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Supporting the interpretation of formulas in physics education through mathematics lessons

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To use formulas in physics education, students have to be able to interpret formulas. This study focuses on how mathematics lessons might support the interpretation of formulas in physics lessons in upper secondary school. Based on literature, a model for interpreting formulas in science is proposed. In a short intervention, grade 11 students were taught to recognize the structure of formulas and to reason with formulas through graphing formulas by hand. In grade 12, four of these students were asked to think aloud while working on physics assignments involving formulas. Results showed that the students recognized the structure of formulas using function families and used similar reasoning as taught in mathematics lessons ten months before. Findings suggested how mathematics lessons might support the use of formulas not only in physics, but also in other natural sciences and in STEM contexts.

Keywords: Mathematical formulas, reasoning, interpreting formulas.

Introduction

Many students in secondary school struggle with mathematical formulas in mathematics and in physics, as they often see formulas as calculation rules (plug and chug) and fail to see its structural role (Arcavi et al., 2017; Redish & Kuo, 2015; Sokolowski, 2020). In physics, mathematical formulas are used to define and to describe relationships between quantities, and to reason about these relationships. When discussing problems using mathematics in physics, teachers often focus is on mathematical techniques (Redish & Kuo, 2015; Uhden et al., 2012). Redish and Kuo (2015) have indicated other possible causes of students' problems, namely, mathematics in physics is not the same as in mathematics. This can be illustrated with the formula: $v_t = v_0 + at$, in which v_t is the velocity (in m/s) after t sec, v_0 the start velocity in m/s, and a the acceleration in m/s^2 . From mathematical perspective, we see, among other things, a linear function with t being the independent variable and v_t the dependent variable and a line with slope a and intercept v_0 . From a physics point of view, we see, among other things, that this formula concerns a movement with constant acceleration, so, speed increases steadily; a structure: base + change, i.e., v_0 as base and acceleration as change per sec; unit analysis gives $[m/s] = [m/s] + [m/s^2 \cdot s]$. So, using mathematical formulas in physics requires not only recognizing the structure of the formula, but also to understand the relationship between the variables in the physics context (Redish & Kuo, 2015).

The current study is about how mathematics education in grades 11 and 12 might support the interpretation of mathematical formulas for reasoning in physics education.

Theoretical models

To describe how mathematics is integrated physics, various models can be found in literature. We discuss two of them. In their physical-mathematical-model (PMM), Uhden et al. (2012) have

described how mathematics is intertwined with physical thinking and working, emphasizing the structural role of mathematics with mathematization (i.e., formulating mathematical formulas), and with interpretation of a formula (i.e., interpreting formulas in terms of physical contexts). They have distinguished the structural and technical skills, i.e., between PPM and the pure mathematics techniques, like solving equations, calculating derivatives and/or integrals. Also, translations between the rest of the world and PMM can be added, resulting in a model with all processes of a modelling cycle, focusing on the structural skills needed for understanding physics through mathematics.

In the current study we focused on one aspect, interpreting formulas. This can be considered as conditional for mathematization, because without interpreting one cannot understand formulas. In their model to verbalize formulas, Geyer and Kuske-Janßen (2021) have formulated steps to interpret formulas. They have assumed that a formula is only understood if one can make a so-called concept image of the formula in everyday language, including, e.g., a set of examples, situations, theory that can be linked to the formula. Figure 1 shows the general model on the left, and an example about resistance on the right. The model outlines two pathways, a physical path (IIA) and a mathematical path (IIB), which are brought together at level III to provide relationships between the variables of the formula in physical language. Then this information is contextualized at level IV, leading to an interpretation in everyday language at level V. The importance of integrating mathematical and physical thinking has also been stressed by others, e.g., Zhao and Schuchardt (2021), who formulated their science-math sensemaking framework, with a science and a math sensemaking dimension.

