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Designing mathematical tasks to promote preservice teachers' algebraic reasoning and argumentation

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The aim of this study is to shed light on one group of preservice teachers' engagement with designed mathematical tasks and in what ways these tasks promote the preservice teachers' argumentation and reasoning. The study took place within a university course on argumentation and reasoning and on how to draw on these mathematical processes within mathematics teaching in upper secondary school. We analyse the preservice teachers' written solutions to the given tasks unfolding within small-group work. The analyses reveal that they used predicted approaches, such as deductive proof and geometrical representation. However, they also used unique approaches, such as manipulating one side of the equation to look like the other, using functions to generalize, proving by example, and using drawings instead of variables.

Keywords: Argumentation, designing tasks, preservice teachers, reasoning.

Introduction

For several years, the mathematical processes of argumentation and reasoning have been emphasized as crucial to learn mathematics (NCTM, 2000). In recent years, these processes have been advocated in the Norwegian curriculum for school mathematics, through its identification of 'core elements', one of which is called 'reasoning and argumentation'. Thus, it is highly relevant to educate preservice mathematics teachers (PSMTs) to make them ready to teach mathematics emphasizing reasoning and argumentation. The aim of this study is to shed light on one group of PSMTs' engagement with designed mathematical tasks and in what ways these promote their reasoning and argumentation.

Kempen and Biehler (2019) investigated preservice mathematics teachers' (PSMTs') proof validation. They found that 3 out of 10 first year PSMTs saw testing of examples as valid proof and failed to recognize algebraic errors in proofs. Ko and Rose (2022) studied 14 preservice mathematics teachers' considerations with respect to self-productions to tasks comprising algebra and number theory, as proofs. They found that the students made a variety of arguments. Some students relied on what they call empirical arguments, i.e. what Balacheff (1988) would call naïve empiricism (see below). Other students relied on unstructured as well as structured arguments. The latter may be aligned with Balacheff's thought experiment (see below).

According to Martinez and Pedemonte (2014), the context of number theory facilitates deductive argumentation, and it provides a paradigmatic exposure to mathematical thinking (Mason, 2006). Since teachers' argumentation in mathematics teaching is important (e.g. Ayalon & Even, 2014; Bersch, 2019), it is relevant to study PSMTs and their engagement with reasoning and argumentation in mathematics. Therefore, we have formulated the following research question: In what ways do designed mathematical tasks promote PSMTs' algebraic reasoning and argumentation?

Algebraic reasoning and argumentation in number theory

As argued above, reasoning and argumentation are seen as vital processes within mathematics teaching and learning. It may even be argued that argumentation is inherent in mathematical activities. It is defined as the process that “produces a logically connected (but not necessarily deductive) discourse about a given subject” (Douek, 1999, p. 127) and as a verbal, social and rational activity aimed at convincing another person about the validity of the arguer’s standpoint (van Eemeren & Grootendorst, 2003). Argumentation consists of one or more arguments that can have various forms (verbal, numerical, etc.) (Douek, 1999).

Douek’s (1999) definition of argumentation aligns with how we see reasoning as a process. Even though the construct of mathematical reasoning, according to Jeannotte and Kieran, lacks consensus with respect to its consistencies, we nevertheless adopt these authors’ definition, that mathematical reasoning is “a process of communication with others or with oneself that allows for inferring mathematical utterances from other mathematical utterances” (2017, p. 7). Aligning the definition of argumentation (cf. Douek, 1999) and this definition of reasoning (cf. Jeannotte & Kieran, 2017), we coin a stipulated definition and view reasoning and argumentation in mathematics as an intertwined mediating process of communication with others that produces a logically connected discourse about a given subject through the inferring of mathematical utterances from other mathematical utterances.

We agree with O’Halloran (2005) that reasoning and argumentation in mathematics is a mediating activity in which three semiotic resources are functioning together. These three resources are language (both oral and written), symbolism (mathematical symbols and other symbols), and visual imagery (drawings). Symbols and drawings are included in what we call inscriptions. Inscriptions are “understood as artifacts such as graphs, drawings, and mathematical symbols used for cognitive, communicative, mathematical, and problem-solving purposes in interactional settings” (Carlsen, 2009, p. 54). As will be seen later, the PSMTs’ written solutions to a designed mathematical task, exemplify in what ways these semiotic resources may function together.

