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► To cite this version:

Vilma Mesa, Eric Khiu, Saba Gerami, Tom Judson. Conceptions of spanning sets and linear independence emerging from examples and student responses to reading questions in an interactive linear algebra textbook. Thirteenth Congress of the European Society for Research in Mathematics Education (CERME13), Alfréd Rényi Institute of Mathematics; Eötvös Loránd University of Budapest, Jul 2023, Budapest, Hungary. hal-04417611

HAL Id: hal-04417611

<https://hal.science/hal-04417611>

Submitted on 25 Jan 2024

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Conceptions of spanning sets and linear independence emerging from examples and student responses to reading questions in an interactive linear algebra textbook

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Using Balacheff's (2013) model of conceptions we analyzed textbook examples in two sections that modeled the mathematical work needed to answer two reading questions and used the intended conceptions to identify control structures in student responses to those questions. Reading questions are an interactive textbook feature meant to entice students to read the textbook before attending the lesson when such ideas will be discussed; as students provide responses in their interactive textbook, the instructors can learn about how students are thinking about the content before a lesson. We found additional control structures, which suggest additional conceptions beyond the ones promoted in the textbook. We discuss implications for designing these types of questions in interactive textbooks.

Keywords: Conceptions, linear algebra, interactive textbooks, spanning sets, linear independence.

We continue our investigation of what can be learned from a set of student responses to reading questions collected via interactive textbooks. In an earlier exploratory analysis (Mesa et al., under review), we worked with two practitioners to classify student responses along two dimensions: the validity of the reasoning exhibited, and the language used. For the analyses presented here, we take a step further and rely on Balacheff's (2013) cKc model of conceptions to investigate conceptions of spanning sets and linear combinations avowed by the textbook and contrast those with the ones emerging from the student responses. This work is undertaken to explore the affordances of analysis of large sets of student responses to open-ended questions that can inform teaching and learning.

Theoretical framing

Balacheff's model assumes that meaning derives from the system that encompasses a milieu and a cognizant subject; the meaning is not inscribed in either of them alone, but is created through the interactions between the two, that is, the actions of the subject on the milieu and the feedback the milieu provides to the subject (Balacheff, 2013; Balacheff & Gaudin, 2009). His work responds to three needs related to investigating student thinking (1) explaining why particular held conceptions of a notion that appear contradictory to an observer may not appear so to the holder of the conceptions; (2) the interest in describing the array of conceptions that can be held by learners, which are the result of historical and pedagogical process of concept development; and (3) understanding the connection between behaviors and knowing. Following this tradition, *savoir* (knowledge), the subject matter developed by a community, is differentiated from *connoisseur* (knowing, as a noun), the knowledge held by an individual, which can be incomplete or even mathematically invalid. Balacheff (2013) asserts that students' conceptions of mathematical notions are tied to the problems in which those conceptions emerge. Thus, understanding the spheres of practice in which specific mathematical ideas are called for is the first step for understanding student conceptions. The model, which has become a

heuristic for identifying conceptions for specific notions, has four components: problems, operators, semiotic and representation systems, and control structures. *Problems* correspond to the “class of the disequilibria the considered conception is able to recover from” (Balacheff & Gaudin, 2009, p. 190); they merge for the sphere of practices in which individual concepts are called for. *Operators* refers to “actions on the milieu” including those needed “to transform and manipulate linguistic, symbolic or graphical representations.” (p. 190). The *semiotic and representation systems* are defined as the “linguistic, graphical or symbolic means which support the interaction between the subject and the milieu” (p. 190). Finally, the *control structures* refer to the “components supporting the monitoring of the equilibrium of the [S - M] system” (p. 190); in other words, the strategies that the cognizant subject relies on to decide whether they had solved the problem and that they had done so correctly. We adopted this model as we have had experience using it to identify potential conceptions that could emerge as students work with different components in textbooks, such as the problems, the examples, or the textbook presentation with different concepts (functions, Mesa, 2004; differential equations, Mesa, 2010; angles, Mesa & Goldstein, 2016). We seek to identify distinct student conceptions about spanning sets and linear independence that can be derived from responses to reading questions embedded in interactive textbooks.

