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# Topological approach to historical analytic narrative in the ‘Game Theory lab’ activities

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*The application of mathematical theories of rational choice, such as individual decision theory, game theory, and social choice theory, to fields like economics, sociology, and politics has been rapidly increasing. Game theory in particular has emerged as the logical and mathematical framework for understanding the conflict and interaction between individuals. The ability to reinterpret past events through game theory models has made history an analytical narrative of events, with mathematics as the primary tool used to formalise universal models to analyse the interactions between agents. In this work we will illustrate an experimental educational laboratory developed with the students of the last classes of the Mathematical High School Project and dedicated to the theory of rational choices.*

*Keywords: Mathematics education, performance technology, game theory, mathematical high school project, analytic narrative.*

## Introduction

The Department of Mathematics of the University of Salerno has unveiled an innovative educational path called the Mathematical High School Project (MHS) and focuses on restoring the interdisciplinary dimension of mathematics in the teaching curriculum. In the effort to foster a greater understanding of the interplay between mathematics and history, researchers have designed and tested educational paths and laboratories that focus on the intersection of these two disciplines. The laboratory described in this paper focuses on mathematics, history and economics, with the goal of helping students gain a holistic view of how mathematics has influenced historical events, as well as how history has sometimes directed the development of mathematics. In fact, mathematics is a powerful tool in the process of event analysis, and is often used as the main formalization tool in the formation of theoretical models and is of aid and support to Game Theory (GT). The laboratory project has been fully developed in accordance with the national guidelines set by the Ministry of Education, with a focus on providing a comprehensive and comprehensive experience for students. The indications state that “[the student] will be able to frame the various mathematical theories studied in the historical context within which they developed and will understand their conceptual meaning. The student will have acquired a historical-critical vision of the relationships between the main themes of mathematical thought and the philosophical, scientific and technological context.”

So, the importance of this project is clearly outlined, emphasizing that it should provide a comprehensive learning experience for students, incorporating aspects of both theoretical and practical knowledge. In addition, the project should also help to ensure that students develop the necessary skills for their future paths.

## Literature

The social sciences differ in the methods used to study their subjects, and economists in particular are distinguished by their belief in methodological individualism, the idea that social phenomena should be explained through the study of individual behaviour. This belief is based on two principles: the first assumes that people have consistent preferences and make choices based on them, while the second principle examines how individual behaviour aggregates to form more complex phenomena, taking individual preferences into account and looking for a consistent way to form collective preferences (Brams, 2011). GT, offers a more general view of how individual behaviour can be aggregated to examine more complex phenomena for analysing interactions between people. The purpose of a game is to provide useful insights into the interaction being studied, to predict possible outcomes of the interaction and to investigate whether there is an equilibrium of the game. Researchers are challenged to choose a model to use that best represents the interaction between agents.

GT is the mathematical study of conflict and cooperation between different agents, referred to as players (Israel & Gasca, 2008). GT concepts provide a mathematical framework for formulating, structuring, analysing and understanding such game scenarios, i.e. they provide useful mathematical models and tools for understanding the possible strategies that agents may follow when competing or cooperating in games. By assuming that individuals are intelligent, rational and motivated to maximise their own welfare, economists can better determine whether social problems are caused by flaws in institutions or by ill-informed individuals or non-rational behaviour. The Nash equilibrium is the formal definition of this hypothesis and provides a framework for predicting the behaviour of players in non-cooperative games.

In the literature for the analysis of historical events, the analytical-narrative approach is widespread, which aims to highlight the logic of the processes that generate the phenomena studied, using rational choice theory and GT to consider the sequence of choices and their importance for the results. According to Bates et al. (1998), some historical events that raise interpretative and explanatory problems cannot be solved with classical narratives and require the use of well-formalised models. Analytical narratives use qualitative data, such as changes in institutional patterns and decisions made by individuals or groups, to understand individual and collective preferences in economic modelling (Mongin, 2010, 2016). Analytical narratives analyse historical events by combining the narrative approach of historians with the analytical tools used by economists and political scientists in rational choice theories. GT is one of the main tools of narrative analytics, which enables the analysis of strategic interactions between the players in the history.

Mongin first used the analytics narrative approach to analyze historical events and to learn research methods. By tracing the historical context, identifying actors, decision points, choices made, and possible strategies, he constructed an analytical narrative that highlighted the logic of the processes that generated the phenomena studied. GT considers the sequence of choices and emphasizes their importance for outcomes, reflecting the influence of history, uncertainty, and people's ability to manipulate and strategize. Analytical narratives use qualitative data to uncover the individual and collective preferences behind economic modeling. This approach is surprising for students as it

represents a departure from the mathematical calculations of strategic interaction and rational choice associated with disciplines in the humanities.