Several aspects of algebraic reasoning set it apart from other forms of mathematical reasoning (Radford, 2012; Reinhardtsen & Carlsen, 2022). For the purposes of this study, we limit our discussion to what we identify as three central aspects when PSMTs engage with problems in number theory. The first is a structural view of numbers, signs and operations that allow for the identification of general properties, as well as the syntactically guided manipulation of symbolic expressions in solving and proving activities. The structural view also involves a recognition of expressions as general solutions to specific problems and as objects in themselves that can be operated on, substituted or classified (Kieran, 2018). The second aspect is an awareness of generality in mathematical activity (Mason, 1996). It involves a flexible movement between looking at and looking through one object or problem situation, noticing, and expressing what is general and what is particular about it in relation to other relevant objects or situations. The third aspect of algebraic reasoning is a deductive way of arguing for general results, as well as their possibilities and constraints (Mason, 1996). Rather than drawing conclusions from examples, or referencing rules, a deductive argument is founded on what is known to be true (axioms, definitions), or characteristic properties of classes of objects identified while exploring open problems (Yopp & Ely, 2015). Elaborating Radford’s (2012) three steps of a deductive argument in pattern generalization to other open problems such as what can be said about the sum of consecutive numbers, or the sum of odd

and even numbers, we include the following steps: 1) identifying commonalities across the objects $p_1, p_2, p_3, \dots, p_k$, regarding operations on and transformations of these objects; 2) extending these commonalities beyond the perceptual field to objects $p_{k+1}, p_{k+2}, p_{k+3}, \dots$; 3) using the commonalities to formulate general expressions, equalities or inequalities concerning the objects and their transformations; 4) manipulating expressions and formulating arguments in a logical sequence.

Open problems in number theory invite students to use arithmetic and algebra interchangeably as they explore conjectures and engage in processes of refutation and validation. Martinez and Pedemonte (2014) argued that there are two complexities involved when engaging in this activity. The first concerns the syntactic difference between arithmetic and algebra. The study of Lee and Wheeler (1989) highlight students' difficulty in aligning these two domains. They found that Grade 10 students tended to explain results by referring to rules rather than known behaviour of numbers, particularly concerning familiar generalities such as the commutative and the distributive laws. The second complexity concerns the different modes of reasoning involved when generalizing and when arguing for the validity of the generalization. The first mode involves inductive reasoning as one works with and studies examples often using numbers, while the second mode involves deductive reasoning using algebraic symbols. Martinez and Pedemonte (2014) showed that high school students' use of algebraic symbols in the exploration and generalization phase, when working with pattern generalization, provided them with the means to construct a deductive proof, thus providing a bridge between the two modes of reasoning.

Methods

The design of the tasks followed the principles described in Healy and Hoyles (2000), an investigation of the characteristics of the arguments that middle school students recognised as proofs, the reasons behind their judgements and the ways they constructed proofs for themselves. Healy and Hoyles coined three types of tasks: 1) tasks that required students' written descriptions of what they considered to be a proof and its purpose; 2) tasks presenting mathematical conjectures and associated arguments of which students were asked to choose the one that was closest to their own approach, as well as the one that would be graded higher by their professor; and 3) tasks that included a specific, empirical argument which required no further elaboration, an argument relying on common properties or a generic case, a narrative argument, and a deductive proof written formally, presenting a logical argument with explicit links between data and conclusion. Designing the tasks for this study, we chose tasks on number theory and algebra from Mason et al. (2005) and designed them in accordance with the second and third types of tasks described by Healy and Hoyles (2000). The tasks that were created can be separated into three categories seen in Table 1.

In the first phase of the analytical process, we looked through all PSMTs' responses. In the second phase, we identified two answers that encompassed a variety of their responses. Thirdly, the two PSMTs' responses were categorised according to Balacheff's four categories of students' attempts to prove a mathematical statement, *naïve empiricism* – “asserting the truth of a result after verifying several cases” (1988, p. 218), *crucial experiment* – “verifying a proposition on an instance which ‘doesn't come for free’, asserting that ‘if it works here, it will always work’” (1988, p. 219), *the generic example* - verifying a statement through a characteristic example that makes explicit the

reasons for why the statement is true, and *thought experiment* – verifying a statement by a statement that detaches “itself from a particular representation” (1988, p. 219).

