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# Exploring students' understanding of multiplication with eye tracking: A study on the use of strategies in array representations

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*A central concern of mathematics education is to develop students' understanding of multiplication. To this end, it is necessary to investigate students' strategies in multiplication tasks. Common representations for multiplication are array representations (AR) (i.e., rectangular arrangement of elements). Previous research has identified strategies in multiplication tasks with AR. However, little is known about students' use of these strategies—which would allow inferences about students' understanding of multiplication. This paper presents an eye-tracking study on strategies of 163 fifth-graders in multiplication tasks with AR, and their use of these strategies. Three strategies were found: In two of them, the students made use of multiplicative structures, while in one they did not (e.g., counting all dots). We found that the fifth graders used a counting strategy in almost one-fifth of the tasks, which suggests difficulties in the understanding of multiplication.*

*Keywords: Multiplication, strategies, eye tracking, array representations.*

## Introduction

Multiplication of natural numbers (e.g.,  $8 \cdot 7$ ) is an important part of mathematics education (KMK, 2022; NCTM, 2000) and serves as the foundation of more advanced concepts of multiplication (e.g., fractions) (Downton & Sullivan, 2017). Although multiplication within 100 is taught in primary school, students in the transition from primary to secondary school may still encounter difficulties in their understanding of multiplication (e.g., Moser Opitz et al., 2017). Since difficulties in understanding of multiplication can cascade into severe problems in later learning, it is important to gain insights into students' understanding of multiplication, particularly their use of strategies (Downton & Sullivan, 2017; Mulligan & Mitchelmore, 2009), to support students appropriately.

For developing an understanding of multiplication, it is crucial for students to grasp various meanings and to build connections between different representations of multiplication. This requires the development of so-called *basic ideas* (“Grundvorstellungen”) of multiplication that transfer between different representations of multiplication (e.g., symbolic numbers and iconic representations) (e.g., Prediger, 2008). One important basic idea of multiplication is the *spatial-simultaneous* basic idea, which is typically supported through the use of array representations (AR) (where elements are arranged in rectangular shapes) due to their visual and spatial nature.

Previous studies have investigated strategies in multiplication tasks with AR (e.g., Barmby et al., 2009; Bolden et al., 2015): Students have been found to solve multiplication tasks with AR in different ways and their strategies have been found to indicate differences in multiplicative understanding, for example, when students draw on properties of multiplication when working on tasks with AR, or when they count all elements of AR, indicating difficulties in their understanding of multiplication (Barmby et al., 2009). However, little is known about how students use strategies

in multiplication tasks, that is, how often students use strategies that reflect a multiplicative understanding and how often they use strategies that reflect difficulties in their multiplicative understanding.

The aim of this study is to investigate students' strategies and their use of these strategies when working on multiplication tasks with AR. We pursued this aim by investigating what strategies are used and how often these strategies are used by students. We conducted a study with 163 fifth graders and used eye tracking (ET), which has been shown to be suitable for collecting large amounts of data (e.g., Simon et al., 2021) and has been shown to be a valuable method for identifying student strategies in mathematics, including multiplication tasks with AR (Bolden et al., 2015).

## **Theoretical background**

### **Multiplication of natural numbers**

For an understanding of multiplication, it is necessary for students to develop basic ideas of multiplication (vom Hofe et al., 2005). Basic ideas carry different meanings of mathematical concepts and build connections between the mathematical world and the individuals' world of thinking (vom Hofe et al., 2005). Main basic ideas of multiplication of natural numbers include (1) spatial-simultaneous interpretation (rectangular array), (2) temporal-successive interpretation (repeated addition/equal groups), (3) multiplicative comparison, and (4) Cartesian product (combinatorial interpretation) (Mulligan & Mitchelmore, 1997; Prediger, 2008; Watanabe, 2003). In this paper, we focus on the spatial-simultaneous basic idea of multiplication. The characteristics of this basic idea are that multiplicative structures can be recognized in spatial representations and that spatial representations represent multiplicative structures. For fostering the spatial-simultaneous understanding of multiplication, AR are often used.