## **Theoretical framework**

The didactic laboratory's activities were designed and implemented by employing the theoretical frameworks of Chevallard, Rabardel, and Verillon. The main objective was to instill new learning behaviors through the use of technology in the students' routine.

Yves Chevallard's anthropological approach highlights mathematics as a social undertaking practiced and propagated by institutions. The activity of mathematics can be explained through praxeologies, a term coined to emphasize the importance of considering both practical and theoretical aspects of a given task (praxis + logos = praxeology). Changes in praxeologies may have significant implications for the meaning of mathematical objects used (Chevallard, 2020).

The process of instrumental genesis, as described by Rabardel and Verillon, plays a crucial role in the transformation of a material object into a functional tool. Instrumentation and instrumentalization are important factors in the successful integration of artifacts into a user's cognitive structure and their effective use in the work environment (Verillon & Rabardel, 1995).

## **The MHS project**

The MHS project is an innovative initiative dedicated to high school students, which seeks to expand learning opportunities by developing interdisciplinary paths that integrate mathematics as a cultural glue between the humanities and scientific disciplines (Rogora & Tortoriello, 2018 and 2021). Currently active in over 160 high schools across Italy, the project involves the collaboration of 26 universities that are dedicated to providing students with cutting edge educational resources and opportunities. Through this project, students are able to gain insight into the role of mathematics in the development of culture, history (Tortoriello & Veronesi, 2021), technology (Tortoriello & Veronesi, 2021), and the social sciences (Bimonte et al., 2022 and 2023). Additionally, students have the opportunity to interact with university professors, creating a stimulating and enriching learning experience.

## **The Game Theory Lab Project**

The Game Theory Lab Project (GTL) is working to gain a better understanding of reality through an interdisciplinary approach. Students from MHS have been carefully selected to bring a comprehensive and in-depth vision of real phenomena, by utilising a multidisciplinary approach. Through the use of mathematics, engineering, and computer science, the students are striving to gain a more profound understanding of the world around them and how the various disciplines interact with each other. The project is hoping to gain insights into the real-world implications of game theory, to better prepare the students for their futures in the professional world. The project is also looking to explore innovative ways to apply GT to create new solutions and strategies, thus furthering our understanding of the world.

At the GTL, we design biennial courses to promote rational thinking for the interpretation of both social and individual choices. Through our courses, we strive to empower students to critically

examine the guiding forces that shape their decisions and the decisions of others. We focus on developing their self-awareness, analytical skills, and their ability to evaluate the consequences of their choices, as well as those of their peers. We also aim to provide students with an understanding of how these choices affect the global community, so they can make informed decisions in the present and in the future. We believe that these courses are essential to the development of our students, and to the future of our society. The rationality of our decisions and thoughts can be undermined by many factors, including misperceptions, cultural influences, and educational levels. These elements can lead us to make decisions that are not based on logic, but rather on assumptions and preconceived notions. For example, individuals may have false beliefs about certain people or cultures, or may lack understanding of complex topics due to educational deficiencies. These factors can lead to flawed decisions that may have far-reaching consequences, so it is important to critically evaluate thoughts and actions to ensure that we are making decisions based on informed and rational considerations.

## **Methodology**

The activities of the workshop are developed in four afternoon meetings of two hours each. Students divided into groups face classic GT problems, analyse possible scenarios and determine the optimal choice in various contexts. The constructivist approach to teaching and learning is based on the idea that students are active participants in their own learning process (Vygotsky, 1962). This methodology emphasizes the role of teachers as facilitators who mediate learning experiences, providing students with the necessary tools and resources to construct their knowledge. In this approach, students are encouraged to engage in meaningful and authentic activities that allow them to explore and discover new concepts and ideas. In constructivist classrooms, teachers use a variety of instructional strategies, such as problem-based learning, inquiry-based learning, and cooperative learning, to create a learning environment that fosters critical thinking, creativity, and collaboration. Through these strategies, students are challenged to think deeply about the content, to apply what they have learned in real-world situations, and to work collaboratively with their peers to solve problems and create new knowledge. Overall, the constructivist approach promotes active, student-centered learning that is grounded in the belief that learning is a social, interactive process that involves the construction of meaning through experience and reflection.