Table 1: Examples of the different categories of tasks

Category	Tasks
Review of students' arguments	<p>Task 1: Tim claims that $2(x + 1) = 2x + 1$ and Charlie claims that $2(x+1)=2x+2$.</p> <p>a) Which student is correct?</p> <p>b) Provide two mathematical arguments (for example symbolic, diagrammatic, etc) to explain why the student you chose in the previous question is correct.</p> <p>c) Provide one mathematical argument to explain why the other student's answer is incorrect.</p>
Exploration	<p>Task 2: The following is a sequence of arithmetic statements: $(3 + 2) \cdot (3 - 2) = 3^2 - 4$, $(4 + 2) \cdot (4 - 2) = 4^2 - 4$, $(5 + 2) \cdot (5 - 2) = 5^2 - 4$, (6...</p> <p>a) Provide a general mathematical expression for this sequence.</p> <p>b) Explain how you came up with the expression in question a by using two different arguments (for example symbolic, diagrammatic, etc).</p>
Production of their own argument	<p>Task 7: a) Can the number $2^3 \cdot 3^2 \cdot 5 \cdot 7$ be divided by 105? Provide an argument to explain why or why not.</p> <p>b) Is the number 105 divisible by 15? Provide an argument to explain why or why not.</p>

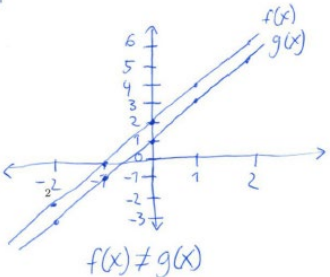
Results

We will present and analyse the solutions for Task 1 (see Table 1), provided by two participants: Eric and Gavin. Based on our analysis of all the PSMTs' responses, these two stood out as being typical and illustrative, documenting the PSMTs' responses to one of the designed tasks. Their answers can be seen in Table 2 and Table 3 below.

Eric used several forms of argumentation in his response to Task 1. Regarding question *a*, he referred to a rule to explain his choice in a narrative argument. When asked to provide two arguments for why the transformation is correct, question *b*, Eric first provided an argument by performing the same transformation on a more general expression. According to Eric, the transformation of the more general expression (with a cloud instead of 1) showed that $2x+2$ is correct as it is a special case of the more general expression. Although the argumentation is deductive in nature, it used the transformation to be validated as a foundation for the argument and thus it failed in proving it.

Eric's second argument involved specializing, or in other words naïve empiricism, as he evaluated both sides of the equal sign for multiple values of x , skilfully choosing different numbers (zero, negative numbers, fraction), recognizable strategies for exploring patterns used in the course. However, using these did not bring Eric closer to proving that the transformation is correct. In order to show that the transformation $2(x+1) = 2x+1$ is incorrect, question *c*, he provided an argument that exhausts all possible values for x , by using a graphical representation, as the two expressions form two parallel lines (see Table 2), not considering the simple solution of providing one counterexample.