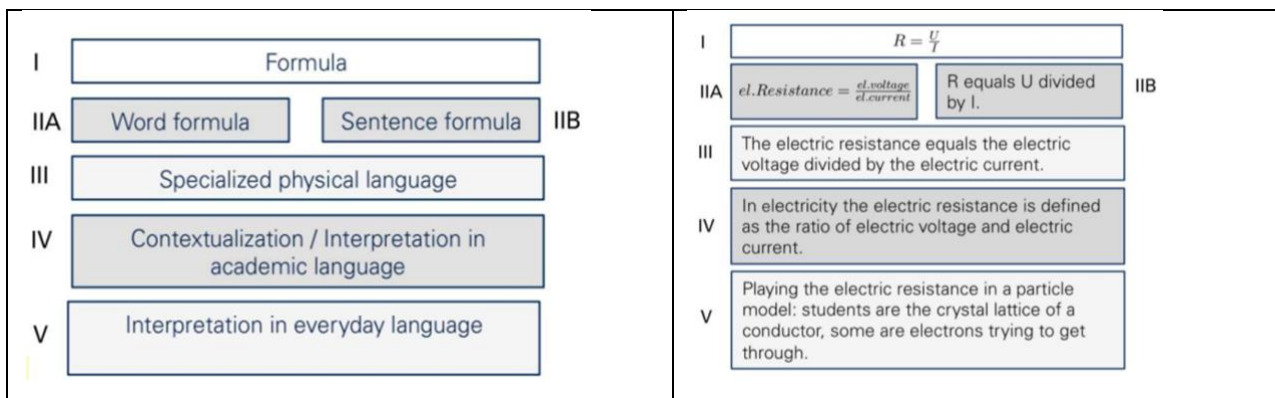


Figure 1: Interpreting formulas model (Geyer & Kuske-Janßen, 2021)

The role of mathematics in the Geyer and Kuske-Janßen model is very limited. In the example in Figure 2, it concerns only the translation into an arithmetic operation (R equals U divided by I). We suggest adjusting the model, so that the role of mathematics would become more prominent. Many, including Redish (2021) and Sokolowski (2020), have stressed the importance of recognizing structures of formulas, reasoning about covariation, and functional thinking for blending science and mathematics knowledge. These are aspects of symbol sense (Arcavi et al., 2017). Research by Kop et al. (2021) has shown how these aspects of symbol sense can be taught by graphing formulas by hand. They showed how grade 11 students improved their insight in formulas through a series of lessons, which focused on a repertoire of basic function families for recognition and qualitative reasoning, i.e., reasoning about global features of a function/graph in terms like in/decreasing, positive/negative outcomes, infinity behavior. Based on this literature, we suggest adjusting the

Interpreting formulas model: instead of only ‘Sentence formula’ at IIB in the general model, we suggest describing the mathematical path with: IIB ‘recognizing the formula structure in terms of basic function families’ and IIIB ‘describing covariation, using functional thinking, including function features.’ In the resistance example, the mathematical path could consist of IIB: R is proportional to U , and inversely proportional to I , with function-families $y = a \cdot x$ and $y = \frac{a}{x}$; at IIIB, covariation could be described via graphs of both relationships. Levels IV and V could be extended with this information. See Figure 2.


$R = \frac{U}{I}$	
IIB Word formula: $Resistance = \frac{voltage}{current}$	IIB Recognizing the formula structure: R is proportional to U , and inversely proportional to I ($y = a \cdot x$ and $y = \frac{a}{x}$)
III Specialized physical language: Electric resistance equals electric voltage divided by electric current	IIIB Covariation; rough sketches 
IV Contextualization: In electricity the electric resistance is defined as the ratio of electric voltage and electric current, i.e., If electric voltage becomes twice as large, then resistance has to be twice as large (if current is kept constant); if electric current is twice as large then resistance has to be twice as small (if voltage is kept constant)	
V Concept image of the formula, including descriptions of covariation, rough graphs, problem situations, models, metaphors (e.g., if pressure increases then flow will increase, if resistance is constant, etc.	

Figure 2: Adjusted Interpreting formulas model (based on Geyer & Kuske-Janßen, 2021)

The adjusted model, with a more prominent role of mathematical reasoning, shows how mathematics lessons might support students interpreting formulas in physics contexts, namely learning them to recognize the structure of formulas (level IIB) and to reason with formulas and link them to graphs (level IIIB). This leads to the research question for the current study: To what extent can the teaching in graphing formulas in mathematics lessons foster students’ abilities to recognize the structure of formulas and to reason about covariation in physics contexts?

Method

Participants

In this study we investigated how four grade 12 students at a regular secondary school in the Netherlands worked on four tasks in a physics lesson. Ten months before, the four students were in grade 11 and worked in a series of lessons on graphing formulas by hand. All were in a Math B class of 21 students. Math B prepares for studies in engineering, mathematics, physics, etc. Based on their performances on other mathematics tests during the year, student P can be considered as a high achieving student, student Q above average, and students S and T slightly below average. The students worked in pairs P, Q (group 1) and S, T (group 2) on the tasks in the physics lesson.