### **Array representations**

AR are rectangular grids of discrete elements (e.g., dots) arranged in rows and columns. AR are important to develop students' understanding of multiplication since they represent multiplicative structures: The rows and columns are representatives of the multiplier and the multiplicand, which allows multiplicative interpretations (Barmby et al., 2009; Jacob & Mulligan, 2014). Furthermore, AR allow all dots to be represented simultaneously. A widely used AR both in school and mathematics education research is the 100-dot field (Figure 1; e.g., Schindler et al., 2019). The 100-dot field is a 10-by-10 grid with a bigger space between the fifth and sixth column/row. These structural spacings are designed to enable students to perceive the number of dots in rows and columns without having to count, by using structures (i.e., quasi-simultaneously) (Barmby et al., 2009). By presenting parts of the 100-dot field, all multiplication tasks with factors from 1 to 10 can be represented (see Figure 1 for examples).

### **Research on strategies when working on multiplication tasks with array representations**

Previous studies have investigated strategies used by students in the transition phase between primary and secondary school when working on multiplication tasks with AR (with and without structural spacings between the fifth and sixth column/row): For arrays with structural spacings, Barmby et al. (2009) analysed audio-visual recordings of students from grade 4 and 6. They found that students

used counting strategies (in ones or small groups), distributive strategies (distributive properties based on groups of 25), rearranging strategies (moving parts of the array to make the calculation easier), and completing strategies (first complete a bigger array then subtract) to solve the multiplication tasks. In a study using ET, Bolden et al. (2015) analysed eye movements of nine fifth graders when working on tasks with AR without structural spacings. They found that the fifth graders either counted all elements in the AR, or they counted only the elements in the rows and/or columns. Some of the identified strategies in these studies (e.g., counting all elements in the AR) indicated that students had not yet understood the spatial-simultaneous basic idea of multiplication sufficiently.

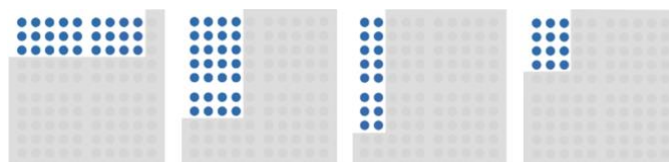
To the best of our knowledge, studies on students' strategies when working on multiplication tasks with AR tend to have small sample sizes (e.g., Barmby et al., 2009). Therefore, little is known about how students use strategies in multiplication tasks with AR, that is, how often students use strategies that reflect a multiplicative understanding and, more importantly, that reflect difficulties in their multiplicative understanding. Insights into the use of strategies are, however, necessary for determining students' need for support. ET is useful for identifying student strategies in multiplication tasks with AR (Bolden et al., 2015) and examining student strategies since it enables insights into student strategies for a variety of students (e.g., Schindler & Lilienthal, 2018).

Based on the current state of research, this study aims to investigate students' strategies and their use of strategies when working on multiplication tasks with AR through the following two research questions: (1) *What strategies can be identified in students' work on multiplication tasks with AR among fifth graders using ET?*, and (2) *How often are these strategies used by fifth graders for different multiplication tasks with AR?*

## Method

### Participants, multiplication tasks, procedure, and devices

Data were collected in a German comprehensive school with 163 fifth graders aged from 9.8 to 12.8 years ( $M = 10.6$ ,  $SD = 0.5$ ) in their first month of secondary school. The students worked individually on four multiplication tasks on a computer:  $3 \cdot 9$ ,  $7 \cdot 4$ ,  $8 \cdot 2$ , and  $4 \cdot 3$  (Figure 1). Multiplication tasks were presented on the 100-dot field, which was partly covered by a semi-transparent grey shape, leaving a rectangular arrangement of blue-coloured dots visible.



**Figure 1: Multiplication tasks with array representation**

Since the 100-dot field is a frequently used representation in German primary schools and is widely used in primary school mathematics textbooks as well, most students are used to working with the 100-dot field. However, since we could not rule out that individual students had not worked with the 100-dot field before or, more importantly, had forgotten about it, we presented a video to all students individually introducing the 100-dot field as well as the grey shape and how it can be used to represent multiplication tasks on the 100-dot field. This ensured that all students understood the representation.