Methods

The interactive feature, *reading questions*, was added to three undergraduate mathematics PreTeXt textbooks in calculus, linear algebra, and abstract algebra, to entice students into reading the material in the textbooks prior to coming to class. The feature collects students’ responses directly in the textbook and delivers them to teachers in real time; perusing the responses allows teachers to possibly alter their lesson plans. We worked with all the responses students provided to six reading questions in two sections of the linear algebra textbook used in the project (Beezer, 2021). For this analysis we worked with the responses provided by 76 students who answered Reading Question 1 in *Spanning Sets* (RQ1-SS, Figure 1a) and 61 students who answered Reading Question 1 in *Linear Independence and Spanning Sets* (RQ1-LISS, Figure 1b). These students were distributed in six classes taught by six instructors at different institutions across the U.S. Sample responses are provided in the findings.

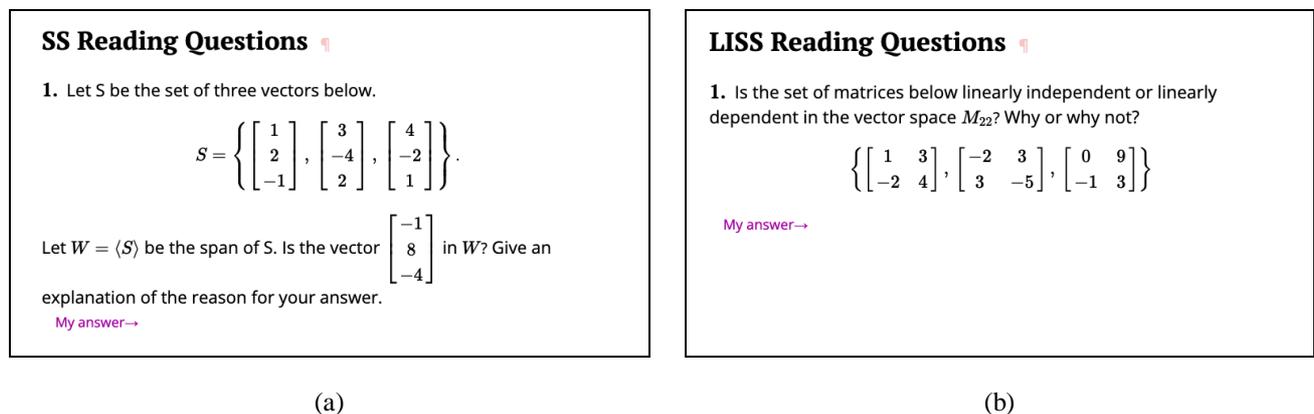


Figure 1: (a) RQ1 in Spanning Sets (RQ1-SS); (b) RQ1 in Linear Independence and Spanning Sets (RQ1-LISS)

The analysis proceeded in four stages. We first began by reading the accompanying textbook content to familiarize ourselves with the authors' intentions. Next, the second and third authors independently read the text accompanying the reading questions to identify the problems that could be associated with answering the reading questions; they found examples directly connected to these problems, and identified operators, representations, and control structures that were avowed by the author. In parallel, the first and fourth authors separately read the answers to RQ1-SS and RQ1-LISS to identify the operators, the representations, and the control structures evident in the responses. Third, we discussed the difficulties we faced, clarified the meaning of the elements in the model, and wrote a codebook, which we used to code the textbook examples (second, third and fourth authors) and the student responses to the two reading questions (first and second authors). Because in many of the responses the operators were implied (e.g., "No, because the matrix has infinite solutions when we need a unique solution." SS, #11¹) whereas the justification for their conclusions was always explicitly stated, we decided to code only for the control structures adding new ones as needed.; this was done, independently, by the two first authors. Each response could have none (e.g., "No, it's not in the set at all." SS-RQ1, #71), or one or more control structures. Both coders agreed in 54 of the 76 SS responses (89%) and in 41 of the 60 of the LISS responses (67%). Disagreements were resolved by consensus. In the fourth and final stage, we investigated the correctness of the conclusions through the response to see whether the student (1) arrived at the correct conclusion by using the appropriate interpretation in applying the control structures available (CO1), (2) had inaccurate intermediate interpretations but nevertheless arrived at a correct conclusion (CO2), (3) had accurate intermediate interpretations but arrived at an incorrect conclusion (CO3), and (4) had inaccurate intermediate interpretations and arrived at an incorrect conclusion (CO4). CO0 was used when no control structure was evident, or the information was uninterpretable. Each response was coded by at least two authors (2nd, 3rd, and 4th); we found seven disagreements in SS and 10 disagreements in LISS, which were resolved through consensus and resulted in refinements of the code definitions.²

