



HAL
open science

The work done when solving linear equations: Altering experiences due to change of numbers

Anna Holmlund

► **To cite this version:**

Anna Holmlund. The work done when solving linear equations: Altering experiences due to change of numbers. Thirteenth Congress of the European Society for Research in Mathematics Education (CERME13), Alfréd Rényi Institute of Mathematics; Eötvös Loránd University of Budapest, Jul 2023, Budapest, Hungary. hal-04417698

HAL Id: hal-04417698

<https://hal.science/hal-04417698v1>

Submitted on 25 Jan 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

The work done when solving linear equations: Altering experiences due to change of numbers

Anna Holmlund

The University of Gothenburg and Chalmers University of Technology, Sweden;

elanna@chalmers.se

This paper investigates how one student solves linear equations with a focus on how different numbers (e.g., natural numbers, numbers with decimals, and negatives) in the equations influence reasoning. The choice of coefficients in equations is not usually the focus in research nor in teaching. However, as students are used to taking numbers into consideration, the influence of coefficients on equation solving is an important topic to address. In a previous study, a model for how students experience equations was suggested. Here, the model is used to analyse one interview with a student in upper-secondary school, solving several similar equations but with numbers from varying number domains. The results show the decisive role that numbers play in how this student perceives the equations. Using a wide range of numbers also helped to challenge the student to explore the properties of the equations' structure. Possible applications in the classroom are discussed.

Keywords: Linear equations, mathematics education, coefficients, phenomenography.

Introduction

This paper emanates from a research project investigating equation solving and especially students' attention to numbers in their reasoning about equations. The aim is to investigate how students work with interpreting equations based on their previous experiences and what influence the numbers in the equation have on their reasoning. Underlining the role of numbers in equation solving, a perspective that is not as often fronted (Kieran, 2022; Vlassis & Demonty, 2022), can provide teachers with essential knowledge on how to support students' learning of algebra.

Background

In developing algebraic thinking, students should learn to see through mathematical objects, such as numbers, to distinguish patterns (Kieran, 2018). Mason (1996) suggested that students need to shift their way of seeing the numbers as particular quantities to compute and rather anticipate the properties of numbers. To encourage this way of thinking, one possible way is to avoid the initial use of numbers – as not to encourage calculation – but to promote thinking of structures (Tuominen et al., 2018). Another possibility is to use the choice of numbers systematically. Zazkis (2001) used numbers beyond the range of mental calculation to make a student reason about structure instead of calculating the result. It is, therefore, interesting to consider how the choice of numbers influences students in their equation solving and whether numbers with decimals and negatives could also be used by teachers to develop students' awareness of structure, form, and relations.

Concerning equation solving, it has been documented that the kind of numbers used for coefficients influence how equations are perceived by students (Holmlund, 2022). Numbers affect what students can imagine that the equations represent (Hackenberg & Lee, 2015; Vlassis, 2002, 2008) or what transformations they see as possible (Linchevski & Livneh, 1999; Vlassis & Demonty, 2022). Despite

this, the role of numbers in equations usually lies in the background in research concerning equation solving (Kieran, 2022; Vlassis & Demonty, 2022). Instead, the choice of strategy is a common topic in research (Vlassis, 2002). However, studies show that students may struggle in algebra even though they are comfortable with a strategy such as “doing the same things on both sides” (Andrews & Öhman, 2019). Perhaps an analytic strategy only gives students a way to start reasoning (Threlfall, 2002), but students also need to control and justify the solution process (Balacheff, 2001). Transforming an equation can be seen as a semantic process – thinking of a concrete representation or model for what happens – or a syntactic process – with a focus on the symbols that are operated on (Balacheff, 2001; Carraher & Schliemann, 2007). Seeing the relations semantically or syntactically will reflect the structure of the equation, i.e. an internal order between quantities and operations that can be seen before or after transformation (Hoch & Dreyfus, 2004). Either with a semantic or syntactic meaning, previous experiences of numbers and operations can influence how equations are perceived (Holmlund, n.d.).

To further explore the idea of numbers as relevant in algebraic thinking in the setting of equation solving, the following research questions are posed:

- What work is performed by a student when solving equations? How do learnt strategies and meaning-making interplay when coefficients are changed?
- How can a wide range of numbers contribute to the student’s ability to discern structure in a more qualitative way?

