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# ON THE DYNAMICS OF THE LINE OPERATOR $\Lambda_{\{2\},\{3\}}$ ON SOME ARRANGEMENTS OF SIX LINES

XAVIER ROULLEAU

ABSTRACT. The operator  $\Lambda_{\{2\},\{3\}}$  acting on line arrangements is defined by associating to a line arrangement  $\mathcal{A}$ , the line arrangement which is the union of the lines containing exactly three points among the double points of  $\mathcal{A}$ . We say that six lines not tangent to a conic form an *unassuming* arrangement if the singularities of their union are only double points, but the dual line arrangement has six triple points, six 5-points and 27 double points. The moduli space of unassuming arrangements is the union of a point and a line. The image by the operator  $\Lambda_{\{2\},\{3\}}$  of an unassuming arrangement is again an unassuming arrangement. We study the dynamics of the operator  $\Lambda_{\{2\},\{3\}}$  on these arrangements and we obtain that the periodic arrangements are related to the Ceva arrangements of lines.

## 1. INTRODUCTION

In [12], we give the formal definitions of some operators acting on line arrangements and configurations of points in the projective plane. The aim of that paper was to show that these operators appear quite naturally and ubiquitously when constructing or studying line arrangements with remarkable properties (such as reflexion line arrangements, simplicial arrangements, or Sylvester-Gallai line arrangements, i.e., line arrangements without double points), and therefore one should start to study these operators for themselves, as well as for finding interesting line arrangements. The present paper is about the operator  $\Lambda_{\{2\},\{3\}}$  and certain arrangements of six lines we define below. The study of the operator  $\Lambda_{\{2\},\{3\}}$  enabled us to find these simple looking arrangements (see Figure 2.1), which (to our knowledge) have not been noticed before.

Recall that for  $k \geq 2$ , we denote by  $\mathcal{P}_{\{k\}}$  the *point operator*, which to a line arrangement  $\mathcal{A}$ , associates the set  $\mathcal{P}_{\{k\}}(\mathcal{A})$  of  $k$ -points of  $\mathcal{A}$  i.e., the points where exactly  $k$  lines of  $\mathcal{A}$  meet. The order of  $\mathcal{P}_{\{k\}}(\mathcal{A})$  is denoted by  $t_k(\mathcal{A}) = |\mathcal{P}_{\{k\}}(\mathcal{A})|$ . For  $k \geq 2$ , we denote by  $\mathcal{L}_{\{k\}}$  (resp.  $\mathcal{L}_k$ ) the *line operator* which to a finite set of points  $P$  associates the set  $\mathcal{L}_{\{k\}}(P)$  (resp.  $\mathcal{L}_k(P)$ ) of lines containing exactly (resp. at least)  $k$  points in  $P$ . The operator  $\Lambda_{\{m\},\{n\}}$  is defined by  $\Lambda_{\{m\},\{n\}} = \mathcal{L}_{\{n\}} \circ \mathcal{P}_{\{m\}}$ , and we also define  $\Psi_{\{m\},\{n\}} = \mathcal{P}_{\{n\}} \circ \mathcal{L}_{\{m\}}$ . We denote by  $\mathcal{D}$  the dual operator, which to a set of points (respectively

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lines) associate the corresponding set of lines (respectively points) in the dual projective plane.

The operator  $\Lambda_{\{2\},\{3\}}$  is therefore the map which to a line arrangement  $\mathcal{A}$  returns the union of all the lines containing exactly three double points of  $\mathcal{A}$ .

A line arrangement  $\mathcal{A}$  of six lines in general position has 15 double points and the line arrangement  $\check{\mathcal{A}} = \mathcal{L}_2(\mathcal{D}(\mathcal{A}))$  (the union of the lines going through at least two points in the dual set  $\mathcal{D}(\mathcal{A})$ ), is the union of 15 lines with 45 double points and six 5-points (the points in  $\mathcal{D}(\mathcal{A})$ ), ie. we have  $t_2(\check{\mathcal{A}}) = 45$ ,  $t_5(\check{\mathcal{A}}) = 6$ . That leads us to the following definition:

**Definition.** We say that an arrangements  $\mathcal{C}_0$  of six lines in the projective plane is *unassuming* if the singularities of  $\mathcal{C}_0$  are 15 double points and the line arrangement  $\check{\mathcal{C}}_0 = \mathcal{L}_2(\mathcal{D}(\mathcal{C}_0))$  has singularities

$$t_2(\check{\mathcal{C}}_0) = 27, t_3(\check{\mathcal{C}}_0) = 6, t_5(\check{\mathcal{C}}_0) = 6,$$

we require moreover that the six 5-points of  $\check{\mathcal{C}}_0$  are not contained on a conic.

