



HAL
open science

An introduction to TWG3: Algebraic thinking

Maria Chimoni, Cecilia Kilhamn, Luis Radford, Jorunn Reinhardtzen

► **To cite this version:**

Maria Chimoni, Cecilia Kilhamn, Luis Radford, Jorunn Reinhardtzen. An introduction to TWG3: Algebraic thinking. Thirteenth Congress of the European Society for Research in Mathematics Education (CERME13), Alfréd Rényi Institute of Mathematics; Eötvös Loránd University of Budapest, Jul 2023, Budapest, Hungary. hal-04417012

HAL Id: hal-04417012

<https://hal.science/hal-04417012>

Submitted on 25 Jan 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

An introduction to TWG3: Algebraic thinking

Maria Chimoni¹, Cecilia Kilhamn², Luis Radford³ and Jorunn Reinhardtson⁴

¹University of Cyprus, Cyprus; mchimo01@ucy.ac.cy

²University of Gothenburg, Sweden

³Laurentian University, Canada

⁴University of Agder, Norway

Overview of themes

In the Thematic Working Group 3 on “Algebraic Thinking”, 21 papers and 2 posters were presented with a total of 26 participants coming from Europe and other continents. The contributions reflected recent evolutions with respect to the nature of algebraic thinking and the routes by which it might be fostered among students of different ages and educational backgrounds, as well as pre-service and in-service teachers. The papers were presented and discussed in working sessions under four themes: (1) Algebraic thinking, (2) Patterns and functions, (3) Equal sign and equivalence, and (4) Variables and equation solving. Considering that these themes are intertwined, the assignment of papers to themes was guided by a “best fit” approach. We discuss each of these themes below. The report concludes with key issues and challenges, as these were extracted from the discussions during the sessions.

Algebraic thinking from different perspectives

This theme included five papers related to the role of various factors in supporting the development of algebraic thinking: research translated into practice, rich tasks, teachers’ subject matter and pedagogical content knowledge, and cognitive processes. The paper by Day presented a large project that aimed to the development of a learning trajectory for algebraic thinking among middle school students, based on existing research. This was aligned with formative assessment tasks to identify where students are on their learning journey, as well as explicit teaching modules. Fred et al. introduced the term “algebraic wicked problems” to characterize rich tasks that can lead students of 7-9 years old to engage with core algebraic ideas, and at the same time reflect on environmental and societal issues. It was concluded that the design of these tasks and their successful implementation remain challenging.

The other two papers in this theme focused on pre-service teachers and ways of helping them to enhance their own algebraic thinking abilities and knowledge related to students’ typical misconceptions, errors, and strategies. The study by Kaloutsis et al. presented the way tasks that involve properties of numbers and operations were devised to elicit pre-service teachers’ argumentation skills. The results revealed that pre-service teachers exhibited various types of arguments ranging from naïve to formal. Cabral proposed an alternative activity that assisted pre-service teachers in deepening their understanding of core dimensions of early algebraic thinking. Tasks that addressed relational and functional thinking were solved by both the pre-service teachers and children/students of 4-10 years old. The pre-service teachers were then asked to compare their own solutions to those provided by the students and notice students’ approaches.

The last study in this theme by Chimoni et al. examined algebraic thinking of students from Grades 4 to 7 from a cognitive perspective. The results indicated that a significant predictor of algebraic thinking abilities in all grades was working memory. This implies that algebraic achievement relates to the ability for selecting, maintaining, and using information, indicating that teachers need to be aware of the cognitive effort imposed by students.

Patterns and functions

The five papers in this theme addressed the role of tasks and models that help students to explore functional relationships. Figural patterns seem to remain a dynamic tool that can be used for different purposes within algebra lessons. Nyman et al., investigated the experience of Grade 5 students with a set of figural pattern tasks and identified a 3-level hierarchical structure in students' approaches, varying from the view of singular discrete images to understanding that the images become larger in a systematic way to the extraction of rules. It was highlighted that teachers need to be able to assess the level implied in students' responses—albeit some of them do not involve formal algebraic symbolism. Figural pattern tasks were also used in the study of Mora Badilla et al. as a tool for identifying potential gifted students in the primary grades. A framework was formulated that encompasses all different categories of behaviors that students may exhibit when they deal with figural patterns and further a set of “discriminants” was extracted which reflect potential gifted students. Goñi-Cervera et al. presented the way a 9-year-old student with autism spectrum disorder was able to engage with figural pattern tasks. The advanced spatial abilities of this student were considered to support the interpretation of the spatial structure within the figural pattern and a movement from unsuccessful approaches to the identification of recursive relationships. Similar results were also shown by Gregorio who worked with a student in Grade 9 diagnosed with dyslexia and dyscalculia. Figural patterns were found to motivate this student to see relationships between the terms of the pattern, despite the problems the student had with counting and algorithms. Lócska et al.'s study questioned the usefulness of using a bar model for representing the functional relationship embedded in a word problem. The results showed that some Grade 8 students preferred to use the bar model method while others preferred to model the problem directly with an equation. The discussion of these papers altogether raised deep issues, including how effective different models or representations are stimulating conceptual understanding and assisting students to move to abstract contexts.

The two posters presented in TWG3 also dealt with generalization and functional relationships. Palm Kaplan presented the design of a study that aims to provide a research overview of studies related to generalization, thus teasing out what we already know about this much discussed issue. Šabaková presented the preliminary results of an empirical study about the role of graphical representations in helping students to discuss that changing the scale may change some visual properties of the function without changing the properties of the function itself.