After this video, students were given a sample task to ensure that they correctly understood the instruction and task. After the sample task, the four multiplication tasks mentioned above were given. For every task, students were asked to enter answers using a number keypad connected to the computer (e.g., 27 for the task  $3 \cdot 9$ ). Each task was shown until the students pressed the enter key on the number keypad. They were then asked to enter the number of blue dots visible in the respective task. Students moved on to the next task without receiving feedback about if the answer was correct. From one task to the next, the screen showed a colour-filled star in the upper-left corner, which the students needed to look at, to ensure that the gazes started from the same point for all students. The eye movements were recorded with the screen-based eye tracker Tobii Pro X3-120 (infrared, binocular, 120 Hz) which was attached to the bottom of the computer screen. A five-point calibration and a four-point validation were conducted with each student. Students' heads were about 60–65 cm away from the computer screen. The ET data showed an average accuracy of  $1.1^\circ$  ( $SD = 0.9^\circ$ ), which corresponded to an error of 1.2–1.3 cm on the 24" screen (53 x 30 cm, 60 Hz, 1920 x 1080 pixels).

### **Qualitative analysis: Identification of strategies when working on multiplication tasks with AR**

To identify students' strategies when working on multiplication tasks with AR, we used gaze-overlaid videos (gazes displayed as a semi-transparent moving dot) produced with the Tobii Pro Lab software. We used all ET data (i.e., analyses were not limited to, e.g., fixations) to analyse student strategies in as much detail as possible (e.g., Schindler et al., 2019). All of the student's videos were analysed, regardless of whether the children answered correctly or not, to avoid exclusion of videos displaying strategies that might be associated with (typical) errors. Student's responses were not included in the analysis of strategies. Videos of tasks were marked as data loss if they met one or more below criteria: gazes were displayed as flickering and not as a consecutive path; gazes started from the middle of the AR rather than the top-left corner, which was set as a starting point. In total, 87 of 652 videos were excluded, resulting in 565 videos remaining. Qualitative content analysis of gaze-overlaid videos was conducted (Mayring, 2000). The inductive category development consisted of three stages:

*First stage:* The first author of this paper watched all videos and wrote down descriptions of the gazes. Similar descriptions were grouped into the same categories, that is, strategies when working on multiplication tasks on a 100-dot field. This resulted in a preliminary category system. All videos were then coded based on this category system. *Second stage:* The second author of this paper coded approximately 11% (18 out of 163) of randomly selected students' videos based on the preliminary category system. Coding results were compared and different assignments of codes between the two researchers were discussed. Descriptions of the eye movements were then revised, and the category system was updated. *Third stage:* With the revised category system, all videos were again coded by the first author. About 26% (43 out of 163) of the videos were coded independently by the second author. Cohen's Kappa ( $\kappa = 0.874$ , 95% CI [.798, .949]) indicated a strong interrater agreement.

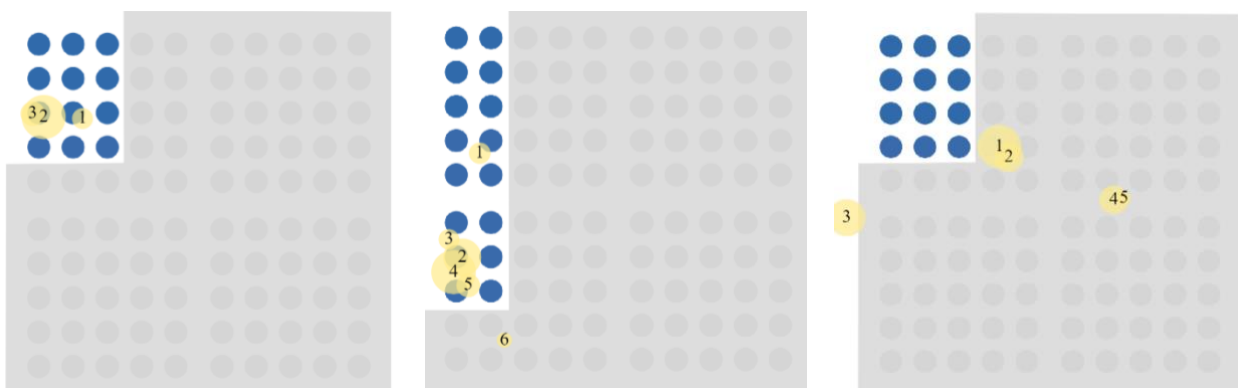
## **Results**

Analysing gaze-overlaid videos revealed three strategies fifth graders used when working on multiplication tasks with AR. Examples of student gaze patterns for each strategy are presented in Figures 2 and 3 and in an example video provided online (see Lai et al., 2023). We use gaze plots for presenting the strategies in this paper, although we used gaze-overlaid videos for data analysis.

