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Stories of congruent triangles in two geometry textbooks

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In this paper, we report findings from a comparative analysis of the introduction and development of congruent triangles in two textbooks: a high school geometry textbook from the U.S. and an eighth-grade mathematics textbook from China. Using the Mathematics Curriculum as A Story framework developed by Dietiker (2015) as well as additional coding schemes, we illustrate how each textbook design its congruent triangle unit with the goal to provide students a fertile ground for the development of geometry reasoning and proof. Through analyzing the characters, actions, settings, acts and plots of each story, we identify both similarities and differences. The findings demonstrate the illuminating power of the Curriculum as a Story framework to fill the gap of current textbook analysis research.

Keywords: Congruent triangles, geometry, textbook analysis.

Introduction

The concept of congruency is an important topic in geometry. Euclid’s Elements included three conditions for triangle congruency: side-angle-side (SAS), side-side-side (SSS), side and any two angles (SAA) as propositions 4, 8, and 26. These conditions were then used to prove many subsequent propositions. The concepts of congruent triangles have many real-life applications with the most notable one being the indirect measurements. Learning triangles builds foundation approaches for students to further learn more complex polygons and circles. The idea that a partial set of all angle and side length measures can guarantee congruence between two given polygons is a unique property of triangles, which provides a rich setting for developing students’ geometric intuition and ability to perform deductive proof. These significances make the topic of congruent triangles a constant feature in upper-level geometry curriculum worldwide (Jones & Fujita, 2013).

However, students worldwide appeared to find congruent triangles to be a challenging topic. For example, the international average of the correct responses remains to be 35% for TIMSS 1995 for an item seen in Figure 1. This result indicates a possible lack of understanding of the definition of congruent triangle as well as the concepts of corresponding sides and angles.

![Figure 1. A TIMSS 1995 released item (Beaton et al., 1996, p. 32)](image)

These triangles are congruent. The measures of some of the sides and angles of the triangles are shown. What is the value of x?

Wang and colleagues (Wang et al., 2018) conducted an error analysis of 8th grade Chinese Students’ reasoning and proof of congruent triangles. They developed a rubric for evaluating students’ level of
reasoning and proof with respect to key components needed for performing deductive proof in the context of congruent triangles. They also found that some Chinese 8th grade students had difficulty completing a deductive proof successfully. For example, about 55% of the students in their study were able to solve the following proof problem in Figure 2 successfully.

As shown in Figure 2, in $\triangle ABE$ and $\triangle CDF$, $AB=CD$, $AE=DF$, $CE=FB$.
Prove $AF=DE$.

**Figure 2. A deductive proof item (Wang et al., 2018, p. 117)**

González and Herbst (2009) identified four different student conceptions of congruency while using dynamic geometry software. The perceptual conception of congruency determined two figures were congruent by their look. The measure-preserving conception of congruency checked whether the two objects had the same measures to determine the congruency. The correspondence conception considered two objects as congruent if they are corresponding parts in two triangles that were known to be congruent. The transformation conception of congruency establishes that two objects are congruent if there is a geometric transformation, mapping one to the other while preserving length and angles. Jones and Fujita (2013) developed a framework based on these four conceptions of congruency to analyze the introduction and development of triangle congruency in an eighth-grade Japanese textbook. They found that while the Japanese textbook expected students to use various conceptions of congruency to explore ideas of congruent triangles experientially, the student experiences focused almost exclusively on the correspondence conceptions when constructing formal proofs. This approach, shared by many East Asian countries, was different from the approach adopted by the Common Core State Standards for Mathematics, which used the transformation conception as the basis for defining congruency. However, little is known about how textbooks introduce and develop congruent triangle theorems to their students based on these different approaches. An in-depth analysis of the introduction and development of triangle congruency theorems in the U.S. and China may provide helpful insights that are critical in promoting student learning of triangle congruence theorems.

**Theoretical perspectives**

Mathematics textbooks have long been regarded as a bridge between the intended curriculum, such as national curriculum standards, and the actual teaching in the classroom, i.e., the implemented curriculum (Valverde et al., 2002). The interest in the study of mathematics textbooks in the context of international comparative study has grown since the TIMSS 1995 study. Fan et al. (2013) found that 63% of the literature on mathematics textbook research studies they reviewed focused on textbook analysis and comparison. Typically, findings of the content analyses were descriptive in nature, and frequently accompanied by a table or a diagram that is the result of a systematic coding for easy comprehension. Previous research on mathematics textbooks has made significant contributions to identify and report many textbook features that can be used as the basis for
comparisons. However, most studies on textbook analysis do not include detailed accounts of the progression of learning. Thus, few were able to provide a sense of the changes and flow of the mathematical content throughout a textbook, nor are they able to account for the aesthetic of sequencing.