Table 2: Eric's responses to Task 1

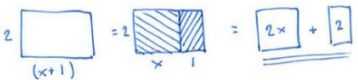
Eric's response	Interpretative explanation																																							
<p>a) Charlie har rett da to'eren utenfor parentesen skal multipliseres med begge ledd inni parentesen. Tim har kun multiplisert det ene leddet med to.</p> <p>b) $2(x+1) = 2 \cdot x + 2 \cdot 1 = 2x + 2$ som tilsvarer $2x + 2$ $f(x) = 2(x+1)$, $g(x) = 2x + 2$ utregning eks: $f(\frac{3}{2}) = 2(\frac{3}{2} + 1) = 2(\frac{3}{2} + \frac{2}{2}) = 2(\frac{5}{2}) = \frac{10}{2} = 5$ $g(\frac{3}{2}) = 2 \cdot \frac{3}{2} + 2 = \frac{6}{2} + \frac{4}{2} = \frac{10}{2} = 5$</p> <table border="1" data-bbox="172 593 507 689"> <tr><td>x</td><td>-2</td><td>-1</td><td>0</td><td>$\frac{3}{2}$</td><td>4</td><td>2,5</td></tr> <tr><td>f(x)</td><td>-2</td><td>0</td><td>2</td><td>5</td><td>10</td><td>52</td></tr> <tr><td>g(x)</td><td>-2</td><td>0</td><td>2</td><td>5</td><td>10</td><td>52</td></tr> </table> <p>c) $f(x) = 2(x+1)$, $g(x) = 2x + 1$</p> <table border="1" data-bbox="183 784 395 896"> <tr><td>x</td><td>-2</td><td>-1</td><td>0</td><td>1</td><td>2</td></tr> <tr><td>f(x)</td><td>-2</td><td>0</td><td>2</td><td>4</td><td>6</td></tr> <tr><td>g(x)</td><td>-3</td><td>-1</td><td>1</td><td>3</td><td>5</td></tr> </table> 	x	-2	-1	0	$\frac{3}{2}$	4	2,5	f(x)	-2	0	2	5	10	52	g(x)	-2	0	2	5	10	52	x	-2	-1	0	1	2	f(x)	-2	0	2	4	6	g(x)	-3	-1	1	3	5	<p>a) Eric chose the student he agrees with and explained his choice. He explained that the transformation $2(x+1) = 2x+2$ is correct by referring to rules of operations concerning multiplication and parenthesis. He rejected the transformation $2(x+1) = 2x+1$, explaining that the second part of the expression in the parenthesis was not multiplied by 2. For Erik, a calculational rule was sufficient to explain his choice.</p> <p>b) Eric provided two arguments. In the first, he changed the expression replacing 1 with a sign (a cloud) and transformed the expression. Then he wrote that the result corresponds to $2x+2$ (we interpret that Eric by this statement see $2x+2$ as a special case of $2x+2(\text{cloud})$). The second argument comprises functional representations.</p> <p>c) Eric did the same with the transformation he rejected and created a graph of the two functions.</p>
x	-2	-1	0	$\frac{3}{2}$	4	2,5																																		
f(x)	-2	0	2	5	10	52																																		
g(x)	-2	0	2	5	10	52																																		
x	-2	-1	0	1	2																																			
f(x)	-2	0	2	4	6																																			
g(x)	-3	-1	1	3	5																																			

Gavin used various representations in his arguments. His approaches included algebraic and geometrical arguments and throughout his response, he mostly maintained a generalized approach and used deductive reasoning. In question *a*, he used symbolic representations to explain why Charlie gave the correct answer, seemingly basing his argument on the distributive law.

However, his argument does not reveal why the distributive law works. For question *b*, Gavin provided a deductive argument that falls in the category of a thought experiment. By visualizing the expression as an area, he was able to separate it into two different parts and show how the expression can be transformed. This argument was explanatory of the connections between the elements in the operation and followed the same logical steps as his argument in question *a*.

For his second argument, he used deductive proof recognizing a connection between the operations. He was able to move from the structure of multiplication to that of addition and he proved the statement using properties. For question *c*, he used a different approach. He used a formal representation as well, without following deductive reasoning. He interpreted the equivalence sign to mean that the two sides of the equation should be equal to be written in the same way. While he used algebra to show that the two sides cannot be written in the same way, he did not realize that he used the distributive law in his calculations rendering his argument incorrect. At the same time, even though he was comfortable with working in generalized expressions throughout his response, he could not go into specification to prove that the expression is incorrect. However, he was able to generalize and think in an algebraic way and did not resort to the use of examples in his answer.

Table 3: Gavin's responses to Task 1

Gavin's response	Interpretative explanation
<p>a) Jeg regner ut for å se hvem jeg er enig med</p> $2(x+1)$ $2 \cdot (x+1)$ $= 2x+2$ <p><u>Charlie har rett</u></p> <p>b) Utrykket som et areal</p>  <p>Aritmetisk</p> $2(x+1) = (x+1) + (x+1) = x+x+1+1 = 2x+2$ <p>c) faktorisering 2x+1, med 2 uten for parentes</p> $2x+1 = 2(x+\frac{1}{2})$ $2(x+\frac{1}{2}) \neq 2(x+1)$ <p>Dvs</p> <p><u>2x+1 ikke er en måte å skrive 2(x+1) på</u></p>	<p>Gavin used arrows to emphasize that the number outside the parenthesis had to be multiplied with both elements inside the parenthesis.</p> <p>For his first argument, he drew a rectangle and wrote “the expression as an area”. He proceeded to show that it can be split in two smaller rectangles.</p> <p>He wrote “factorizing 2x+1 with 2 outside of the parenthesis” and shows that it cannot be written as 2x+1. Therefore, he reached the conclusion that Tim’s answer is not correct.</p>