We use the word “unassuming” in order to describe something that is deceptively simple (here 15 nodal points) but has hidden qualities or advantages (the dual configuration is not generic). As it will be clear, one needs the value  $t_3 = 6$ , because we are dealing with 6 lines: the dual of these 6 triple points creates six new lines. We work over the complex numbers; we obtain the following result.

**Theorem 1.** *If  $\mathcal{C}_0$  is a generic unassuming arrangement, then  $\Lambda_{\{2\},\{3\}}(\mathcal{C}_0)$  is also an unassuming arrangement. The (closure of the) moduli space of unassuming arrangements has two irreducible components, one is isomorphic to  $\mathbb{P}^1$ , the other one is a point.*

We construct explicitly the unassuming arrangements  $\mathcal{C}_0 = \mathcal{C}_0(z_o)$ , which are parametrized by  $z_o$  in  $\mathbb{P}^1 \setminus S$ , where  $S = \{\infty, -1, 0, 1, \pm 2 \pm \sqrt{5}\}$ . The genericity assumption in Theorem 1 comes from the fact that for finitely many  $z_o \notin S$ , the image  $\Lambda_{\{2\},\{3\}}(\mathcal{C}_0(z_o))$  may correspond to a point in  $S$ .

From the above result, starting with an unassuming arrangement  $\mathcal{C}_0 = \mathcal{C}_0(z_o)$ , one can define by induction a sequence  $(\mathcal{C}_k)_{k \in \mathbb{N}}$  of unassuming arrangements by setting  $\mathcal{C}_{k+1} = \Lambda_{\{2\},\{3\}}(\mathcal{C}_k)$ . It is then natural to study these sequences  $(\mathcal{C}_k)_{k \in \mathbb{N}}$  and to expect interesting configurations from their union, in particular when the sequence is periodic. We obtain the following result.

**Theorem 2.** *For fixed  $z_o$  in the complement of the unit circle and any integers  $k \neq k'$ , the unassuming arrangements  $\mathcal{C}_k, \mathcal{C}_{k'}$  are not projectively equivalent.*

*For  $z_o$  a primitive  $n$ -root of unity with  $n > 1$  odd, the associated sequence  $(\mathcal{C}_k)_{k \in \mathbb{N}}$  with  $\mathcal{C}_0 = \mathcal{C}_0(z_o)$  is periodic. For  $n \in \{3, 5, \dots, 21, 23\}$ , the union of the periodic line arrangements  $(\mathcal{C}_k)_{k \in \mathbb{N}}$  is a Ceva line arrangement or is contained in a Ceva line arrangement.*

The line operator  $\Lambda_{\{2\},\{3\}}$  acts on the moduli space of unassuming arrangement (isomorphic to  $\mathbb{P}^1(z)$ ) through the second Tchebychev polynomial map  $z \rightarrow 2z^2 - 1$ .

We recall that the Ceva( $n$ ) line arrangement is (up to projective equivalence) the union of the following  $3n$  lines:

$$(x^n - y^n)(x^n - z^n)(y^n - z^n) = 0$$

in the projective plane; this is a rather prominent line arrangement e.g., for the construction of ball quotient surfaces as cover of the plane, or for studying Terao's freeness conjecture. In characteristic 0, this is the unique known infinite family of Sylvester-Gallai line arrangements. We do not know if an  $n$ -periodic unassuming line arrangement is always contained a Ceva line arrangement, (see Section 3.3 where other cases  $n$  are discussed).

By duality, one can rephrase the results in Theorem 1 on the action of  $\Lambda_{\{2\},\{3\}}$  on unassuming arrangements as follows:

**Theorem 3.** *For a set  $P_6 = \{p_1, \dots, p_6\}$  of six points in the plane, consider the following property:*

- (P) *The union of the lines containing two points in  $P_6$  possesses exactly six triple points  $p'_1, \dots, p'_6$ . The points of  $P_6$  are not inscribed in a conic.*

*Suppose that  $P_6$  is generic to satisfy satisfies (P). Then the set of triple points  $P'_6 = \{p'_1, \dots, p'_6\}$  also satisfies (P), moreover if the points in  $P_6$  are real, there exists a unique set of six real points  $P_6^-$  satisfying (P) and such that  $(P_6^-)' = P_6$ .*

See Figure 1.1 for an example of such set of points  $P_6$  (in black) and its associated set  $P'_6 = \Psi_{\{2\},\{3\}}(P_6)$  (in red); joining the red points, one can find the set  $P''_6$  (however, it seems more difficult to find  $P_6^-$  using only geometry). Depending on the situation, for example if the points in  $P_6$  have real coordinates, the sequence  $P_6, P'_6, P''_6$  etc. has only distinct elements. In contrast, for any integer  $n > 1$ , there exist sequences  $P_6, P'_6, P''_6 \dots$  which are periodic with period at least  $n$ .