## **Contents**

GT is the study of situations between competing or cooperating agents, where by agents we mean human beings, entities, cells or machines. It is the science of optimal strategy and decision-making in a strategic context, where the agents' choices depend on the decisions of the other players involved. A game is defined by the number of players,  $N$ , the set of strategies for each player,  $S_i$ , and a pay-off function defined on the cartesian product of the sets of strategies,  $f(s_1, s_2, \dots, s_n)$ . In a positioning game, the plan represents a constraint on the possible strategies of each player.

One of the multifaceted themes that students encounter in their studies is the decision-making process concerning asset allocation. For approaching optimal positioning problems in GT, geometry has been an important tool for students who didn't study analysis for function with severable variables. The utilization of traditional tools such as the compass and ruler has enabled students to initiate the process of constructing viable solutions on the plane. However, it is evident that even with minimal points,

the complexity of problem representation is heightened. To overcome this obstacle, the laboratory has introduced dynamic geometry software that expedites problem-solving significantly. This issue holds significant relevance across disciplinary boundaries, as it applies to a diverse range of contexts, including but not limited to the storage of hazardous waste, the placement of nuclear power plants, and the siting of critical facilities such as hospitals or emergency response centers. Additionally, an exploration of this topic from a historical perspective can yield valuable insights into the outcomes of pivotal conflicts and the fields on which they were fought. From a mathematical perspective, it appears that students face considerable difficulties in resolving problems that involve optimizing constraints across multiple variables, primarily due to a dearth of analytical tools necessary for the task. However, the application of geometric principles to these problems can enable students to construct a comprehensive range of feasible choices and ultimately identify the optimal position, leveraging key mathematical concepts such as

The Voronoi tessellation which, given  $N$  points of a space, divides the space into domains containing the points with the characteristic that in each domain  $K$  there are all the points that have from point  $k$  generating tessellation shorter distance than all other generators

The Delunay triangulation which, given  $N$  points of a space, constructs a lattice of triangles having vertices on the points with the characteristic of having the maximum minimum angle, that is, among all possible triangulations considers the one from the greatest value of the smallest angle present and, with reference to the activities of the laboratory, allows to calculate the circumcenter from the maximum area.

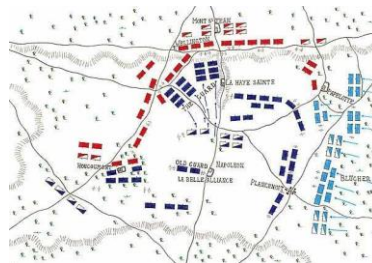
The use of Geogebra software facilitates tessellation of space and enables students to search for the optimal point based on distance, expounding on their understanding of the subject matter. Additionally, they are presented with Zhao's theorem of 2008, which illustrates that the point obtained is precisely the Nash equilibrium point, typically computed analytically, further enhancing their knowledge base.

### **An example - The activity “Waterloo battlefield”**

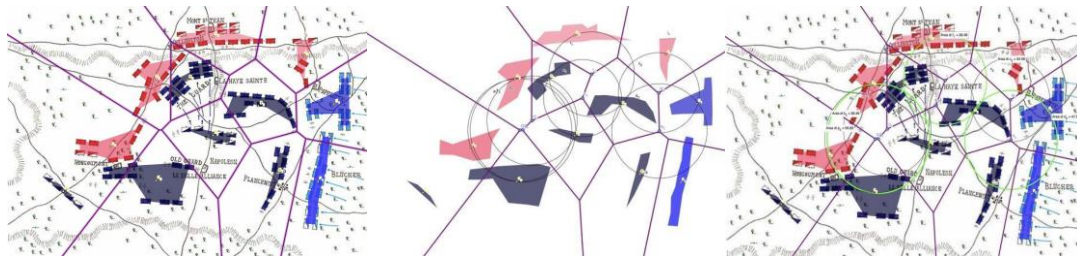
During the workshop, the group of students were presented with abstract and simplified scenarios. Their task was to identify the optimal positioning of the scenarios proposed on educational cards. The workshop also demonstrated how positioning plays a significant role in various contexts such as waste storage, hospitals, nuclear power plants, and military strategies. The activities of the laboratory are not limited to the academic and theoretical study of battles and wars. Space was also dedicated to the interactive and engaging construction and modelling of famous battles such as the battle of Waterloo.

