use the approximation property and the massive harmonicity of  $\mathbf{e}$  to prove Lemma 6.4.

*Remark A.3.* This proof is a shortcut: it avoids all direct computations which are hidden in the massive harmonicity of the discrete exponential. A direct computation is also possible but lengthy.

*Proof.* Let  $x \in \mathbf{V}_\delta^\circ$  be fixed. We introduce a function  $e^{M, \bar{u}} : y \in \mathbb{C} \rightarrow \exp(2M \langle e^{i\bar{u}}, y - x \rangle)$ . On the one hand, by Lemma 6.3, for  $y \sim x$ ,

$$\mathbf{e}_{(x, y)}(u - 2K - 2iK'|k) = \exp(2M \langle e^{i\bar{u}}, y - x \rangle) + O(\delta^3) = e^{M, \bar{u}}(y) + O(\delta^3)$$

so by linearity of  $\Delta_\delta^k$  and massive harmonicity of  $\mathbf{e}$ ,

$$(\Delta_\delta^k e^{M, \bar{u}})(x) = O(\delta^3).$$

On the other hand,  $e^{M, \bar{u}}$  is (continuous) *massive harmonic with mass  $4M^2$* :  $\Delta e^{M, \bar{u}} = 4M^2 e^{M, \bar{u}}$ . By the approximation property applied at  $x$  to the function  $e^{M, \bar{u}}$  (and since by definition,  $e^{M, \bar{u}}(x) = 1$ ):

$$m^2(x|k) = 4M^2 \delta^2 \mu_\delta(x) + O(\delta^3) \quad \square$$

## A.2 Proof of the uniform crossing estimate

**The weak Beurling estimate for the critical Laplacian.** We recall the notation and result of [CS11]. For a complex number  $z \in \mathbb{C} \setminus \{0\}$ , denote by  $\arg(z)$  its argument in  $(-\pi, \pi]$ . For an open euclidean ball  $B \subset \mathbb{C}$  and  $\delta > 0$ , denote by  $B_\delta$  the *discretization* of  $B$ : it is the subgraph induced by  $V_\delta^\infty \cap B$ . Note that for  $r/\delta \geq C_0$  some absolute constant depending only on the bounding angle assumption, it is connected and coincides with the definition of [CS11], and moreover  $\partial B_\delta$  is at distance  $\leq 2\delta$  of  $\partial B$ : this might not hold for more complicated sets. For  $a, b \in \partial B$ , resp.  $a_\delta, b_\delta \in \partial B_\delta$ , we denote by  $(a, b) \subset \partial B$ , resp.  $(a_\delta, b_\delta) \subset \partial B_\delta$ , the points of  $\partial B$ , resp. vertices of  $\partial B_\delta$ , encountered when following the boundary counterclockwise. Denote by  $X_\delta$  the RW with conductances  $\tan(\bar{\theta}_{xy})$ , that is the RW associated with the critical Laplacian  $\Delta_\delta$ .

**Lemma A.4** (Lemma 2.10 of [CS11]). *For  $o_\delta \in V_\delta^\infty$ ,  $r > C_0\delta$ , let  $B(o_\delta, 2r)$  be the euclidean ball of radius  $2r$ ,  $B_\delta(o_\delta, 2r)$  its discretization and  $\tau_\delta$  the escape time of  $B_\delta(o_\delta, 2r)$  by  $X_\delta$ . Let  $a_\delta, b_\delta \in \partial B_\delta(o_\delta, 2r)$  such that  $\arg(b_\delta - o_\delta) - \arg(a_\delta - o_\delta) \geq \pi/4$ . Then,*

$$\forall x_\delta \in B_\delta(o_\delta, r), \mathbb{P}_{x_\delta}((X_\delta)_{\tau_\delta} \in (a_\delta, b_\delta)) \geq \eta_0 \geq 0,$$

where  $\eta_0$  is an absolute constant (independent of the isoradial grid,  $r$ ,  $o_\delta, a_\delta, b_\delta$ ).

**The uniform crossing estimate for the near-critical Laplacian.** We prove Proposition 6.10. We only do the proof for  $\text{Cross}_\delta^k(r, z)$ : the proof for  $\text{Cross}_\delta^k(r, z)'$  is exactly the same.

*Proof.* Fix  $R > 0$ , and let  $C_0\delta \leq r \leq R$ . The remaining case  $r < C_0\delta$  will be considered at the end of the proof.