Our proof of Theorem 3 is analytic. For a discussion about Theorem 3 and classical geometry, such as Pascal's hexagon Theorem, see [12].

In [14], Schwartz defines the pentagram map: it is the map which to a polygon of  $m \geq 5$  lines i.e., a  $m$ -tuple of lines  $(\ell_1, \dots, \ell_m)$ , associate the polygon  $(\ell'_1, \dots, \ell'_m)$ , where  $\ell'_k$  is the line containing the points  $p_k, p_{k+2}$ , for  $p_k = \ell_k \cap \ell_{k+1}$  (the indices being taken mod  $m$ ). That rational self-map on the space of  $m$ -gons is of interest in classical projective geometry, algebraic combinatorics, moduli spaces, cluster algebras, and it is an example of a Liouville-Arnold discrete integrable system [4], [11]. Our paper is an example of another type of operators acting on line arrangements.

Let us now review the structure of the paper, and some of the content not mentioned in the above Theorems.

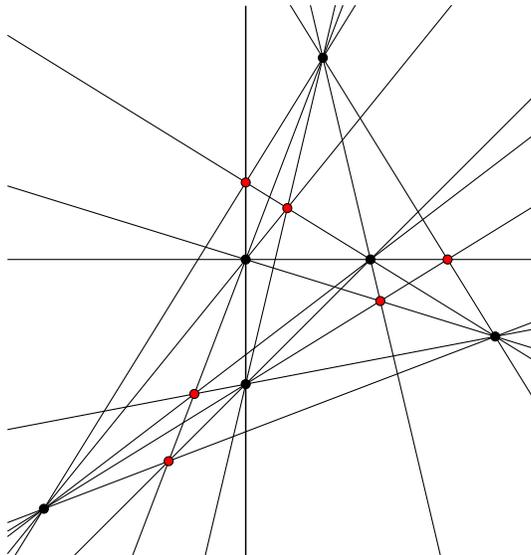


FIGURE 1.1. The points in  $P_6$  and the six associated triple points

In Section 2, we construct the moduli space of labelled (or marked) unassuming arrangements. We obtain that the image by the operator  $\Lambda_{\{2\},\{3\}}$  of an unassuming arrangement is again an unassuming arrangement, and we compute the action of the operator  $\Lambda_{\{2\},\{3\}}$  on the moduli space without marking. Since we are interested in the lines arrangements rather than their class modulo projective equivalence, we compute the formulas for the equations of the lines of the successive images of the operator  $\Lambda_{\{2\},\{3\}}$ . If one starts from the configuration  $\mathcal{C}_0 = \mathcal{C}_0(z_o)$  (for some fixed  $z_o \in \mathbb{P}^1$ ) and inductively defines the sequence  $(\mathcal{C}_k)_{k \in \mathbb{N}}$  by  $\mathcal{C}_{k+1} = \Lambda_{\{2\},\{3\}}(\mathcal{C}_k)$ , the coordinates of the line arrangement  $\mathcal{C}_k$  are controlled by the powers  $z_o^{\nu_k}$ , where  $(\nu_k)_{k \geq 0}$  is the sequence beginning by 0, 1, 1, 3, 5, 11... and defined by

$$\nu_0 = 0, \quad \nu_{2k+1} = 2\nu_{2k} + 1, \quad \nu_{2k} = 2\nu_{2k-1} - 1.$$

By contrast, the action of  $\Lambda_{\{2\},\{3\}}$  on the moduli space of unmarked line arrangements up to projectivity is through iterate of the Tchebychev map  $F_\Lambda : z_o \rightarrow 2z_o^2 - 1$ . We discuss also in this section if the action of  $\Lambda_{\{2\},\{3\}}$  on unassuming line arrangement can be obtained by a Cremona transformation.

In Section 3, after recalling some definitions on dynamical system, we study the periodic configurations. The sequence  $(\mathcal{C}_k)_{k \in \mathbb{N}}$  starting with  $\mathcal{C}_0 = \mathcal{C}_0(z_o)$  is periodic with period dividing  $n > 2$  if  $z_o^{\nu_n} = 1$ . We give examples of periodic line arrangements, and show that their union is strongly related to the Ceva arrangements. We also discuss the Julia and Fatou domains of  $F_\Lambda$  and their meaning for  $\Lambda_{\{2\},\{3\}}$ . We then study the isolated 0 dimensional moduli space of unassuming arrangements. We obtain that the corresponding line arrangement is related to the famous Hesse arrangement of 12 lines.