**(A) Enumerating dots in rows and/or columns (quasi-)simultaneously:** We identified three kinds of approaches in this category:

*(A1) Simultaneous enumeration:* In this strategy, gazes went to one or two dots of rows/columns. This indicates that the number of dots in the rows/columns was perceived simultaneously (without counting). Since only a limited number of elements can be perceived simultaneously, students used this strategy only when the number of dots in one row or column was less than five. *(A2) Using structures and enumerating further:* Gazes went to one or two dots within the first five dots. Then the gazes moved to the dots after the structural spacing between the fifth and sixth row/column. These dots were either looked at one by one or perceived simultaneously. This indicates that the first five dots in rows/columns were enumerated quasi-simultaneously and further dots in rows/columns were either enumerated by counting or perceived simultaneously. *(A3) Using structures and enumerating backwards:* Gazes went directly either to the last blue dot of the row/column or to dots covered with the grey shape. This indicates that the number of dots was perceived using the structure of ten when enumerating backwards to nine or eight, or using the structure of five when enumerating backwards to four or three.

Gazes of strategies (A2) and (A3) indicate that students enumerated the number of dots in rows and columns using structural hints of the 100-dot field (i.e., the structure of 5 and structure of 10).



*(A1) Simultaneous enumeration*

*(A2) Using structures and enumerating further*

*(A3) Using structures and enumerating backwards*

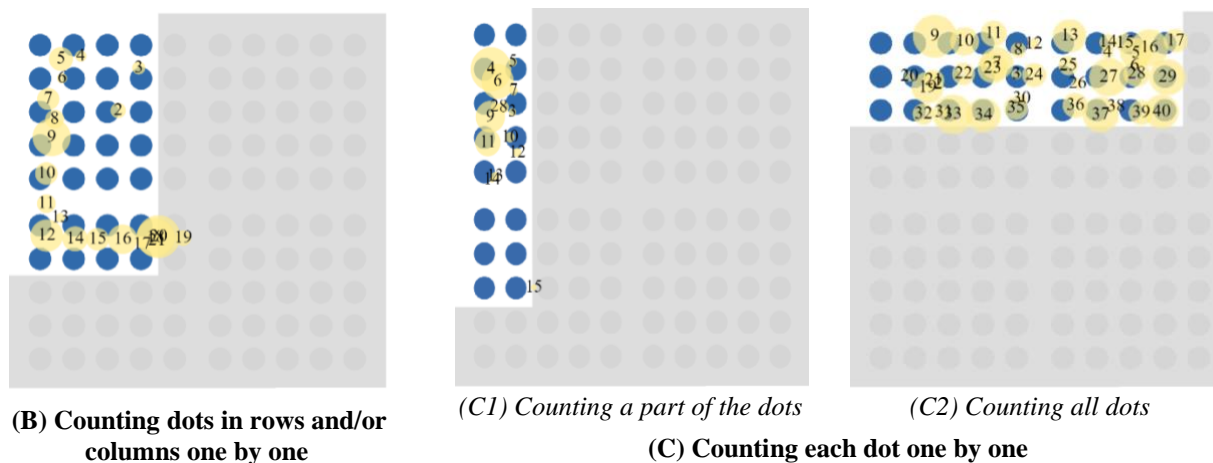
**(A) Enumerating dots in rows and/or columns (quasi-)simultaneously**

**Figure 2: Gaze plots of strategy (A)**

**(B) Counting dots in rows and/or columns one by one:** Gazes went to every dot in one row or/and one column. This indicates that students counted dots in row and/or column one by one to solve presented multiplication tasks.

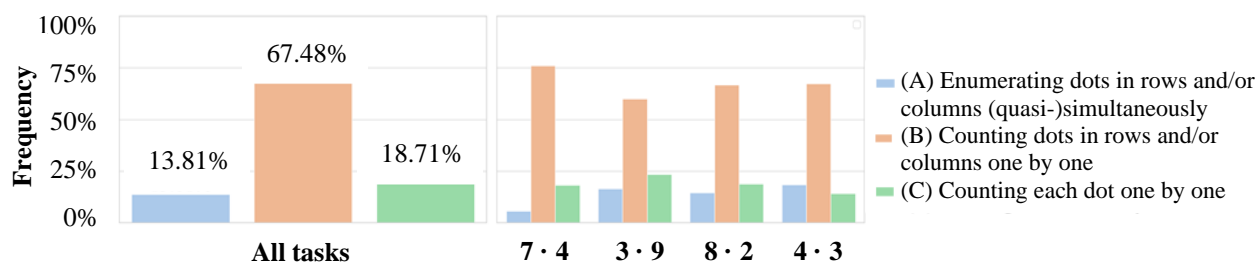
**(C) Counting each dot one by one:** There were two kinds of approaches in this category.