Recognizing this limitation, Dietiker (2015) proposed a conceptualization of the mathematics textbook as a story based on the narrative perspectives that includes four main elements. Mathematical characters are “figures” that were brought into existence by descriptive naming that could be in a variety of forms, such as words, graphs, tables, or symbolic forms. Mathematical actions are manipulations taken on a mathematical character that result in a mathematical change that can move a mathematical story forward. Mathematical setting is the space in which the mathematical character emerged and developed via mathematical actions. For example, a “function” character that lives in a mathematical setting of multiple representations affords richer learning opportunities for students to know this character. A mathematical plot is the “potential temporal dynamics of the story that encourages (or discourages) a reader to continue to advance through the mathematical story” (Dietiker, 2015, p. 299).

Using this framework, Dietiker and Richman (2021) investigated the following research question, “In what ways do high school geometry textbooks encourage or discourage sustained student inquiry” (p. 302). Focusing on a single lesson contains the discussion of SSA from five US high school geometry textbooks, their analyses identified several characteristics of lesson structures that promoted sustained student inquiry. For example, lessons that were able to raise and suspended students’ curiosity of important mathematics ideas throughout a significant portion of the story tended to have more questions that were not immediately answered. Furthermore, the average number of unanswered questions tended to peak at the middle of the lesson, followed by gradual resolutions throughout the remaining lessons. While this study demonstrates the power of using this framework in answering a significant question that was not answered before, the feasibility of using this framework to analyze a unit instead of a single lesson remains unknow. Analyzing an entire unit of congruent triangles in two textbooks as this study intended to do will give a fuller picture of the potential of this framework. This study intends to produce a dense narrative and interpretations that answer the central research question, “What are the similarities and differences between the U.S. and Chinese textbooks in introducing and developing the criteria of triangle congruence?”

**Methodology**

The design of the study was informed by the theoretical perspectives and framing outline above and the analysis was conducted by the constant comparative analysis (Stake, 2000).

**Data set**

An eighth-grade math textbook published by People’s Education Press (PEP, 2013) in China (typical eighth graders in China are 13-14 years old) and the *Eureka Math Geometry* published by (Great Minds, 2015) in the U.S. were selected for this analysis. In the U.S. typical high school students take a year-long geometry class during the tenth grade (typically, 15-16 years old). People’s Education Press has the longest history in developing K-12 curriculum in all subjects, and their mathematics textbooks were most widely analyzed (e.g., Cai & Jiang, 2017). The Eureka Math was the commercial
version of the EngageNY, originally created and maintained by the New York State Education Department as an open source, completely Common Core Curriculum Standards aligned K-12 curriculum that have been downloaded more than 66 million times by educators all across the country since the website launched in 2011 (Kaufman et al., 2017).

Each textbook devoted four lessons to the five congruent triangle theorems: SSS, SAS, ASA, AAS, and HL. Focusing on the same content and the same number of lessons in each textbook made the comparison more compatible despite the grade-level difference due to the different curriculum standards in each country. Right before the unit of congruent triangle, the general definitions of congruence were discussed in both textbooks. Student edition of each textbook was used as the primary source of data. Our analysis focused on expository texts, explorations, worked out examples, as well as exercises for additional practice and short assessments, items that are intended for in-class uses as specified by the teacher edition.

**Data analysis and procedure**

Initial analyses of the congruent triangle units were carried out jointly by the three authors based on the narrative framework developed by Dietiker (2015). Each of the authors first read the textbook unit independently, then wrote a story summary based on our notes taken during this first-read with an eye toward plot shifts and developments. We shared and discussed our interpretations multiple times to meet a consensus. During the process, we realized that additional coding schemes were needed in order to capture the nuances of the characters and actions. Constant comparisons (Stake, 2000) were used to develop these coding schemes. Then we met on-line to develop the first joint story summary by combining all the distinct details as well as resolving any differences in our own reading of the story and coding. The development of coding schemes for characters and actions based on our readings of the stories will be discussed as reporting our findings.