Discussion

We set out in this study to come up with answers to the research question: In what ways do designed mathematical tasks promote PSMTs’ algebraic reasoning and argumentation? As revealed in the analyses, we observe that the PSMTs use several argumentations in their engagement with the designed task. These argumentations were indeed promoted by the designed task, due to its formulation and the context in which it occurred.

The designed tasks elicited various arguments including referencing rules, naïve empiricism, deduction and thought experiment (cf. Balacheff, 1988), as well as the use of narratives, geometric, functional, and algebraic representations. Thus, the PSMTs took advantage of these semiotic resources to make them function together (O’Halloran, 2005). These arguments provide some evidence of PSMTs’ algebraic reasoning or lack thereof (cf. Jeannotte & Kieran, 2017). Eric showed a structural view in that he went from considering an equality to considering the expressions as equal functions. However, the transformation under consideration is founded on properties of operations he does not recognize. In contrast, Gavin manages to make a connection between a multiplicative and an additive structure. Both PSMTs stayed within the algebraic domain and did not resort to arithmetic although that would be a more efficient approach for refuting the incorrect transformation. However, Gavin and Eric’s approaches agree with the results of Lee and Wheeler (1989), who also found that students quoted a rule while trying to explain why an algebraic statement was true.

Through this task, Eric’s difficulty crossing from arithmetic to algebra became evident. He realizes the need for generality, but uses an unorthodox way to pursue it, raising the question if he perceives algebraic proof as generalized arithmetic (cf. Martinez & Pedemonte, 2014). In both PSMTs’ cases, it becomes clear that they do not explicitly question the mathematical rules. On the contrary, they use

them in place of a deductive argument (cf. Mason, 1996; Radford, 2012) as a way of validation. This result is different from what Martinez and Pedemonte (2014) found in their study. The high school students in their study used algebraic symbols as means to construct a deductive proof, thus providing a bridge between the arithmetic and the algebraic modes of reasoning. As revealed in the analyses, the bridging between these modes of reasoning was blurry in the preservice students' responses.

The students used approaches that were predicted, such as deductive proof and geometrical representation. However, they used unique approaches as well, such as manipulating one side of the equation to look like the other, using functions as a way to generalize using inscriptions instead of variables and proving by example. The latter was also prevalent in the findings of both Ko and Rose (2021) and Kempen and Biehler (2019). In terms of aspects of algebraic reasoning, steps 2, 3 and 4 are evident in the students' answers. Eric used examples to identify commonalities and generalize. They both try to manipulate the expressions provided in order to formulate logical arguments.

References

- Ayalon, M., & Even, R. (2014). Factors shaping students' opportunities to engage in argumentative activity. *International Journal of Science and Mathematics Education*, 14(3), 575–601. <https://doi.org/10.1007/s10763-014-9584-3>
- Balacheff, N. (1988). Aspects of proof in pupils' practice of school mathematics. In D. Pimm (Ed.), *Mathematics, teachers and children* (pp. 216–235). Hodder & Stoughton.
- Bersch, S. (2019). Teachers' perspectives on mathematical argumentation, reasoning and justifying in calculus classrooms. In U. T. Jankvist, M. Van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Proceedings of the Eleventh Congress of the European Society for Research in Mathematics Education* (pp. 128–135). Utrecht, the Netherlands: Freudenthal Group & Freudenthal Institute, Utrecht University and ERME.
- Carlsen, M. (2009). Reasoning with paper and pencil: The role of inscriptions in student learning of geometric series. *Mathematics Education Research Journal*, 21(1), 54–84. <https://doi.org/10.1007/BF03217538>
- Douek, N. (1999). Some remarks about argumentation and mathematical proof and their educational implications. In I. Schwank (Ed.), *Proceedings of the First Conference of the European Society for Research in Mathematics Education (CERME 1, August 1998)* (pp. 125–139). Osnabrück, Germany: Forschungsinstitut für Mathematikdidaktik and ERME.
- Healy, L., & Hoyles, C. (2000). A study of proof conceptions in algebra. *Journal for Research in Mathematics Education*, 31(4), 396–428. <https://doi.org/10.2307/749651>
- Jeannotte, D., & Kieran, C. (2017). A conceptual model of mathematical reasoning for school mathematics. *Educational Studies in Mathematics*, 96(1), 1–16. <https://doi.org/10.1007/s10649-017-9761-8>
- Kempen, L., & Biehler, R. (2019). Pre-service teachers' benefits from an inquiry-based transition-to-proof course with a focus on generic proofs. *International Journal of Research in Undergraduate Mathematics Education*, 5, 27–55. <https://doi.org/10.1007/s40753-018-0082-9>