Method

The data was gathered in the autumn and winter of 2020/21. As the purpose concerned the influence of numbers as students solve equations, the idea was to let students solve similar equations but with different coefficients – with the help of a calculator – in interviews. 111 students in their first year in vocational education (when they turn 16 years old) in a Swedish school took a test and a few weeks later 23 of these students were interviewed – selected to represent a wide range of ways of experiencing equations based on their test answers. Solving linear equations is a mandatory learning goal in compulsory school, so all the students had practised this topic earlier. The test was arranged with 10 equations to act as a selection instrument for the interviews. The equations had three different structures ($a = b \cdot x$, $a = b + x$, and $a = b \cdot x + c$) and included either only natural numbers, numbers with decimals, or negative numbers. Some of these equations were then used in interviews, presented on a piece of paper to probe students’ ways of experiencing the equations, as described further by Holmlund (n.d.). In both the test and interviews, the students had a calculator and had on both occasions shown that they could use it. In this paper, one of these student interviews is analysed. This interview was chosen as the student, who will be called Emil, expressed his thinking rather clearly and was one of the students who exhibited greater variation in ways of experiencing equations of similar structure. The interviews were recorded with a video camera, only catching the writing and recording the voices. The recordings were then transcribed verbatim and translated from Swedish to English. The process of translation was guided by the intent of Emil’s utterances.

In the analysis of the interview, a framework with five categories – describing students’ ways of experiencing equations in qualitatively different ways – is used. This framework was created in a

phenomenographic analysis of the same data as used in this paper (Holmlund, n.d.). The phenomenographic research tradition focuses on the second-order perspective, how people perceive phenomena (Marton, 1981; Marton & Booth, 1997). Research has shown that there is a limited number of ways to experience a phenomenon within a group of people. By describing how these ways of experiencing are constituted, differences between conceptions can be identified and used in teaching to help students discern the phenomenon in a more developed way (Marton, 2014).

The five ways of experiencing equations while solving are displayed in Table 1 (Holmlund, n.d.). Conceptions E and D both describe experiences where the meaning of the equation is unclear. While conception E describes a focus on how the numbers work, conception D focuses on the operation as most important. Conceptions C, B and A describe the equation as a relation. In conception C, the equation is seen as having a concrete representation, a semantic meaning. Experiencing the equation as conception B, the focus is instead on the syntactics of the symbols and numbers – how they can be handled rather than what they represent. The more refined way of seeing the equations (conception A) focus on structure, and numerical values are seen as less important (Holmlund, n.d.).

Table 1: Five ways of experiencing equations while solving (Holmlund, n.d.), where previous experiences of numbers have a decreasing influence (from E to A)

The linear equation experienced as...	Meaning discerned in the structure	Aspect in foreground
A. ...a relation between operations and elements	Syntactic meaning	Structure
B. ...a relation between numbers	Syntactic meaning	Numbers and structure
C. ...a relation that must represent a situation	Semantic meaning	Numbers and structure
D. ...numbers to compute	Searches for a meaning	Operation
E. ...an unfamiliar equation	Searches for a meaning	Numbers

In the analysis of the interview data, the following questions were addressed: What meaning does the student see in the equation, and what does he focus on? How does the student relate his experience of the equation to a potential strategy? How do the numbers in the equation impact this work? The interview was analysed repeatedly and compared to the conceptions in Table 1. In the analysis below, the reference for how the student is presently experiencing the equation will be written in parentheses. References to lines in the transcript will also be written in parentheses.

Results

First, Emil's solution to $0.12 = 0.4 \cdot x$ in the test, is displayed in Figure 1. It shows a clear distinction between how he, at that moment, interpreted a multiplicative equation including numbers with decimals differently from other equations. Emil consequently converted numbers with decimals in the test before dividing them, despite having a calculator. Multiplicative equations with natural

numbers respectively negative numbers were solved with division directly on all but one occasion. We can see that Emil knows how to balance an equation by doing the same on both sides.

The image shows three handwritten equations in a box. The first equation is $0,12 = 0,4 \cdot x$ with a -10 written below the 0,12 and another -10 written below the 0,4. The second equation is $1,2 = 4 \cdot x$. The third equation is $\frac{1}{2} = \frac{4x}{4}$. Below these equations, the solution $x = 0,3$ is written.

Figure 1: Emil consequently transformed numbers with decimals in multiplicative equations in the test

In the interview, Emil does not come to think of multiplying the equation with ten as he did when he took the test but approaches the equation in another way.

First part of the interview: guided by a learnt strategy

Seeing the equations, Emil initially comes to think of a *strategy* for how to approach the equation but ignores the operation. He can not explain why his strategy would work and gets confused.