The six points in  $P_6$  which are dual to an unassuming arrangement are not contained in a conic, thus by blowing up these points, we can associate to  $P_6$  a smooth cubic surface  $X$ . Then the triple points of  $\mathcal{L}_2(P_6)$  have a geometric meaning for  $X$ : these points are Eckardt points, i.e., they are isolated fixed points of involutions acting on  $X$ . Using the classification of the automorphisms group of cubic surfaces, we then show that we obtained all unassuming arrangements.

Finally, let us mention that the K3 surfaces obtained as double cover of the plane ramified over an unassuming arrangement have Picard number  $\geq 19$ ; these surfaces and how they are related through the operator  $\Lambda_{\{2\},\{3\}}$  are the object of a paper in preparation [13].

We used the computer algebra system Magma [5]; the Magma code for dealing with matroids, the unassuming line arrangements and the various operators can be found on arXiv, in the ancillary file of this paper.

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## 2. UNASSUMING ARRANGEMENTS AND THE $\Lambda_{\{2\},\{3\}}$ -OPERATOR

**2.1. Matroids and moduli space of labelled unassuming arrangements.** In the present section, we construct three irreducible components of the moduli space of unassuming arrangements with a labeling, by which we mean that the elements of the moduli are 6-tuples

$$\mathcal{A} = (\ell_1, \dots, \ell_6)$$

of lines. One irreducible component, which we denote by  $\mathfrak{B}_6$ , is one dimensional and for any arrangement  $\mathcal{C}_0$  in  $\mathfrak{B}_6$ , and up to automorphism of  $\mathbb{P}^2$ , there exists  $z_o \in \mathbb{P}^1(\mathbb{C})$  with  $z_o \notin S = \{\infty, -1, 0, 1, \pm 2 \pm \sqrt{5}\}$  such that the normals of the six lines  $\ell_1, \dots, \ell_6$  of  $\mathcal{C}_0 = \mathcal{C}_0(z_o)$  are given by the six columns of the following matrix

$$(2.1) \quad M_{z_o} = \begin{pmatrix} 1 & 0 & 0 & 1 & z_o + 1 & -z_o + 1 \\ 0 & 1 & 0 & 1 & -z_o + 1 & z_o + 1 \\ 0 & 0 & 1 & 1 & 2 & 2 \end{pmatrix}.$$

Case  $z_o \in \{\infty, 0\}$  is degenerate: the arrangement associated to  $M_{z_o}$  has 5 lines with one triple point. Case  $z_o = \pm 1$  gives the complete quadrilateral: the six lines going through 4 points in general position. The line arrangement with  $z_o \in \{\pm 2 \pm \sqrt{5}\}$  given by matrix  $M_{z_o}$  has six lines with 15 nodes but is not an unassuming arrangement; we discuss that case in the last Section.

As we show below, the two other irreducible components of the labeled moduli space are points which are conjugated under the Galois action, and the two associated line arrangements are projectively equivalent when one forgets the labelings.

In order to construct the moduli space  $\mathfrak{B}_6$ , we will use some vocabulary and elementary technics of matroid theory. A matroid is a pair  $(E, \mathcal{B})$ , where  $E$  is a finite set and the elements of  $\mathcal{B}$  are subsets of  $E$ , called *basis*, subject to the following properties:

- $\mathcal{B}$  is non-empty
- if  $A$  and  $B$  are distinct members of  $\mathcal{B}$  and  $a \in A \setminus B$ , then there exists  $b \in B \setminus A$  such that  $(A \setminus \{a\}) \cup \{b\} \in \mathcal{B}$ .

The bases have the same order  $n$ , called the rank of  $(E, \mathcal{B})$ . Order  $n$  subsets of  $E$  that are not bases are called *non-bases*. We identify  $E$  with  $\{1, \dots, m\}$ . A realization (over some field) of the matroid  $(E, \mathcal{B})$  is the data of a size  $n \times m$  matrix  $M$ , with columns  $C_1, \dots, C_m$ , such that any order  $n$  subset  $\{i_1, \dots, i_n\}$  of  $E$  is a non-base if and only if the size  $n$  minor  $|C_{i_1}, \dots, C_{i_n}|$  is zero.