This is possible through the use of dynamic geometry software, which allows us to create a detailed and accurate topology of the battlefield as in Figure 1. The decision to examine the Battle of Waterloo is connected to the research undertaken by Mongin through the analytical narrative technique (Mongin 2018). Mongin opted for a probabilistic approach, whereas in our laboratory, we adopted a geometric approach that utilized the students' existing knowledge of topology. The search for the ideal strategic position for Napoleon to take advantage of in battle was made possible by using topological techniques. With the students of the laboratory, it was analyzed in particular with the Voronoi tessellation of the plane and the Delaunay triangulation. This analysis was supported by the

implementation of technological solutions, such as computer-generated simulations, to determine the best course of action for a given conflict. First, with Voronoi, the plan of war action was partitioned by attributing a domain of action to the space surrounding each troop. This domain of action was calculated by determining a Voronoi diagram, which is a partitioning of a plane into regions based on the distance to points in a specific set of points as seen in Figure 2a.



**Figure 1: Waterloo battlefield**



**Figure 2: a) Voronoi tessellation, b) and c) Circumcircles by Delaunay triangulation**

Subsequently, utilising the Delaunay triangulation, the centres of the circumcenters concerned were identified and the maximum area circumcenter was accurately calculated as seen in Figure 2b. This area of the circumcenter was then used to find the best possible solution for the problem at hand.

The centre of the maximum circumscribed circle (Figure 2c) turns out to be a Nash equilibrium point and is the point that topology indicates as a strategic optimal for Napoleon.

## Observations

During the activities, students engaged in geometric studies to calculate the point of optimum. Additionally, they were presented with historical documentation indicating that this same point was identified by military strategists who analysed the battle from a strictly war-oriented perspective and determined it to be the best strategic position for Napoleon's troops. By applying a mathematical perspective to the analysis of the battle, it became clear that Napoleon's strategic moves were less effective in attaining the optimal position due to his failure to consider the positioning of certain troops in the field. It is noteworthy to observe the utility of mathematics in providing insights into the battlefield tactics of renowned historical figures like Napoleon.

The students engaged in an activity pertaining to military strategies, showcasing a comprehensive grasp of mathematical concepts including the topology of the plane and the Euclidean properties of distance. Additionally, they demonstrated proficiency in advanced mathematical concepts such as Voronoi tessellation and Delaunay triangulation, which are typically beyond the scope of secondary education. The students confidently applied these concepts with the use of dynamic geometry software, both in theoretical scenarios such as the placement of a retailer in a given area and in

practical mapping exercises. The students' final reports indicate an appreciation for the real-world applications of mathematics, while their worksheets demonstrate their ability to convert abstract concepts into concrete solutions. The activity was a vivid demonstration of the power of interdisciplinarity. The students took on the challenge of navigating through a variety of topics, such as history, economics and mathematics, without viewing them as isolated or separate. Instead, the students were able to integrate and combine their knowledge of each discipline to create their own unique path. It was remarkable to witness the students' determination and aptitude for approaching the challenge from different angles and integrating ideas from a multitude of disciplines. This truly highlights the profound potential of interdisciplinary collaboration and serves as a reminder of the importance of considering multiple perspectives when tackling complex problems.

## **Conclusions**

The concept of GT has a long history, with its roots stretching back centuries. It has become increasingly important in the modern era as its mathematically and logically systematic formulation has become necessary for explaining situations of conflict or interaction between multiple individuals or entities. It has become an invaluable tool for analysing decision-making processes, and has been used to great effect in a variety of fields, from economics to political science. GT has provided us with a way to understand the strategies and strategies of decision-making in a variety of contexts, from sports and business to international relations. It is a powerful tool for analysing the behaviour of people and organisations in conflict, and for understanding how different strategies can be employed for the best possible outcome.

Through its various manifestations, mathematics has become a universal key for understanding and interpreting the relationships between various agents. This makes it a vital component of analytical narratives of events, as mathematical models provide a logical and consistent framework for interpreting the behaviour of agents, and the implications of their actions. Through the use of mathematics, analysts can develop detailed, accurate, and insightful models of the events, allowing them to gain a deeper understanding of the overall dynamics of the particular situations.

In our analysis of some key historical events, we identified essential elements of history to construct a comprehensive analytical narrative. This narrative allowed us to examine the equilibrium of the model, and gain greater insight into the underlying dynamics at play. By delving into the historical context of the events and exploring the multiple facets of the model, we were able to gain a deeper understanding of the history and its implications for the present. Through this process, we were able to recognize the importance of history and its lessons, and to build an informed model for the future.

The laboratory, as discussed in the paper, was specifically designed for the MHS and catered to students who held a keen interest in the discipline. It is possible that this targeted approach played a role in the positive outcomes observed in the analysis of student assessment questionnaires. In order to examine the impact of this factor, we intend to replicate the same activity in a conventional scientific high school, with the aim of conducting a comparative evaluation of student feedback.



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