Equal sign and equivalence

The first two papers in this theme explored tasks and tools that may facilitate understanding of the equal sign as a symbol of equivalence. Mestre described a task that aimed to engage Grade 1 students in using quantitative relationships for finding the value of an unknown quantity. The

task involved a word problem related to the equivalence between the number of blue and red marbles, as well as a visual representation. The results showed that the task did not successfully trigger the students to think in a relational way, since they insisted on using counting. It was discussed that students may need more systematic work and time with these kinds of tasks. Venkat et al. presented a large-scale project designed in the context of South Africa, involving the development of a wide range of teaching and assessment materials that promise the successful combination of algebraic and arithmetic goals in teaching and learning early mathematics. The key features of these materials included open number sentences with a focus on equivalence, verbalisations of equivalence, and attention to patterns in calculations.

The other two papers assigned to this theme showed that older students have perspectives on the equals sign and equivalence that do not mirror those that researchers would like to see in the secondary grades and beyond. Petterson et al. showed the gap in Grade 9 students' strategies when solving a simple linear equation task and a linear expression task that required proportional reasoning. Students used either algebraic or arithmetical strategies to solve the linear equation task. However, arithmetical strategies proved not to be powerful enough for solving the proportional task. Wladis et al., explored how college students conceptualize properties of operations. Their findings suggested that students' "descriptive conceptions" about the properties of operations, i.e., the conceptions that they developed did not fully align with the "prescriptive conceptions", i.e., the conceptions that were intended for them to acquire from instruction. There were cases where students used the properties of operations to transform algebraic expressions and others where they did not notice their relevance.

Variables and equation solving

The papers assigned to this theme discussed how students learn and perceive the concept of variable within different pedagogical contexts and educational levels. Radford's study presented a teaching-learning activity that aimed to facilitate Grade 4 students to move from a manipulative-based context for solving equations that used cards and envelopes to a context that involved alphanumeric symbols. The findings indicated that there is a gradual shift from "denotative" to "connotative" meanings, i.e., a shift from understanding a letter as meaning something else (e.g., envelopes) to understanding that a letter acquires a value within a semiotic system that uses specific signs. The study presented by Klingbeil investigated the potential of a technological tool, the SMART online test, to automatically diagnose misconceptions related to the concept of variable (e.g., the "Letter as Object" misconception) among Grade 7-8 students. The findings showed that the multiple-choice items allowed teachers to detect certain misconceptions, thus nourishing the practice of formative assessment.

Kontreff proposed the use of mathematical narratives as a means that supports students of Grades 8-9 to become aware of the meanings of variable as an unknown and as a generalizer. The results of this study showed that posing and comparing mathematical narratives helped students to understand the differences between these meanings. Oldenburg also discussed the different conceptions of variables through the lenses of the Anthropological Theory of the Didactics (ATD). This theoretical paper contrasted the conceptualization of variable at three levels: university, school, and the didactics of mathematics, indicating an inconsistency between university and school which could be mediated through the didactics of mathematics.

The other papers in this theme focused on strategies and errors of secondary school students when solving linear equations. The idea of algebra as a “gatekeeper” in school mathematics was revisited by Sjöstrand et al. who presented an empirical study about the effect of algebraic thinking abilities to mathematical achievement. This study presented evidence that there exist students even in the upper grades who use arithmetical strategies rather than algebraic methods for solving linear equations. Moreover, the results verified that students who used algebraic strategies for solving linear equations prior to a mathematics course were more successful in the post-assessment test. Holmlund’s study focused on solving linear equations that involve different numbers as coefficients, for example positive whole numbers, negative numbers, and decimals. The results showed that the ability to successfully solve the task is affected by the numbers involved. It was discussed that the use of numbers other than whole numbers can challenge the students to reflect on the structure of the equation. Finally, Çelikdemir and Koştur presented an empirical study that linked algebraic thinking with proportional reasoning. They found that Grade 8 students were able to adapt the strategy they used according to the type of the proportional reasoning problem they had to solve. The formulation of an equation that models the cross-product algorithm was not the most preferable strategy, indicating a flexibility in analyzing the quantitative relationships involved in these problems.

Key issues from discussions and challenges

This section outlines the main issues discussed during the last session and challenges that may feed future research. The design of rich tasks that promote algebraic thinking among students and teachers was frequently discussed. We feel that understanding of what a task should do is an important area for future work, specifically considering purposes such as providing students with opportunities to develop algebraic arguments, discuss societal problems, or learn in cooperation with the teacher and the other students. The need to better understand algebraic thinking in special student populations was a feature in several papers, and we feel that more work could be focused on how the distinctive way in which these populations analyze situations can be linked with algebraic processes.

The different conceptions of variable and their educational implications were revisited, and more research could be done around the processes that underpin these concepts and their use (e.g., linguistic, symbolic, and cognitive). The transition to algebraic symbolism was also at the core of discussions, and we agreed that we still need to better understand the interface between concrete and abstract semiotic systems. Should we assume a continuity or a rupture in the corresponding ways of thinking? Is it worthwhile teaching with concrete materials and models? Another issue extensively discussed was the role of language. The importance of naming mathematical objects and relationships to be able to talk about them, and hence, think about them was highlighted. What kinds of language uses, as described by different theoretical stances, are important to algebraic thinking and affect its quality? As communication of ideas, as an expression of thought, or as a social dialogical means of signifying self and knowledge? The link between arithmetic thinking and algebraic thinking was once again debated. We feel that future work needs to address the differences more explicitly between arithmetic and algebraic thinking, as well as their evolving complementarities. This understanding has implications in teachers’ practices and curriculum development.