For our problem, we identify the lines of the searched arrangement to  $E = \{1, \dots, m\}$ . Since we search for line arrangements in  $\mathbb{P}^2$ , one has  $n = 3$ : the columns of the matrix  $M$  are the normals of the lines. Three columns  $(i, j, k)$  are non-base if and only if the lines  $i, j, k$  meet at a point, indeed if  $(x_{1i}, x_{2i}, x_{3i}), \dots, (x_{1k}, x_{2k}, x_{3k})$  are the normals of lines  $i, j, k$ , the lines  $(i, j, k)$  meet at point  $(v_1 : v_2 : v_3)$  implies that the matrix  $(x_{s,t})_{1 \leq s \leq 3, t \in \{i, j, k\}}$  has a non-trivial kernel, and conversely.

We are looking for line arrangements of 15 lines with singularities  $t_2 = 27$ ,  $t_3 = 6$ ,  $t_5 = 6$ . More precisely, we impose that the following elements are non-bases

$$(2.2) \quad \text{NB}_1 : (1, 5, 10), (2, 4, 13), (3, 9, 13), (5, 7, 8), (6, 11, 12), (12, 14, 15)$$

(these are the triple of lines meeting at the same points), moreover any triple of lines among the following quintuples

$$(2.3) \quad \begin{aligned} &(1, 2, 8, 9, 12), (1, 6, 7, 13, 15), (2, 3, 5, 6, 14), \\ &(3, 4, 7, 10, 12), (4, 5, 9, 11, 15), (8, 10, 11, 13, 14), \end{aligned}$$

must be non-basis (these are the six quintuples of lines meeting at the 5-points). In order to obtain the moduli space of such line arrangement, one takes a  $3 \times 15$  matrix  $T$ , with entries as unknowns. One can reduce the number of variables, by taking the four first columns to be the canonical basis of  $\mathbb{P}^2$ , ie  $C_1 = (1, 0, 0)$ ,  $C_2 = (0, 1, 0)$ ,  $C_3 = (0, 0, 1)$ ,  $C_4 = (1, 1, 1)$  (taking care that the four minors of  $(1, 2, 3, 4)$  are bases of the matroid). Since the columns are the normals of the lines, one can suppose that in the remaining columns there is always a 1 as follows: If the set  $(2, 3, i)$  is a basis, then the first entry in row  $i$  must be non-zero, so we assume it's 1. If not, the first-row entry in column  $i$  must be zero and we fix the second-row entry to be 1 (it cannot be 0 since we would then have a repetition of the normal  $(0, 0, 1)$ ). The ideal generated by the 66 minors corresponding to the non-bases, defines a variety in  $\mathbb{A}^{19}$ , which is our moduli space of 15 lines arrangements with singularities  $t_2 = 27$ ,  $t_3 = 6$ ,  $t_5 = 6$ . One then takes the dual of the six 5-points and obtain (using the computer algebra

system Magma) the moduli space  $\mathfrak{B}_6$  (isomorphic to  $\mathbb{P}^1$ ), with the normals in (2.1), which is what we call the (labelled) moduli space of unassuming arrangements. The entries of the matrix  $T$  are rational functions in the variable  $z_o$ . One can compute the determinants of the  $3 \times 3$  minors of it, their numerators (resp. denominators) are powers and products of elements in  $\{z_o, z_o \pm 1, z_o^2 \pm 4z_o - 1\}$  (resp.  $\{z_o, z_o \pm 1\}$ ). The vanishing of such additional minors create other singularities, or gives a degeneration of the arrangement i.e., an arrangement with repetition of some lines, thus the condition  $z_o \notin \{\infty, -1, 0, 1, \pm 2 \pm \sqrt{5}\}$ .

A difficulty (which does not appear in this paper) in computing the moduli space of line arrangements was to find the appropriate parametrization of it  $\gamma : \mathbb{P}^1 \xrightarrow{\cong} \mathfrak{B}_6$ , so that one can understand clearly the action of the line operator  $\Lambda_{\{2\},\{3\}}$  on  $\mathfrak{B}_6$  and on the plane, and so that one can understand the associated dynamical systems. For example, the well-known Techbychev map  $F_\Lambda : z \rightarrow 2z^2 - 1$  will appear, but our first parametrization gave instead the map  $\tilde{F} : z \rightarrow \frac{z^2 - 16}{4z^2}$ . Using the theory of multipliers of rational functions (see e.g., [15]), we recognized the dynamic of  $F_\Lambda$  and found another parametrization  $\phi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ , so that  $\phi^{-1} \circ \tilde{F} \circ \phi$  equals to  $F_\Lambda$ . The moduli space  $\mathfrak{B}_6$  is defined over  $\mathbb{Q}$ .