Series of lessons

In grade 11, the students worked on graphing formulas by hand during the mathematics lessons. In two 90-minute lessons and one 30-minute lesson, students worked in pairs on five tasks: a matching

task for recognizing basic functions (match a formula to a graph), a task about transformations of basic functions (how are the graphs of $y = -3^x$ and $y = 3^{-x}$ the same/different?), a task on recognizing some graph features like zeros, extreme values, and vertical asymptotes (what features can be simply read from formulas like $y = (x - 4)^2 - 1$ and $y = (x - 5)(x - 3)$?), and a task about reasoning about infinity behavior and increasing/decreasing (e.g., $y = 19 - 13 \cdot 0.78^x$, and $H = 2 - \frac{2}{(x-2)^2}$). These tasks were also used by Kop et al. (2021). A short task about functional reasoning was added (e.g., the difference between the graphs of $f(a) = a \cdot x^2$ and $f(x) = a \cdot x^2$). After these graphing tasks, students took a formative test with sketching ten formulas like $y(x) = 2\sqrt{6-x} + 3$; $T = 60 \cdot 0.8^x + 10$; $y = -2x(x-2)(x+4)$; $D = 2 \log(v-2)$; $F(R) = \frac{50m}{R}$; $F(m) = \frac{50m}{R}$; $E(Q) = c - d \cdot \ln(Q)$.

Data and data analysis

In this study, we were interested to what extent students were able to use their skills in graphing formulas to interpret formulas in physics contexts. A 30-minute formative test was administered to the two pairs of students (group 1 and group 2) in a physics lesson. The students were asked to think aloud and were videotaped while working on the tasks. The first three tasks were taken from physics textbooks. The tasks were discussed with the physics teacher and five other experienced physics teachers. All of them considered these tasks challenging and at a high level for grade 12 students.

The video recordings were transcribed and divided into meaningful units of the solution process and then coded with a similar codebook that was used in Kop et al. (2021), because it gives more detailed analyses of the student's reasoning at level IIB and IIIB.

Codebook: If the function family was recognized and used: code FF (level IIB). If a reasoning was used like 'if ... increases, then ...': code Rg; If a reasoning was used like 'if ... is close to ... then ...': code Ra; If a reasoning was used like 'if ... becomes very large then ...': code Ro (all level IIIB). If a pseudo variable was used (a random chosen number was used): code Pv; If a formula was manipulated (e.g., factorized): code M; If a calculation was performed: code C. Repetition in the transcripts and confirmations were ignored in the coding. Also, a score was given: 0 if not correct; 0.3 if some correct ideas; 0.7 if largely correct; and 1 if correct.

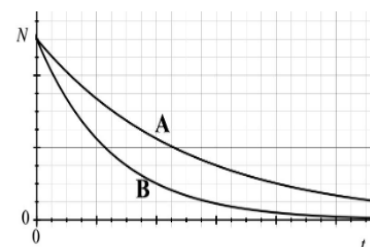
Results

We describe the tasks and strategies of both groups.

Task 1

For radioactive material, the number of atoms decreases according to the formula $N(t) = N(0)e^{-\frac{t}{a}}$, in which $N(0)$ the number of atoms at $t=0$, a is a measure of how fast that reduction goes, and t the time in years.

At $t=0$ two different elements A and B start with the same number of atoms. The graphs show the decay of both elements over time. Reason from the formula which element (A or B) has a greater value of a .



In group 1 both students have their own solution method in mind. The student P barely responds to Q's remarks. P starts reasoning about t/a via "if a is larger than the exponent is smaller" (Rg) and uses the pseudo variable $t=2$ in his explanation (Pv). Because of the minus-sign in the exponent, he rewrites the formula (M) to $1/e^{\frac{t}{a}} \cdot N(0)$ after which they focus on the denominator $e^{\frac{t}{a}}$. However, they forget that they talk about the denominator and thus arrive at the wrong conclusion. Q remarks at the beginning that "The greater to the power of minus number, the smaller the e." Q seems to recognize the function family e^{-x} as a decreasing function (FF). But P does not respond to that. Later Q brings it up again "No, because it is still e-to-the-power minus." But because at that moment P is busy explaining why a division arises, Q agrees ("oh yes I see") (score 0.7)

In group 2 we also see both strategies. S starts by looking at exponent, which becomes smaller as a is larger (Rg); the minus sign seems to be omitted. This brings S to graph B. T seems to recognize the function family e^{-x} (FF) with "it is $-t$ and the larger a is the smaller the exponent is, so the less the change is" (Rg). S responds with "Oh, you can do that too; yes right, it is negative; so, if it is greater value than the number above is smaller." However, they are still unable to complete the task. They continue to consider manipulating the formula (M) with " e^{-t} means to the power -1 ; so, 1 divided by ...; so, it is actually a fraction". But S takes over again and still omits the minus sign in exponent "if a is larger than e in the power a smaller number and so it becomes smaller" (Rg). T: "I still think that A has greater value." They leave it for a while and later T takes the graphic calculator (GC) and makes two graphs; they arrive at the final answer "The larger a is, the smaller the power of e and therefore the less steeply the graph of $N(t)$ decreases. With element A, therefore, a is greater than with B." (score 0.3, because of using GC).