**Findings**

**Plots**

The main mathematical plot in PEP Math opens with the following question: “Is it possible to identify various subsets of these six conditions (i.e., three pairs of congruent sides and three pairs of congruent angles) that will also make two triangles congruent?” The investigation started with the minimal number of criteria, one pair of congruent side or one pair of congruent angles, then move to two pairs of congruent angles, two pairs of congruent sides, and one pair each of congruent angles and sides. The story asks the students to draw two triangles that fit each condition, then check to see if they are congruent. It then systematically goes through the cases with three pairs of congruent sides (SSS), two pairs of congruent sides and one pair of congruent angles (SAS and SSA), one pair of congruent sides and two pairs of congruent angles (ASA and AAS), three pairs of congruent angles (AAA), and ends with the special cases on right triangles (HL). There is a side plot between SSS and ASA where the construction for duplicating an angle is discussed via SSS. This new construction along with the construction of duplicating a line segment are then used to prove ASA. Throughout the story compass-and-straightedge constructions are used to prove SSS, SAS and ASA while previous congruent criteria learned properties of triangle were used to prove SSA and HL. Counterexamples are used to show that AAS and AAA are not congruent criteria.
The main mathematical plot of congruent triangles in Eureka Math opens with the following question, “Do you think it is possible to know whether there is a rigid motion that maps one triangle to another without actually showing the particular rigid motion? Why or why not?” It then provides step-by-step instructions to prove that there exists a sequence of transformations that can map one triangle to another triangle that has two pair of corresponding congruent sides and one pair of corresponding congruent including angles (SAS). In this process, the story also demonstrates that with the SAS condition, any pairs of disjoint triangles can be transformed to a pair of two connected triangles with the only intersection being a shared side. It then goes through all the conditions that guarantee congruent triangles first in the following order: ASA, SSS, AAS, and HL. Then it goes through all the conditions that do not guarantee congruent triangles: SSA and AAA. There is a side plot between SAS and ASA where the story proves that the base angles of any isosceles triangles are congruent. This new property alone with transformations are then used to prove ASA. Similar to PEP Math, throughout the entire story, transformations are used to prove SSS, SAS and ASA while previous congruent criteria learned properties of triangle were used to prove SSA and HL. Counterexamples are used to show that AAS and AAA are not congruent criteria.

Characters

At the most basic level, the characters of both lessons are pairs of triangles. However, we soon realized that this approach did not add any additional insight into the story analyses. After some pondering, we defined the main characters of the congruent triangle story as a pair of triangles that were represented in verbal form with or without accompanying graphs. All the characters can be divided into two main groups: simple or complex ones. Simple characters include a pair of triangles that are presented as two disjoint triangles, a pair of triangles with a shared vertex or a pair of collinear sides, or intersecting yet two distinct triangles. For example, Figure 3a shows an example from PEP Math with an intersecting point not labeled. So, there are only two triangles that can be named from the figure: $\Delta ABF$ and $\Delta DCE$. Complex characters included 3 or more triangles in an embedded figure. Figure 3b includes an example of complex character Eureka Math that asks students to identify two pairs of congruent triangles: $\Delta ABY$ and $\Delta XYB$ as well as $\Delta ARY$ and $\Delta XRB$. All characters in PEP Math included in our analysis are simple. About 23% of the characters in Eureka Math are complex characters.

<table>
<thead>
<tr>
<th>Simple Characters</th>
<th>Complex Characters</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta ABF$ and $\Delta DCE$</td>
<td>$\Delta ABY$ and $\Delta XYB$, $\Delta ARY$ and $\Delta XRB$</td>
</tr>
</tbody>
</table>

Based on the information provided, determine whether a congruence exists between triangles. If a congruence exists between triangles or if multiple congruencies exist, state the congruencies and the criteria used to determine them.

Given: $RY=RB$, $AR=RX$

Figure 3. Examples of simple and complex characters
The analyses discussed above provided an initial glimpse into the nature of the characters introduced by these two textbooks. A more complete picture requires additional analyses of the actions on these characters and the setting these characters lived in.