There are two other moduli spaces of labeled unassuming arrangements, namely, one can take for non-bases the following triple (which correspond to the six triple points):

$$(2.4) \quad \text{NB}_2 \quad (1, 3, 11), (1, 4, 14), (3, 8, 15), (4, 6, 8), (6, 11, 12), (12, 14, 15)$$

and the same 60 triple coming from equation (2.3). One obtains that the associated moduli space, the realization space of that matroid, is the union of two points. We will discuss that case in Section 3.5 and in Section 3.6.

We will prove in Section 3.6 that there are no other irreducible components of the moduli space of unassuming arrangements, or in other words that we can always label an unassuming arrangement in a way that leads to Equations (2.3) and (2.2) or (2.4). We used the computer algebra system Magma [5] for the computations of these moduli spaces and double-checked using the Homalg package [10] developed in Julia/Oscar.

We explain in Section 3.7.2 how we found these two matroids.

## 2.2. Image of an unassuming arrangement by the $\Lambda_{\{2\},\{3\}}$ -operator.

Let us fix the parameter  $z_o$  and apply the  $\Lambda_{\{2\},\{3\}}$ -operator on the line arrangement  $\mathcal{C}_0(z_o) \in \mathfrak{B}_6$ . The 15 nodal points of  $\mathcal{C}_0$  are given by the columns of the following matrix:

$$\begin{pmatrix} 1 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & -2 & 2 & z_o - 1 & 0 & z_o + 1 & z_o - 1 & z_o + 1 \\ 0 & 1 & 0 & -1 & 0 & 1 & -1 & -2 & 0 & 0 & z_o + 1 & 2 & z_o - 1 & z_o + 1 & z_o - 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & z_o + 1 & z_o + 1 & z_o - 1 & 0 & z_o - 1 & -2z_o & -2z_o & 0 \end{pmatrix},$$

these columns are also the normals of the 15 lines in the dual configuration. The image of  $\mathcal{C}_0$  by the point operator  $\mathcal{P}_{\{2\}}$  is the union of the above 15 nodal

points of  $\mathcal{C}_0$ . By construction of  $\mathcal{C}_0$ , the line arrangement  $\check{\mathcal{C}}_0 = \mathcal{L}_2\mathcal{D}(\mathcal{C}_0)$  has six triple points, dually that means that there are exactly six lines in  $\mathbb{P}^2$  such that each line contain exactly three points in  $\mathcal{P}_{\{2\}}(\mathcal{C}_0)$ , therefore

$$\mathcal{C}_1 = \mathcal{L}_{\{3\}}\mathcal{P}_{\{2\}}(\mathcal{C}_0) = \Lambda_{\{2\},\{3\}}(\mathcal{C}_0)$$

is a line arrangement of six lines. The interesting fact is that the situation is self-similar:  $\mathcal{C}_1$  is also an unassuming arrangement. Indeed, a direct computation shows that the normals of the line arrangement  $\mathcal{C}_1 = \Lambda_{\{2\},\{3\}}(\mathcal{C}_0)$  are

$$\begin{pmatrix} 1+z_o & 1-z_o & -z_o+1 & z_o+1 & 1 & 0 \\ 1-z_o & 1+z_o & -z_o+1 & z_o+1 & 0 & 1 \\ 0 & 0 & 2 & 2 & 1 & 1 \end{pmatrix},$$

moreover there exists a unique projective transformation such that the six lines of  $\mathcal{C}_1$  are sent respectively to the lines with normals

$$(2.5) \quad \begin{pmatrix} 1 & 0 & 0 & 1 & z_o^2+1 & z_o^2-1 \\ 0 & 1 & 0 & 1 & z_o^2-1 & z_o^2+1 \\ 0 & 0 & 1 & 1 & 2z_o^2 & 2z_o^2 \end{pmatrix},$$

which is the line arrangement  $\mathcal{C}_0(\frac{-1}{z_o})$  in  $\mathfrak{B}_6$  (compare with matrix  $M_{z_o}$  in (2.1)). From that description, we see that.

**Corollary 4.** *For  $z_o \notin S = \{\infty, -1, 0, 1, \pm 2 \pm \sqrt{5}\}$  such that  $-\frac{1}{z_o} \notin S$ , the image of the unassuming arrangement  $\mathcal{C}_0(z_o)$  by  $\Lambda_{\{2\},\{3\}}$  is an unassuming arrangement. The operator  $\Lambda_{\{2\},\{3\}}$  acts on the moduli of labelled line arrangements  $\mathfrak{B}_6$  by the map  $\mathcal{C}_0(z_o) \rightarrow \mathcal{C}_0(-\frac{1}{z_o})$ .*

One may generalize the first part of Corollary 4, since any unassuming line arrangement  $\mathcal{C}$  is always projectively equivalent to a line arrangement  $\mathcal{C}_0(z_o)$ ,  $z_o \notin S$ . Also, as we will see in Section 2, the image of an unassuming arrangement in the moduli with two points is also an unassuming arrangement, and in the same family.