Task 2

In Einstein's theory of relativity, the formula $p = \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}}$ is used to calculate the impulse. In this formula p is the

impulse in Ns (or kg m/s), m the mass in kg, v the velocity in m/s, and c the speed of light in m/s.

This formula $p = \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}}$ can be simplified if v is much less than c , as is the case in ordinary life.

- Give the simplified formula.
- Reason from the formula what happens with the impulse if the velocity v is in the neighborhood of the speed of light c .

Task 2a

Both groups use a reasoning like 'if $v \rightarrow 0$ then $1 - \frac{v^2}{c^2} \rightarrow 1$ ' (Ra) and find a correct answer (score 1).

Task 2b

In group 1, P starts with a Ra-reasoning: " $(\frac{v^2}{c^2})$ becomes almost 1" and then: "No, it is about the square root; $(\frac{v^2}{c^2})$ becomes max 1 because v cannot be larger than c ". Then P remembers features of the root function family (FF): "wait; roots become larger at small numbers, in the neighborhood of 0" and they explore the behavior of root function in the neighborhood of 0: "in root function it works different, because $0.1 \cdot 0.1 = 0.01$, so, $\sqrt{0.01} = 0.1$, so, it becomes larger" (Pv). Then they use again a Ra-reasoning, that gives a correct answer. Q has still some doubts, but P confirms " $\sqrt{0} = 0$, so, it goes to ∞ " (score 1).

In group 2, S immediately starts with Ra-reasoning: “Oh, then this becomes 1, that is not possible because then you have $\frac{mv}{0}$ ”. Both discuss the speed of light c : “it is almost 1; still smaller than this (c)”. T says that p becomes very large, and it is an asymptote. S wants to emphasize this “can never be the same (as c).” They consider how to indicate this. Then they come up with 1-0.9999... and the answer: $p = \frac{mv}{\sqrt{1-0.9999\dots}}$, so, p becomes very large (score 1).

Task 3

In an experiment, a plank is fixed to one side. If the force is exerted on the other side, the plank will bend. For small deflections u (in m), the formula for deflection is:

$$u = \frac{4}{E \cdot b} \cdot \left(\frac{l}{d}\right)^3 \cdot F$$

In which E is the modulus of elasticity, b the width of the plank in m, l the length of the free part in m, d the thickness of the plank in m, F the force in N.

Suppose that l becomes twice as large, and that the deflection must remain the same.

- How will the thickness d have to change?
- Sketch a graph of the relationship between u and d .

Task 3a

Both groups use a Ra-reasoning and discuss “Just times two; was that the question? It is just a proportion” (score 1).

Task 3b

Group 1 translate to x and y : “Let’s say that l is 1; then we get $y = \text{number} \cdot \frac{1}{d^3}$; so, $y = a \cdot \frac{1}{d^3}$ ”. Q adds “ $y = a \cdot d^{-3}$ ” (FF). Then a Ra- and Ro-reasoning follow: “So, a divided by d^3 ; we learned this during the mathematics lessons last year. If d goes to 0 then we get ‘divided by 0’. So, ...an asymptote at 0. If d goes to infinity, then it goes to 0 (indicate the x -axis). About the graph: “The exponent is odd, and $1/x$ is odd too”. They consider graphing the negative x -axis and y -axis, but “deflection can never become negative; only positive; and thickness is also not negative; did you ever saw a plank with a negative thickness?” (score 1).

Group 2 decided to use the graphic calculator. They choose to use x and y and choose random numbers for the other variables. They get the graph of $y = \frac{4}{7 \cdot 8} \cdot \left(\frac{3}{x}\right)^3 \cdot 25$ as a prototypical graph. They consider negative values and reject these (score 0.3).

Task 4 (biological context)

In this table you see the length of an unborn child is on average at different times during pregnancy.

Pregnancy in weeks (Z)	Length in cm (L)
0	0.02
4	0.4
8	2
12	8
14	18
18	25
22	30
26	35
30	40
34	45
38	50

Which type of formula would fit best the data in the table? Explain your choice.