**Actions**

Dietiker (2015) defined mathematical actions as manipulations taken on a mathematical character that results in a mathematical change or creating a new mathematical character. The stories of triangle congruency in both textbooks include six common mathematical actions: 1) “applying” a previously established property such as the sum of all interior angle measures of any triangle is 180 degree when establishing the equivalence between SAA and ASA; 2) “corresponding” to establish the correct correspondences via manipulating the orientations of the shapes, 3) “constructing” refers to the compass & straightedge construction; 4) “drawing and comparing” that is mainly used for testing the conjecture or creating counterexample; 5) “jointing or separating” line segments or angles to create a new segment or angle that is part of a triangle, and 6) “translating” statement such as “$D$ is a mid-point of line segment $AC$ to congruent line segments of $AD$ and $CD$. In addition, Eureka Math included two additional mathematical actions: adding an auxiliary line and rigid transformations (i.e., translation, rotation, and reflection) which did not appear in PEP Math. Table 1 below shows the partial statics of the frequency and percentage of the top four actions from each textbook.

<table>
<thead>
<tr>
<th>Action Types</th>
<th>% in Eureka Math</th>
<th>% in PEP Math</th>
</tr>
</thead>
<tbody>
<tr>
<td>Applying</td>
<td>33.9%</td>
<td>22.8%</td>
</tr>
<tr>
<td>Corresponding</td>
<td>16.5%</td>
<td>21.1%</td>
</tr>
<tr>
<td>Drawing and comparing</td>
<td>2.5%</td>
<td>15.8%</td>
</tr>
<tr>
<td>Transformation</td>
<td>17.4%</td>
<td>0%</td>
</tr>
<tr>
<td>Translating</td>
<td>19.0%</td>
<td>28.1%</td>
</tr>
</tbody>
</table>

**Settings**

The story of PEP Math includes two distinct types of settings: deductive proof and problems embedded in a real-life context as seen in Figure 3a and Figure 4 below. The setting of deductive proof provides students with opportunities to apply the newly learned congruent triangle condition to prove the existence of congruent triangles, sides, or angles. While problems embedded in a real-life context provide students with opportunities to perform the same actions on similar characters as the deductive proofs, they could also enrich students’ understanding of congruent triangles in real-life applications. An example of an application of indirect measure can be seen in Figure 4 below.
To measure the distance between two points $A$ and $B$ from two sides of the pond, you can draw a line segment $BF$ perpendicular to line $AB$. Identify two points $C$ and $D$ from $BF$ such that $BC=CD$. Then draw a line segment $DE$ perpendicular to $BF$ such that points $A$, $C$ and $E$ are on the same line. Then $DE$ and $AB$ have the same length. Why?

**Figure 4. An example of real-life applications from PEP Math (p. 38)**

Different from the story of PEP Math, the story of Eureka Math contains no real-life application. Rather, all discussions, examples, and exercises are presented with formal mathematical languages and in an abstract setting.

**Discussion and conclusion**

As an old Chinese fable says, it would not be possible for a group of blind people to get an idea of what an elephant looks like by just sharing their individual experiences obtained from touching parts of the real elephant. The story’s plots provide a rough sketch of the elephant, while the characters, actions, and settings add rich details to make the picture of the elephant more complete. Through the storytelling, we can see both the similarities and the differences of these stories in a connected way.

These two stories used different conceptions to define congruent triangles and had different intended unit goal. The differences in plot, characters, actions and settings can all be understood by their choices in these two aspects. Because Eureka Math uses transformation conception, it has to start with SAS as it allows the reader to simplify the subsequence proofs to the special case of two triangles with one shared side. Consequently, the actions used by Eureka Math show a higher percentage on transformations. Since compass-and-straightedge construction can only duplicate line segments, PEP Math has to start with SSS and use more drawing and comparison actions. Notice that both stories are high in applying, corresponding and translating actions which indicates that these actions are important for both stories.

With respect to the unit goal, Eureka Math considers congruent triangles as a fertile site for developing students’ ability in writing deductive proof exclusively. Therefore, their story employs only one setting but introduces a lot more complicated characters within that setting than PEP Math. The story of PEP Math includes an additional goal of helping students to see the connection between the abstract mathematical ideas and real life by providing many activities and exercises in the setting of real-life application which is absent in the Eureka Math.

The narrative framework proposed by Dietiker (2015) as well as the additional coding schemes developed by us have highlighted both the similarities and differences between one U.S. and one Chinese textbooks in their introducing and developing the criteria of triangle congruence that went beyond the traditional coding approaches. The information uncovered by our analysis makes it possible to develop assessment items that are more sensitive to the textbook designs which contributes to the future investigation of the potential link between the textbook and student learning, as suggested by Fan et al. (2013) as a research area that needs more work.
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