See Figure 2.1 for a picture of an unassuming arrangement  $\mathcal{C}_0$  (the lines in black) and its image  $\mathcal{C}_1$  (the blue lines) by  $\Lambda_{\{2\},\{3\}}$ . It is the image of an unassuming arrangement  $\mathcal{C}_0(z_o)$  by the projective transformation sending two of the three base points (see Definition 10) to infinity.

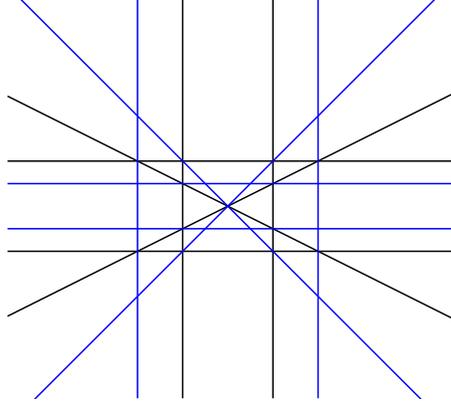
It is also interesting to consider the dual configuration  $\mathcal{D}(\mathcal{C}_0)$ . The arrangement  $\check{\mathcal{C}}_0 = \mathcal{L}_2(\mathcal{D}(\mathcal{C}_0))$ , obtained by taking the union of all lines through the six points of  $\mathcal{D}(\mathcal{C}_0)$  contains 15 lines, and has singularities

$$t_2 = 27, t_3 = 6, t_5 = 6.$$

The arrangement  $\check{\mathcal{C}}_1 = \mathcal{L}_2(\mathcal{D}(\mathcal{C}_1))$  has the same properties, moreover the following relation holds

$$\check{\mathcal{C}}_1 = \Lambda_{\{3\},\{2\}}(\check{\mathcal{C}}_0).$$

For  $k \in \{0, 1\}$ , the six 5-points of  $\check{\mathcal{C}}_k$  correspond by duality to the six lines of  $\mathcal{C}_k$ , and the 5-points of  $\check{\mathcal{C}}_1$  are the 3-points of the arrangement  $\check{\mathcal{C}}_0$ . In Figure 1.1, the points  $p_1, \dots, p_6$  of  $\mathcal{D}(\mathcal{C}_0)$  are in black (these are the 5-points of  $\check{\mathcal{C}}_0$ ),

FIGURE 2.1. An unassuming arrangement and its image by  $\Lambda_{\{2\},\{3\}}$ 

the triple points of  $\check{\mathcal{C}}_0$  are in red (these are the 5-points of  $\check{\mathcal{C}}_1$ ), the 15 black lines are the lines of  $\check{\mathcal{C}}_0$  (here for some fixed value  $z_o$ ).

The matrix  $M_{z_o}$  in (2.1) giving the normals of the six lines attributes implicitly a marking, i.e., an order to the six lines of the arrangement  $\mathcal{C}_0 = \mathcal{C}_0(z_o)$  and the parameter space  $\mathfrak{B}_6 \simeq \mathbb{P}^1$  is a moduli space for marked line arrangements  $(\mathcal{C}_0(z_o), \ell_1, \dots, \ell_6)$ . In the next section we describe the forgetful map  $(\mathcal{C}_0(z_o), \ell_1, \dots, \ell_6) \rightarrow \mathcal{C}_0(z_o)$ , which takes the quotient of  $\mathfrak{B}_6$  by the permutation group  $S_6$  of six elements.

**2.3. The moduli space of unassuming arrangements without marking.** Let  $(\mathcal{C}_0(z_o), \ell_1, \dots, \ell_6) \in \mathfrak{B}_6$ . Consider another order  $\ell_{\sigma(1)}, \dots, \ell_{\sigma(6)}$ , for  $\sigma \in S_6$  a permutation. Since the lines  $\ell_i$  are in general position, the lines  $\ell_{\sigma(1)}, \dots, \ell_{\sigma(4)}$  form a basis in the projective space, and by using an appropriate base-change, the normal of the lines  $\ell_{\sigma(1)}, \dots, \ell_{\sigma(6)}$  are the six columns of the matrix

$$M_\sigma(z_o) = \begin{pmatrix} 1 & 0 & 0 & 1 & \alpha_{1,5} & \alpha_{1,6} \\ 0 & 1 & 0 & 1 & \alpha_{2,5} & \alpha_{2,6} \\ 0 & 0 & 1 & 1 & \alpha_{3,5} & \alpha_{3,6} \end{pmatrix}$$

where the  $\alpha_{i,j}$  are fractions in  $z_o$ . One can then define a morphism

$$\psi_\sigma : z_o \in \mathbb{P}^1 \rightarrow ((\alpha_{1,5} : \alpha_{2,5} : \alpha_{3,5}), (\alpha_{1,6} : \alpha_{2,6} : \alpha_{3,6})) \in \mathbb{P}^2 \times \mathbb{P}^2.$$