*First step: the uniform crossing estimate for the RW with no killing.* For all  $0 \leq i \leq 4$ , define  $B^i = B((1/2 + i/2, 1/2), 1/4)$ ,  $A^i = B((1/2 + i/2, 1/2), 1/2)$  and note that  $B^0 = B^s$ ,  $B^4 = B^t$ . Let  $0 \leq i \leq 4$  and define  $B^i(r, z)$ ,  $A^i(r, z)$  for the scaled and translated balls (as in the statement of Proposition 6.10) and  $B_\delta^i(r, z)$ ,  $A_\delta^i(r, z)$  for their discretizations. Denote by  $o^i(r, z) = r(1/2 + i, 1/2) + z$  the common center of the  $i$ -th small and big balls. Let  $a^i, b^i \in \partial A^i(r, z)$  such that

$$\arg(a^i - o^i(r, z)) = -\frac{\pi}{7} \quad ; \quad \arg(b^i - o^i(r, z)) = \frac{\pi}{7}.$$

where  $\arg$  denotes the argument in  $(-\pi, \pi]$ . Let  $a_\delta^i, b_\delta^i$  be the closest points in  $\partial A_\delta^i(r, z)$  from  $a^i$  and  $b^i$  respectively: they are at distance  $\leq 2\delta$  from  $a_i$  and  $b_i$ , so upon increasing  $C_0$ ,  $\arg(b_\delta^i - o^i(r, z)) - \arg(a_\delta^i - o^i(r, z)) \geq \pi/4$  for all  $r \geq C_0\delta$ . The weak Beurling estimate Lemma A.4 implies that

$$\forall x_\delta \in B_\delta^i(r, z), \mathbb{P}_{x_\delta}((S_\delta)_{\tau_\delta^i} \in (a_\delta^i, b_\delta^i)) \geq \eta_0. \quad (50)$$

Now, we observe that for all  $0 \leq i < 4$ , for all  $y \in (a^i, b^i) \subset \partial A^i(r, z)$ , denoting  $\alpha = \arg(y - o^i(r, z)) \in [-\pi/7, \pi/7]$ ,

$$|y - o^{i+1}(r, z)| = \left| \frac{r}{2} \sin(\alpha) \right| \leq \frac{r}{2} \sin(\pi/7).$$

Since  $\sin(\pi/7)/2 < 1$ ,  $(a_i, b_i) \subset \partial B^i(r, z)$  is a strict subset of  $B^{i+1}(r, z)$ . Since  $(a_\delta^i, b_\delta^i)$  is within  $O(\delta)$  of  $(a^i, b^i)$ , upon increasing  $C$ , we obtain that for all  $r \geq C\delta$ , for all  $0 \leq i \leq 3$ ,  $(a_\delta^i, b_\delta^i) \subset B_\delta^{i+1}(r, z)$ . We deal with the remaining case  $r \leq C\delta$  at the end of the proof.

Let  $\tau_\delta^i$  denote the first escape time of  $A_\delta^i(r, z)$  by  $X_\delta$ . Since we also have  $A^i(r, z) \subset \mathcal{R}(r, z)$ , on the event  $(X_\delta)_{\tau_\delta^i} \in (a_\delta^i, b_\delta^i)$  the RW enters  $B_\delta^{i+1}(r, z)$  before leaving  $\mathcal{R}(r, z)$ : the crossing event  $\text{Cross}_\delta(r, z) := \left( \tau_\delta(B_\delta^t(r, z)) < \tau_\delta(V_\delta^\infty \setminus \mathcal{R}(r, z)) \right)$  for the RW  $X_\delta$  satisfies

$$\bigcap_{i=0}^3 \left( (X_\delta)_{\tau_\delta^i} \in (a_\delta^i, b_\delta^i) \right) \subset \text{Cross}_\delta(r, z).$$

Applying four times the strong Markov property (with the stopping times  $\tau_\delta^i$  for  $0 \leq i \leq 3$ ) and using Equation (50) gives the uniform crossing estimate for the critical RW  $X_\delta$ :

$$\forall x_\delta \in B_\delta^s(r, z), \mathbb{P}_{x_\delta}(\text{Cross}_\delta(r, z)) \geq \eta_0^4. \quad (51)$$



*Second step: upper bound on the escape time for the RW with no killing.* Let  $z \in \mathbf{V}_\delta^\infty$  and  $x_\delta \in B_\delta^s(r, z)$  a starting point. Recall that  $(\xi_\delta)_n = (X_\delta)_{n+1} - (X_\delta)_n$  denotes the increments of the RW. The law  $\xi_\delta(y_\delta)$  of  $(\xi_\delta)_n$  depends only on the value  $y_\delta = (X_\delta)_n \in \mathbf{V}_\delta^\infty$ . Denote by  $M_\delta = \Re(X_\delta)$ . From Equation (42), we see that  $M_\delta$  is a martingale in the canonical filtration  $(\mathcal{F}_n)_{n \in \mathbb{N}}$  of  $X_\delta$  with increments satisfying for all  $y_\delta \in \mathbf{V}_\delta^\infty$ ,  $\mathbb{E}(\Re(\xi_\delta(y_\delta))) = 0$  and  $\mathbb{E}(\Re(\xi_\delta(y_\delta))^2) \geq \eta_1 \delta^2$  for some absolute constant  $\eta_1$ . Hence the quadratic variation  $\langle M_\delta \rangle$  of  $M_\delta$  satisfies for all  $n \geq 0$ ,  $\langle M_\delta \rangle_n \geq \eta_1 n \delta^2$ . Let  $\sigma_\delta = \inf\{n \geq 0, |(M_\delta)_n - \Re(x_\delta)| \geq 3R\}$ . By the monotone convergence theorem and the optional stopping theorem applied to the bounded martingales  $((M_\delta)_{\cdot \wedge n} - \Re(x_\delta))^2 - \langle M_\delta \rangle_{\cdot \wedge n}$  at stopping time  $\sigma_\delta$ ,