Findings

We organize our findings by textbook section; within each section we present the analysis of the textbook examples followed by the analysis of the control structures present in the student responses to the reading question and the correctness of their conclusions.

Spanning sets

Inferred conceptions from examples in textbook. This section is in the Vectors chapter (the second chapter after Systems of Linear Equations) and includes two subsections: Span of a Set of Vectors and Spanning Sets of Null spaces. The first subsection has three examples, *A basic span* (ABS), *Span of the columns of Archetype A* (SCAA, Archetype A refers to a linear system with three equations, three unknowns, with a singular coefficient matrix with dimension 1 null space.), and *Span of the columns of Archetype B* (SCAB, Archetype B refers to a system with three equations, three unknowns,

¹ We identify each response by the section, SS or LISS and by a consecutive number. The response ID do not represent the same student across sections.

² Definitions, examples, and frequencies of each of these available upon request or at <https://tinyurl.com/mrx8sch9>.

with a nonsingular coefficient matrix) that addressed the problem: “To decide whether a given vector is in the span of a set of vectors.” The three examples rely on similar symbolic representations, vector space in R^4 for example ABS and vector space in R^3 for the other two examples and RQ1-SS. Each example gives a list of vectors in a set, followed by two vectors to be tested. Across these examples we identified seven operators, but they were used in two distinct solution paths, with the last operator being optional (Figure 2).

Problem	Determine whether a given vector is in the span of a spanning set	Operators: O1: Construct a system of equations of the vector as a linear combination of the vectors in the spanning set where the scalars are to be determined. O2: (By Theorem SLSLC, Solutions to Linear Systems are Linear Combinations) Form an augmented matrix from the system. O3: Perform row reduction on the augmented matrix. O4.0: Inspect the resulting row-reduced echelon form (RREF) matrix O4.1: (By Theorem RCLS, Recognizing Consistency of a Linear System) Determine whether the system is consistent. O4.2: Determine the number of solutions to the system. O5: Optional: (<i>If consistent/ unique solution</i>) Solve the system completely and express the vector as a linear combination of the vectors in the spanning set. O6: Optional: (<i>If infinite solutions</i>) Solve the system completely, assign a value to the free variable and solve for other scalars to work as a linear combination.
Semiotic systems	Vector space in R^3 and R^4	
Operators	A. O1→O2→O3→O4.0→O4.1 B. O1→O2→O3→O4.0→O4.2 (optional in the text) A*: A (<i>if O4.1 is consistent</i>) → O5 B*: B (<i>if O4.2 is unique</i>) → O5 B**: B (<i>if O4.2 is infinite</i>) → O6	
Control structure	CS1, for path A CS2, for path B CS3, for paths A*, B*, and B**	Control Structure: CS1: if system is consistent→ the vector in span CS2: if the system has a unique solution or infinitely many solutions→vector in span CS3: the found scalars work in a linear combination

Figure 2: Inferred conceptions from three examples in the section Spanning Sets. Italics indicate optional operations