- $L = a \cdot g^Z$ with $0 < g < 1$
- $L = a \cdot g^Z$ with $g > 1$
- $L = a \cdot Z^b$ with $0 < b < 1$
- $L = a \cdot Z^b$ with $b > 1$
- $L = a + b \cdot Z$ with $b < 0$
- $L = a + b \cdot Z$ with $b > 0$
- $L = \frac{a}{1+b \cdot g^Z}$ with $0 < g < 1$
- $L = \frac{a}{1+b \cdot g^Z}$ with $g > 1$

Group 1 wonders whether g refers to gravitation but decide that it is about a base. “If it is an exponential function then $(\frac{8}{2})^{\frac{1}{4}} = (\frac{25}{8})^{\frac{1}{4}}$ would hold (C); so, it is definitely not exponential.” Which excludes a and b. Option c and d are rejected because at $Z=0$ L is not equal to 0 (C). After randomly choosing $g=0.9$, they choose for option g, because “h is not possible: if g larger than 1 then L decreases and the baby would shrink” (Ro) (score 0.7)

Group 2 starts with making a table with $Z=14,18,22,26,30$, the corresponding L values, and the differences between L values. T initially chooses d. Option b is rejected because “when Z increases then it would increase very fast at the end, but in the table very fast increasing is in the beginning” (Rg). S supposes option c. Because they are not sure, also option h is discussed and rejected: “if Z increases then the denominator becomes larger and the fraction smaller; so, it is not h (Ro). S gestures how the graph should look like (increasing, concave down). They decide “So, g has to be smaller than 1; so, it is c” (score 0.3)

Conclusion and discussion

Our aim was to show how the role of mathematics in the model of Geyer & Kuske-Janßen for interpreting formulas might be expanded and to investigate whether mathematics lessons on graphing formulas might support students’ abilities to recognize the structure of formulas (level IIB in the adjusted model) and to reason with formulas about covariation (level IIIB). To this end, we administered a test with formulas in physics contexts to four students who had learned to graph formulas ten month before. The think aloud transcripts in the result section showed that these students worked at level IIB and IIIB of the adjusted model. They recognized and used function families like $y = e^{-x}$, $y = e^x$, $y = \frac{1}{a^3}$, $y = \frac{1}{x}$, $L = a \cdot g^Z$ (IIB). They also showed their reasoning on level IIIB: “if x becomes very large then ...” (Ro) was used by group 1 in task 3b; “if x is greater than ...” (Rg) was used by both groups in task 1 and 4; “if x approaches to” (Ra) was used by both groups in task 2a and 2b. During their reasoning, the students sometimes referred to the physics contexts, e.g., when the velocity cannot be larger than the speed of light (group 1), and that a baby cannot shrink (group 2).

The reasoning of the four students was similar to the reasoning taught in the graphing formulas lessons in grade 11. Several aspects of the teaching of graphing formulas by hand might explain why these students were able to work in this test at level IIB and IIIB. First, the focus is on qualitative reasoning, i.e., reasoning about global features of a function/graph, using natural language (in/decreasing, infinity behavior, etc.). This is in line with Geyer and Kuske-Janßen (2019), who emphasized the need of repeated opportunities to translate disciplinary understandings into natural language. Secondly, graphing formulas focuses on the relationship between variables, which is important as students often use formulas to do calculations instead to see formulas as relationships between quantities (Sokolowski, 2020). The adjusted model is also in accordance with Redish’s suggestion to use graphs as intermediary between physics contexts and equations. When students learn to interpret what information the graph is telling about the physics system, and learn to connect graphs to formulas, then graphs can bridge the physics systems and formulas (Redish, 2023). This idea is represented in level IV (and V) of the adjusted model.

Naturally, this study had its limitations. Only four students thought aloud while working on only four tasks, with predominantly physics contexts (only one biology context). Another issue is the long period between the mathematics lessons on graphing formulas and the test. And, although we found in the test similar reasoning as in the graphing formulas lessons, which is not obvious at all, it is somewhat speculative to attribute these performances merely to these lessons. Further research is needed, also to validate level IV and V of the adjusted model, i.e., to investigate students' abilities to blend their mathematical and their physics knowledge to construct rich concept images of formulas.

This study proposes the Adjusted Interpretation formulas model as an expansion of the Geyer and Kuske-Janßen model for interpreting formulas. The findings of this study show that the students were able to recognize the structure of formulas and to reason about covariation and graph features. This suggests that graphing formulas by hand in mathematics lessons might support the use of mathematical formulas in physics, in other natural science lessons, and in STEM contexts.

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