$$\eta_1 \delta^2 \mathbb{E}(\sigma_\delta) \leq \mathbb{E}(\langle M_\delta \rangle_{\sigma_\delta}) = \lim_{n \rightarrow \infty} \mathbb{E}(\langle M_\delta \rangle_{\sigma_\delta \wedge n}) = \lim_{n \rightarrow \infty} \mathbb{E}((M_\delta)_{\sigma_\delta \wedge n} - \Re(x_\delta))^2 \leq (3R)^2.$$

By Markov's inequality and since  $\tau_\delta(\mathbf{V}_\delta^\infty \setminus \mathcal{R}(r, z)) \leq \sigma_\delta$ , this implies for the critical RW  $X_\delta$ :

$$\forall \lambda > 0, \mathbb{P}_{x_\delta} \left( \tau_\delta(\mathbf{V}_\delta^\infty \setminus \mathcal{R}(r, z)) \geq \lambda \frac{R^2}{\delta^2} \right) \leq \mathbb{P} \left( \sigma_\delta \geq \lambda \frac{R^2}{\delta^2} \right) \leq \frac{9}{\eta_1 \lambda}. \quad (52)$$

*Third step: coupling the KRW and the RW.* From Lemma 6.4 and Equation (45) we know that the mass satisfies  $m^2(x_\delta | k) = O(\delta^2)$  and that the conductances  $\text{sc}(\theta_{xy} | k)$  of the near critical Laplacian  $\Delta_\delta^k$  are within  $O(\delta^2)$  of the conductances  $\tan(\bar{\theta}_{xy})$  of  $\Delta_\delta$  (after multiplying all conductances by a common factor  $1 + 2M\delta + O(\delta^2)$  which does not change the probabilities). Hence the KRW  $S_\delta^k$  and the RW  $X_\delta$  can be coupled with a probability of failing  $\leq \eta_2 \delta^2$  at each step.

*Fourth step: concluding the proof when  $C_0 \delta \leq r \leq R$ .* We choose  $\lambda$  such that  $9/(\eta_1 \lambda) = \eta_0^4/2$ : Equations (51) and (52) imply that the crossing event  $\text{Cross}_\delta(r, z)$  for the critical RW  $X_\delta$  happens with positive probability at a time of order  $\lambda R^2/\delta^2$  or less:

$$\forall x_\delta \in B_\delta^s(r, z), \mathbb{P}_{x_\delta} \left( \tau_\delta(B_\delta^t(r, z)) < \tau_\delta(\mathbf{V}_\delta^\infty \setminus \mathcal{R}(r, z)) < \lambda \frac{R^2}{\delta^2} \right) \geq \frac{\eta_0^4}{2}.$$

On the event  $\left( \tau_\delta(B_\delta^t(r, z)) < \tau_\delta(\mathbf{V}_\delta^\infty \setminus \mathcal{R}(r, z)) < \lambda R^2/\delta^2 \right)$ , the coupling fails with probability at most  $(1 - \eta_2 \delta^2)^{\lambda R^2/\delta^2} \leq 1 - \eta_3 < 1$  for some  $\eta_3 > 0$  depending only on  $R$  and some universal constants. Since on the event  $\text{Cross}_\delta(r, z)$ , when the coupling succeeds until  $\tau_\delta(\mathbf{V}_\delta^\infty \setminus \mathcal{R}(r, z))$ , the event  $\text{Cross}_\delta^k(r, z)$  happens, we conclude that

$$\forall x_\delta \in B_\delta^s(r, z), \mathbb{P}_{x_\delta}(\text{Cross}_\delta^k(r, z)) \geq (1 - \eta_3) \eta_0^4/2.$$

*Fifth step: when  $r \leq C_0 \delta$ .* In this case, the rectangle  $\mathcal{R}(r, z)$  has a bounded diameter for the graph distance: by the bounded angle assumption, we can always find a path with at most  $C$  vertices from any point in the starting ball to the target ball. The probability that the walk follows this path and survives is uniformly bounded away from 0 by a constant of the form  $\eta^C$  for some absolute constant  $\eta > 0$ . □

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