We identified three different control structures for this type of problem, the *consistency of the system* (CS1): “we see that the system is consistent (...) This is enough evidence to say that $u \in \langle S \rangle$ ” (Beezer, 2021, Example ABS); the *number of solutions to the system* (CS2), either unique, “This system has a unique solution ..., so we are convinced that z really is in $\langle R \rangle$ ”, Example SCAB) or infinitely many solutions (“This system has infinitely many solutions, but all we need is one solution vector, (...) so we are convinced that w really is in $\langle S \rangle$ ”, Example SCAA); and *finding values of scalars* (CS3) that will satisfy the dependence relation. CS1 and CS2 rely on the inspection of the resulting RREF matrix, so their use is predicated on having created an augmented matrix and used a row-reduced echelon transformation of the matrix. As the presentation of the examples progresses, some operators are only alluded to rather than explicitly described. O4.1 and O4.2 mapped to CS1 and CS2 respectively. O5, and therefore CS3 is described as optional in ABS: “If we wished further evidence, we *could compute an actual solution* [emphasis added]” (Beezer, 2021) but it is present in the other examples in the cases where the system is consistent. From this analysis we inferred three distinct possible conceptions that could emerge (see Figure 2) determined by the three different control structures used to support the conclusion of belonging to the span of a vector set; paths A and B are the main paths, but once a decision is made (vector in the span or not), there are three distinct means to reach a conclusion.

Control structures in student responses. Each of the three control structures from the textbook were mentioned in 22 (30%) of the responses. The response “Yes, the vector is in W . The reduced row-echelon form of the set to the vector shows that the system of equations is consistent. This is enough evidence to say that the vector is in W ” (SS, #43) was coded as CS1. Across the 76 responses we

identified five additional control structures: *CS4: Existence of a linear combination* (e.g., “Yes, it is. The three vectors in S and the fourth vector represent a linear combination. This last vector is a solution to the three.” SS, #4), *CS5: SAGE*, *CS6: Proportionality*, *CS7: Number of free variables* (e.g., “Yes, because when you do RREF you don't get values that equal to no solution ($0=1$). you do indeed get 2 free variables and 2 dependent variables when you do RREF.” SS, #8), *CS8: number of free columns*, and *CS9: Number of pivots*. We used CS0 for three cases in which there was no control (e.g., “No, it's not in the set at all.” SS, #71). Except for CS4 that was mentioned in 15 responses, the other control structures were mentioned in four or less responses. While many of the responses (57) mentioned only one control structure, 17 used two control structures, and two mentioned three.

Correctness of student responses. Most of the responses (58) showed correct use of the control structures throughout the solution. Among the remaining responses, the most frequent type of conclusion was CO4 (8 responses):

When the augmented matrix is put into row-reduced echelon form, we get: $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. The bottom row of this matrix tells us there is no solution, thus the given vector is not in the span of S .” (SS, #18).

The student uses CS2 to justify the incorrect conclusion that the vector is not in the span. But the RREF of the augmented matrix is inaccurate so perhaps there was a computational error that led to this conclusion. The response “The vector $[-1, 8, -4]$ is in W with the x_1 and x_2 being equations of the first two vectors.” (SS#16) was coded as CO2; the conclusion is correct, but the language used is not appropriate. Notably, all the responses with two or more control structures (19 responses) arrived at correct conclusions.

Linear independence and spanning sets

Inferred conceptions from examples in textbook. This section is in the Vector Spaces chapter (the fourth chapter after Matrices) and includes three subsections: Linear Independence and Spanning Sets and Vector Representation. The first subsection has three examples related to linear independence; we focus on the example *Linear independence in M_{32}* (LIM32) because it is the most directly connected to RQ1. This example addresses one problem: “to decide if a set of vectors is linearly independent or dependent.” The example relies on symbolic representations, specifically, 3×2 matrices. In LIM32 we identified seven operators that were used in two distinct solution paths, with one operator being optional (see Figure 3). There were three control structures: *trivial relation*, if there is a nontrivial relation, then the system is linearly dependent (CS1); *infinitely many solutions*, when the system has infinitely many solutions and hence is linearly dependent (CS2); *not-all-zero values of scalars* that will satisfy a linear combination to work as a nontrivial relation to the homogeneous system (CS3). From this analysis we inferred three different conceptions, tied to the three paths as seen in Figure 3.