- 1 Interviewer What do you think when you see this? [shows $0.12 = 0.4 \cdot x$]
- 2 Student Hm, I think that I want to get x alone here, so I take minus 0.4 there and minus 0.4 here. Then I get x alone and then it is x equals 0.8.
- 3 Interviewer All right, and how do you know that you should take minus?
- 4 Student Eh, but I have learned that you should try to get x alone on one side. So, that's why actually. How come that is, I'm not sure...

[Next the interviewer shows $2.01 = 0.434 + x$]

- 8 Student Eh, then I think the same as the last one, or no, now it is plus instead of times. My gosh, eh... Then I don't really know how I think.
- 9 Student [...] My first thought was really that I should remove that [points at 0.434] but I don't know if it is okey for plus. For it is plus... No, I'm not sure.

[Emil takes a while to sit and think about the equation. Finally, he decides to try subtraction on both sides, even though he is not sure.]

The first equation is first seen as unfamiliar (Conception E) as he has no justifications as to why the procedures are performed. His focus is on the numbers and most of all on the x. The operation comes to the foreground when an additive equation is presented after the multiplicative, but still Emil is not sure how the operations work with the strategy to “remove” the coefficient (9) (Conception D).

On several occasions Emil mentions that he has a strategy to “get x alone”/”try to remove that” combined with that he does “not know exactly” (4 and 9). He is convinced that the strategy is the right way to proceed, but he also needs some way of justifying his actions, a meaning. This displays the work Emil is presently engaged in; finding a meaning that aligns with the strategy. The numbers with decimals in the equations do not help him discern such a meaning, they do not hint him of any common number combination or concrete situation.

Second part of the interview: struggling with similarities and differences

Continuing the interview, $42 = 3 \cdot x$ is displayed and now Emil discerns a meaning in the right-hand side of the equation:

- 10 Student Mhm

- 11 Interviewer What are you thinking now?
- 12 Student I did the other one wrong, I also noticed.
- 13 Interviewer Mhm.
- 14 Student For now I remembered something, or no. What I learned last year in school was that when it says ‘three times x’ then it is three x:es. And then you can take three x divided by three to remove the three. Then you do the same thing on the other side and then you find what x is.
- [Emil makes the calculation on the calculator, checks that the answer is correct and then turns to $0.12 = 0.4 \cdot x$.]
- 15 Student [...] I was going to do the same as I did on this [points at $42 = 3 \cdot x$] that I take... or wait, now I must think [uses calculator]. Is it zer... or no, it’s one.
- 16 Interviewer Right, you typed 0.4 divided by 0.4.
- 17 Student Mm... No, I don’t want to do that one again. Or do I have to?
- 18 Interviewer Yes, please [...] what did you think when you saw that result?
- 19 Student My brain stopped.
- 20 Interviewer Right, because you got one there.
- [Emil once again considers the same method that he initially used, subtracting both sides with 0.4. After trying out his result 0.8 in the equation he concludes that it is not correct.]
- 21 Student So then you can’t subtract there. Eh, then I must, it feels as if I must divide, or wait, can you take... no, no [thinks a while]
- 22 Interviewer For here you divided [points at $42 = 3 \cdot x$] and now you are a bit insecure what you should do with this [points at $0.12 = 0.4 \cdot x$]. What is it...?
- 23 Student Yes, but I don’t know if I can divide 0.4 with 0.4 because I want to remove 0.4.
- 24 Interviewer Why are you unsure about that?
- 25 Student Mm... I don’t know because I haven’t... It feels as if I haven’t worked with, or, I have done these in school [points at $42 = 3 \cdot x$]. It doesn’t feel as if I have done these really [points at $0.12 = 0.4 \cdot x$] [...] or I might have done it but I don’t remember in that case.
- 26 Interviewer And dividing 0.4 with 0.4, that feels... you don’t want to do that?
- 27 Student No
- 28 Interviewer Why?
- 29 Student For when I typed 0.4 divided with 0.4, then it is one and then I’m unsure if... But if we take three divided by three [types it on the calculator], then that is also one. Could I do something like that maybe? I will try.
- [Emil makes the calculation on the calculator and concludes that it was correct. He also acknowledges the similarities with the other equation.]