- Kieran, C. (2018). Seeking, using, and expressing structure in numbers and numerical operations: a fundamental path to developing early algebraic thinking. In C. Kieran (Ed.), *Teaching and learning algebraic thinking with 5- to 12-year-olds: The global evolution of an emerging field of research and practice* (pp. 79–105). Springer. https://doi.org/10.1007/978-3-319-68351-5_4
- Ko, Y.-Y., & Rose, M. K. (2021). Are self-constructed and student-generated arguments acceptable proofs? *Pre-service secondary mathematics teachers' evaluations*. *Journal of Mathematical Behavior*, 64, 1–15. <https://doi.org/10.1016/j.jmathb.2021.100912>
- Lee, L. & Wheeler, D. (1989). The arithmetic connection. *Educational Studies in Mathematics*, 20, 41–54. <https://doi.org/10.1007/BF00356040>
- Martinez, M. V., & Pedemonte, B. (2014). Relationship between inductive arithmetic argumentation and deductive algebraic proof. *Educational Studies in Mathematics*, 86(1), 125–149. <https://doi.org/10.1007/s10649-013-9530-2>
- Mason, J. (1996). Expressing generality and the roots of algebra. In N. Bednarz, C. Kieran, & L. Lee (Eds.), *Approaches to algebra: Perspectives for research and teaching* (pp. 65–86). Kluwer Academic Publishers. https://doi.org/10.1007/978-94-009-1732-3_5
- Mason, J. (2006). What makes an example exemplary: Pedagogical and didactical issues in appreciating multiplicative structures. In R. Zazkis & S. R. Campbell (Eds.), *Number theory in mathematics education* (pp. 41–68). Routledge.
- Mason, J., Graham, A., & Johnston-Wilder, S. (2005). *Developing thinking in algebra*. Paul Chapman Educational Publishing.
- NCTM (2000). *Principles and standards for school mathematics*. National Council of Teachers of Mathematics.
- O'Halloran, K. (2005). *Mathematical discourse. Language, symbolism and visual imagery*. Continuum.
- Radford, L. (2012). Early algebraic thinking: Epistemological, semiotic, and developmental issues. In S. J. Cho (Ed.), *The Proceedings of the 12th International Congress on Mathematical Education: Intellectual and Attitudinal Challenges* (pp. 209–227). Springer Open eBooks. https://doi.org/10.1007/978-3-319-12688-3_15
- Reinhardtsen, J., & Carlsen, M. (2022). Computational and analytical perspectives in introductory algebra: a theoretical contribution. In J. Hodgen, E. Geraniou, G. Bolondi, & F. Feretti (Eds.), *Proceedings of the Twelfth Congress of the European Society for Research in Mathematics Education (CERME12)* (pp. 596–603). ERME / Free University of Bozen-Bolzano.
- van Eemeren, F. H., & Grootendorst, R. (2003). *A systematic theory of argumentation: the pragma-dialectical approach*. Cambridge University Press. <https://doi.org/10.1017/CBO9780511616389>
- Yopp, D., & Ely, R. (2016). When does an argument use a generic example? *Educational Studies in Mathematics*, 91(1), 37–53. <https://doi.org/10.1007/s10649-015-9633-z>