Problem	Decide if a given set of vectors is linearly dependent or linearly independent	Operators: O1: Write a relationship in which the linear combination of the given vectors with unknown scalars produces the zero matrix (Definition RLD). O2: Transform the left-hand side of the relationship using definitions of vector addition and scalar multiplication so that a single matrix is obtained. O3: Convert the matrix equation into a system into equations that would lead to finding the values of the unknown scalars (Definition ME). O4: Create the coefficient matrix corresponding to the system of equations. O5: Perform row reduction on the matrix. O6.0: Inspect the resulting RREF matrix to see if there are free variables. O6.1: Use definition of linear dependence (Definition LI) to decide if the system is linearly dependent or independent. O6.2: Identify number of free variables to decide if there exists a nontrivial solution O7: Optional: (<i>if exists nontrivial relation</i>) find scalars of the nontrivial linear combination that result in zero vector Control Structure: CS1: if the system has nontrivial solution → Linearly dependent i.e., if the system has <i>only</i> trivial solution → Linearly independent CS2: if the system has infinitely many solutions → Linearly dependent CS3: the found scalars work as a nontrivial relation
Semiotic systems	3x2 matrices	
Operators	A: O1→O2→O3→O4→O5→O6.0→O6.1 B: O1→O2→O3→O4→O5→O6.0→O6.2 (optional in the text) B*: B (<i>if O6.2 says exists</i>) → O7	
Control structure	CS1, for path A CS2, for path B CS3, for path B*	

Figure 3: Inferred conceptions from examples LIM32 in the section Linear Independence and Spanning Sets. Italics indicate optional operations

Inferred conceptions from student responses. Across 61 responses, CS1 was mentioned in 18 (30%) responses (e.g., “By getting the RREF of the set of matrices, we can see that it contains a non-trivial solution. Therefore, the set of matrices is linearly dependent.” LISS, #7), whereas CS2 was mentioned in six responses:

Following Example 2 of this section, if we form a general relation of linear dependence, find the corresponding system of homogeneous equations of each of the matrix entries, create the coefficient matrix and row reduce, we see that the resulting RREF of the coefficient matrix gives us a 4x3 matrix that has one free variable. This says that there are infinitely many solutions for the scalars in the relation of linear dependence, which further means (by Def LI) that the given set of matrices is linearly dependent. (LISS, #15)

In this response we see a description of path B that leads to using that the system has infinitely many solutions to conclude that the system is linearly dependent (Figure 3). We identified eight additional control structures from the student responses: *CS4: Linear combination* (e.g., “The set of matrices are linearly dependent because $2[[1,3], [-2,4]] + [[-2,3],[3,-5]] = [[0,9],[-1,3]]$ thus making one vector a linear combination of the others.” LISS, #1; 14 responses), *CS5: Consistency*, *CS6: Singularity of row-reduced matrix*, *CS7: Number of free variables* (e.g., “Set is linearly dependent. These can be modeled by a system of 4 equations which can be represented in a matrix that, when row-reduced, yields a free variable.” LISS, #17; 9 responses), *CS8: Whether the row-reduced matrix is identity matrix*, *CS9: REMES* (Row-Equivalent Matrices represent Equivalent Systems), *CS10: Number of pivots vs number of vectors* (e.g., “The set of matrices is linearly dependent because as an augmented matrix there are 2 rows and 6 columns therefore the maximum number of pivot columns is 2 which is less than 6.” LISS, #59; 6 responses), and *CS11: Number of equations (rows) vs number variables (columns)*. CS5, CS6, CS8, and CS9 were mentioned in less than six responses. It was interesting to find responses that relied on CS4, a control structure that was not in the textbook. It seems plausible that students can identify that the vectors were linearly dependent by inspection, given that the matrices and the numbers in them are small integers. CS4 could be the result of students working with their current knowledge of the material to answer the question, rather than reading the example

to replicate it in their solution. This would be consistent with our other findings that suggest that students tend to answer the reading questions to check their understanding of the material, before reading (Quiroz et al., 2022).