When Emil comes to think of three x:s as symbolizing three units (14), dividing these units is seen as splitting or sharing. Distributing three x:es in three sections gives no leftovers, which explains how the three is “removed”. The equation now makes semantical sense (conception C) to Emil as he can see the transformation concretely. He wants to transfer this meaning to the first equation ($0.12 = 0.4 \cdot x$). However, this fails as he expects splitting a number of things should result in a zero, not “a one” (15). As the meaning is lost, he still experiences this equation as unknown (Conception E) trying the strategy of removing x by subtraction once again. The fact that the equations are experienced in different ways becomes clear when he refers to the equation as unfamiliar (25). Finally, he realizes that three divided by three also is one, that “removing” the three is in this way not a concrete, but a mathematical action (29) which works for any equation (Conception B).

Emil seems to be encouraged in his exploration by the similarities he notices between the operations in the different equations, mentioning that he can “do the same” (15 and 29). The different ways of experiencing the equations (conceptions E – B) are compared to how well they align with the strategy of “removing” the coefficient. The numbers in the equations have a central role in his work to combine a meaning with the strategy. On the one hand, the numbers keep him from seeing the similarity between the equations because division is seen as sharing, which is not as intuitive for numbers with decimals. On the other hand, it is also the varying range of numbers that challenges his conception of the equations so that it evolves. Comparing his different ways of experiencing the equations, we can see that learnt strategies and meaning interplay to varying degree depending on what coefficients that are in the equation.

Third part of the interview- Ignoring numerical values

Finally, the way the numbers are experienced in the equation evolves.

30 Interviewer [Shows $-24 = 6 \cdot x$] What do you think of this one?

31 Student My gosh, eh... [thinks a while] Okey, so six times something and then it becomes -24. Ehm, [swears], we did this last week when we worked with the minus sign, so... [thinks]. But I think that x... now I don't know if this will work, but I think if I take -24 equals six times x, then I divide with six here and divide by six there. Then I get x equals...

[He performs the calculation on the calculator and obtains a correct answer.]

32 Student Then I actually thought the same way as I did with the others. Because I really just skipped that there was a minus sign there, considering that it should work out anyway I thought. So, then x is negative four.

[In the end of the interview, Emil solves $2.5 = 0.8 \cdot x + 2.1$ correctly without giving extra attention to the numbers.]

At first glance the negative number in $-24 = 6 \cdot x$ comes to the foreground of his attention, making the equation seem unfamiliar (Conception E) (31). However, he soon comes to regard the negative number as any number, he “skipped that there was a minus sign there, considering that it should work out anyway” (32). In using the calculator, the role of the number is reduced, and he experiences as conception B, but also more like conception A, where the numbers are in the background of attention.

In this excerpt, Emil comes to perceive the numerical values as less important, and the strategy and the syntactic meaning he sees fit well together. He has gained confidence in the syntactic meaning, that it should “work out” and he can handle the equations the same way as he did with the others (32).

Discussion

This research presents a student's work while solving equations, especially focusing on the role of numbers. Firstly, the interview shows that the student has learned a strategy to “do the same things on both sides” but struggles to “remove” numbers when they include decimals due to division being interpreted as “sharing units”. Throughout the interview he tries to see how well the strategies align with the meaning of the equation. This confirms Threlfall's (2002) suggestion that a strategy only gives students a way to start a solution. To continue, this strategy needs to align with the meaning experienced (either semantic or syntactic), which could potentially explain why students do not succeed in algebra even though they know strategies for equation solving (Andrews & Öhman, 2019). From the analysis we can also see how the student experiences multiplicative equations in varying

ways depending on what numbers the coefficients are (shifting between E, D, C, E, B, E and B). We can therefore conclude that students' experiences of numbers are relevant in their work with solving equations (Hackenberg & Lee, 2015; Holmlund, n.d.; Vlassis, 2002; Vlassis & Demonty, 2022).

Secondly, the results support that a structural view of numbers can be reached by a systematic variation of numbers (Zazkis, 2001). From the interview we can see that numbers with decimals and negative numbers do not "hint" of a concrete or rehearsed number combination in the same way that low natural numbers do. Instead, this student was encouraged to look for mathematical properties (e.g., any number divided with itself is one) and to ignore the sign of the numbers. Varying numbers in algebraic teaching in a systematic way could be a powerful tool to develop students' ways of experiencing equations and requires further research. This could help students to "see through" the numbers (Kieran, 2018) and experience the structure of equations in a more refined way.