*(C1) Counting a part of the dots:* Gazes went to each dot of a part of the visible blue dots (not covered with the grey shape). The parts were usually separated by the structural spacing between the fifth and sixth row/column. *(C2) Counting all dots:* Gazes went to each dot of the visible blue dots. This indicates that students counted all given blue dots one by one to solve the presented multiplication task.



**Figure 3: Gaze plots of strategies (B) and (C)**

Figure 4 demonstrates the frequencies of the three strategies for all tasks and all students. The bars in the left subplot represent the percentage of each strategy, with strategy (B) *Counting dots in rows and/or columns one by one* being the most often used (67.48%), followed by strategy (C) *Counting each dot one by one* (18.71%). Strategy (A) *Enumerating dots in rows and/or columns (quasi)-simultaneously* was used least often (13.81%). The right subplot shows the use of the strategies in the four different multiplication tasks. Strategy B was used most often in all four tasks. Strategy A, which was the least often used overall, was used more often than strategy C in the task  $4 \cdot 3$ . For the other three tasks, strategy C was used more often than strategy A.



**Figure 4: The use of the multiplication strategies across all students and all tasks**

## Discussion

The aim of our study was (1) to investigate students' strategies and (2) their use of these strategies when working on multiplication tasks with AR. (1) We identified three strategies used by fifth graders when working on multiplication tasks with AR by analysing eye movements: (A) Enumerating dots in rows and/or columns (quasi)-simultaneously, (B) Counting dots in rows and/or columns one by one, and (C) Counting each dot one by one. Strategy (A) with its three subtypes has, to the best of our knowledge, not been identified in previous studies. This strategy was identified particularly because the 100-dot field was chosen as AR, which has structural spacings that allows for (quasi)-simultaneous enumeration. Strategies (B) and (C) were also identified by Bolden et al. (2015). Our findings connect to Bolden et al.'s insights and extend them by differentiating strategy (C) into (C1), Counting a part of the dots, and into (C2), Counting all dots. (2) We investigated how often the identified strategies were used by the fifth graders in this study. This had not been addressed in previous works, since previous studies had small sample sizes. In our study with 163 students, we found that while strategy (B) was most often used, strategy (C), where students did not use

multiplicative structures, was used in almost a fifth (18.71%) of the cases. This is particularly interesting since it indicates that a relevant number of students in grade 5 still appear to show (at least partial) difficulties in their spatial-simultaneous understanding of multiplication.

A potential limitation of this study is that only ET data were used as a data source, which allowed for a study with a large sample size. However, it might also be useful to use additional data sources to identify strategies, such as student interviews, to draw conclusions about student strategies and understanding with greater certainty. For example, the distributive strategy that Barmby et al. (2009) identified for similar AR through analysis of audio-visual data might be difficult to identify through ET. Here, the additional use of children's utterances could show possible links between the strategies we found to those found in previous studies.

What implications do the findings of this study have for the teaching of multiplication? Multiplication up to 100 is a mathematical topic at the primary level. However, fifth graders partially still have difficulties understanding multiplication (e.g., Moser Opitz et al., 2017), which also our study with 163 students indicated: The frequent use of strategies in which dots were counted one by one indicates that some students may not be able to use multiplicative structures in AR, which indicates difficulties in their spatial-simultaneous understanding of multiplication. This suggests that mathematics teaching at the primary level needs to focus even more on developing students' understanding of multiplication. To this end, it is necessary for students to develop basic ideas about multiplication (vom Hofe et al., 2005), to understand various meanings, and to become able to build connections between different representations of multiplication, including AR. Our study provides insight into student strategies and their use when working on multiplication tasks with AR. Given the crucial role of understanding multiplication for further learning of mathematics (e.g., Downton & Sullivan, 2017), our findings highlight the need to support understanding of multiplication, particularly basic ideas of multiplication, even at the beginning of secondary school.

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