Correctness of student responses. Over half of the LISS responses (35) showed correct use of the control structures throughout the solution. Among the remaining responses, 13 were coded as CO4 (incorrect intermediate interpretations and incorrect conclusion), as in the following example: “The matrices above are linearly independent because when you do RREF of every matrix you get the identity matrix, the trivial solution.” (LISS, #37). In this case the student is using two controls, CS8 (identity matrices suggest linearly independence) and CS1 (trivial solution) to justify the incorrect conclusion that the vectors are linearly independent. Several responses were like this one, which suggest that students are considering each matrix independently of each other and after applying some process that yields the identity matrix, are using that to conclude that the solution must be the trivial one. CO3 (incorrect solution but accurate intermediate interpretations) was assigned to five responses, for example: “The set of matrices are linearly independent because there are less pivots than vectors.” Using CS10 (# of pivots) the student correctly identifies that there are fewer pivots than vectors but concludes incorrectly that the matrices are linearly independent. Four responses mentioned three control structures, 10 used two control structures, and the rest, 47 mentioned only one. We noticed that responses with two or more control structures tended to arrive at correct conclusions, although a handful of responses that had two control structures but generated an incorrect conclusion. Although our sample is small, we noticed that CS1 was present in answers with correct and incorrect conclusions, in about similar proportions, and when CS8 was invoked, the conclusion was incorrect. These discrepancies suggest that students are in the process of understanding how the various strategies to decide whether a set of vectors is linearly dependent relate to each other.

Discussion

Our analysis of the student responses to one reading question from two sections in the textbook revealed a much wider set of control structures that students used to solve the given problems than the ones in the textbook examples. This suggests that investigating the answers to these two paradigmatic problems in linear algebra, via responses to reading questions, can be a viable way to learn more about how students interpret the information in the textbook and how they rely on their own prior knowledge to tackle the problems. This fine-grained analysis allows us to further identify potential issues related to the applicability of the various controls that are available to students as they learn the materials. A quick reading of the responses may give instructors a sense that the students are on the right track, but further inspection of control structures reveals that for the students each of these controls are independent entities, each leading to different separate conclusions; thus, recognizing that they are equivalent can be an epistemological obstacle in learning the material. As the textbook doesn't make the equivalence explicitly, we assume that it is the instructors' role to show how and why all these controls are equivalent to each other. It could also be the case that as the course progresses the equivalence of these control structures becomes apparent. Some students didn't use the textbook content to solve the problem of determining the linear dependence of a given set of vectors. Maybe they are reinterpreting the expected goals of the milieu-subject system intended by the reading question. This points to a potential tension in designing the reading questions: to ensure

that students can try the process outlined in the textbook prior to the lesson, the author has chosen a set of vectors that would make the use of the operations and control structures straightforward; but by doing so, the apparatus of linear algebra may be unnecessary: students can solve the problem by inspection. This tension needs to be considered when designing reading questions in interactive textbooks that will get students to read the content prior to class and suggests the need to attend to content sequencing and to the phrasing of the questions that will elicit useful information for the teacher. Finally, while identifying the control structures is very important and useful, our analyses did not indicate when a conception would be in contradiction with the mathematical knowledge. Adding the correctness of the conclusion is a way to address this problem and complements our earlier analyses (Mesa et al., 2023).

Acknowledgment

Funding provided by the National Science Foundation DUE-IUSE Awards 1821706, 1821329, 1821509, 1821114. Opinions, findings, and conclusions or recommendations expressed do not necessarily reflect the views of the Foundation. Thanks to the RTMUS lab and UROP for their support.

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