References

- Andrews, P., & Öhman, S. (2019). Swedish upper secondary students' understanding of linear equations: an enigma? *Acta Didactica Napocensia*, 12(1), 117–129. <https://doi.org/10.24193/adn.12.1.8>
- Balacheff, N. (2001). Symbolic arithmetic vs algebra the core of a didactical dilemma. In R. Sutherland, T. Rojano, A. Bell, & R. Lins (Eds.), *Perspectives on school algebra* (pp. 249–260). Kluwer.
- Carraher, D. W., & Schliemann, A. D. (2007). Early algebra and algebraic reasoning. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics* (Vol. 2, pp. 669–705). Information Age Pub.
- Hackenberg, A. J., & Lee, M. Y. (2015). Relationships between students' fractional knowledge and equation writing. *Journal for Research in Mathematics Education*, 46(2), 196–243. <https://doi.org/10.5951/jresmetheduc.46.2.0196>
- Hoch, M., & Dreyfus, T. (2004). Structure sense in high school algebra: The effect of brackets. In M. J. Høines & A. B. Fuglestad (Eds.), *Proceedings of the 28th Conference of the International Group of Psychology of Mathematics Education* (Vol. 3) (pp. 49–56). PME.
- Holmlund, A. (2022). Different ways of experiencing linear equations with a multiplicative structure. In G. A. Nortvedt, N. F. Buchholtz, J. Fauskanger, M. Hähkiöniemi, B. E. Jessen, M. Naalsund, H. K. Nilsen, G. Pálsdóttir, P. R. i. Portaankorva-Koivisto, J., J. Ö. Sigurjónsson, O. Viirman, & A. Wernberg (Eds.), *Bringing Nordic mathematics education into the future. Proceedings of Norma 20. The ninth Nordic Conference on Mathematics Education, Oslo, 2021* (pp. 89–96). SMDF.
- Holmlund, A. (n.d.). *How numbers influence students solving linear equations*. Unpublished manuscript.
- Kieran, C. (2018). Seeking, using, and expressing structure in numbers and numerical operations: A fundamental path to developing early algebraic thinking. In C. Kieran (Ed.), *Teaching and learning algebraic thinking with 5- to 12-year-olds: The global evolution of an emerging field of research and practice* (pp. 79–105). Springer International Publishing. https://doi.org/10.1007/978-3-319-68351-5_4

- Kieran, C. (2022). The multi-dimensionality of early algebraic thinking: background, overarching dimensions, and new directions. *ZDM – Mathematics Education*, 54(6), 1131–1150. <https://doi.org/10.1007/s11858-022-01435-6>
- Linchevski, L., & Livneh, D. (1999). Structure sense: The relationship between algebraic and numerical contexts. *Educational Studies in Mathematics*, 40(2), 173–196. <https://doi.org/10.1023/A:1003606308064>
- Marton, F. (1981). Phenomenography — Describing conceptions of the world around us. *Instructional Science*, 10(2), 177–200. <https://doi.org/10.1007/BF00132516>
- Marton, F. (2014). *Necessary conditions of learning*. Routledge. <https://doi.org/10.4324/9781315816876>
- Marton, F., & Booth, S. (1997). *Learning and awareness*. Routledge. <https://doi.org/10.4324/9780203053690>
- Mason, J. (1996). Expressing generality and roots of algebra. In N. Bernarz, C. Kieran, & L. Lee (Eds.), *Approaches to algebra: Perspectives for research and teaching* (pp. 65–86). Springer. https://doi.org/10.1007/978-94-009-1732-3_5
- Threlfall, J. (2002). Flexible mental calculation. *Educational Studies in Mathematics*, 50(1), 29–47. <https://doi.org/10.1023/A:1020572803437>
- Tuominen, J., Andersson, C., Bjorklund Boistrup, L., & Eriksson, I. (2018). Relate before calculate: Students' ways of experiencing relationships between quantities. *Didactica Mathematicae*, 40, 5–33.
- Vlassis, J. (2002). The balance model: Hindrance or support for the solving of linear equations with one unknown. *Educational Studies in Mathematics*, 49(3), 341–359. <https://doi.org/10.1023/A:1020229023965>
- Vlassis, J. (2008). The role of mathematical symbols in the development of number conceptualization: The case of the minus sign. *Philosophical Psychology*, 21(4), 555–570. <https://doi.org/10.1080/09515080802285552>
- Vlassis, J., & Demonty, I. (2022). The role of algebraic thinking in dealing with negative numbers. *ZDM – Mathematics Education*, 54(6), 1243–1255. <https://doi.org/10.1007/s11858-022-01402-1>
- Zazkis, R. (2001). From arithmetic to algebra via big numbers. In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *Proceedings of the 12th ICMI Study Conference: The future of the teaching and learning of algebra, 2001*. (Vol. 2, pp. 676–681). The University of Melbourne.