One computes that the images of  $\mathbb{P}^1 = \mathbb{P}^1(z_o)$  by these maps are 15 curves in  $\mathbb{P}^2 \times \mathbb{P}^2$ , among which there is a unique curve with bi-degree  $(1, 1)$ , which we denote by  $L_0$ . There is a sub-group of 48 permutations  $\sigma \in S_6$  such that the image of  $\psi_\sigma$  is the line  $L_0$ . For these 48 permutations, the matrix  $M_\sigma(z_o)$  is equal to either

$$M(z_o), M(-z_o), M\left(\frac{1}{z_o}\right) \text{ or } M\left(-\frac{1}{z_o}\right).$$

TABLE 1.

Let  $G_4$  be the order 4 automorphism group of  $\mathbb{P}^1$  generated by the involutions  $\sigma_1 : z_o \rightarrow -z_o$  and  $\sigma_2 : z_o \rightarrow 1/z_o$ . The degree 4 map

$$(2.6) \quad \mathcal{Y} : \mathbb{P}^1 \rightarrow \mathbb{P}^1, \quad z_o \rightarrow \frac{1}{2}(z_o^2 + z_o^{-2})$$

is invariant by the group  $G_4$  and therefore realizes the quotient of  $\mathbb{P}^1$  by  $G_4$ . The quotient of  $S_6$  by the order 48 group has 15 classes, which classes correspond to the 15 curves contained in  $\mathbb{P}^2 \times \mathbb{P}^2$ . Thus the map  $\mathcal{Y} : \mathbb{P}^1 \rightarrow \mathbb{P}^1$  is also the quotient map  $\phi : \mathfrak{B}_6 \rightarrow \mathfrak{B}_6/S_6$ ; we call it the period map. The images by  $\mathcal{Y}$  of  $\infty, -1, 0, 1$  are respectively  $\infty, 1, \infty, 1$ .

**Definition 5.** The operator  $\Lambda_{\{2\},\{3\}}$  has a natural action on the moduli space  $\mathfrak{B}_6/S_6 = \mathbb{P}^1(z)$ : we denote by  $F_\Lambda : \mathbb{P}^1 \rightarrow \mathbb{P}^1$  the map obtained by sending the point  $z \in \mathfrak{B}_6/S_6$  representing the unassuming arrangement  $\mathcal{C}_0(z)$ , to the point  $F_\Lambda(z) \in \mathfrak{B}_6/S_6$  representing the line arrangement  $\mathcal{C}_1 = \Lambda_{\{2\},\{3\}}(\mathcal{C}_0)$ .

**Theorem 6.** *The action of the operator  $\Lambda_{\{2\},\{3\}}$  on the moduli space  $\mathfrak{B}_6/S_6 = \mathbb{P}^1(z)$  is given by the polynomial map*

$$F_\Lambda : z \rightarrow 2z^2 - 1.$$

*Proof.* From Equation (2.5), there exists a projective transformation  $P$  such that  $P(\mathcal{C}_1(z_o))$  is the labelled arrangement  $\mathcal{C}_0(\frac{-1}{z_o})$ . The image by the quotient map  $\phi$  of the labelled arrangement  $P(\mathcal{C}_1(z_o))$  is the point

$$\phi(\mathcal{C}_0(\frac{-1}{z_o})) = \mathcal{Y}(\frac{-1}{z_o}) = \frac{1}{2}(z_o^4 + z_o^{-4}) = 2\left(\frac{1}{2}(z_o^2 + z_o^{-2})\right)^2 - 1 = F_\Lambda\left(\frac{1}{2}(z_o^2 + z_o^{-2})\right),$$

thus the result.  $\square$

The moduli space  $\mathfrak{B}_6/S_6$  is a moduli space of line arrangement up to projective transformation: there is a projective transformations sending an unassuming arrangement  $\mathcal{C}$  to  $\mathcal{C}'$  if and only if the images of  $\mathcal{C}$  and  $\mathcal{C}'$  in  $\mathfrak{B}_6/S_6$  are equal. That has the consequence:

**Corollary 7.** *There exists unassuming arrangements  $\mathcal{C}_0$  such that the orbit  $(\mathcal{C}_k)_{k \in \mathbb{N}}$  of  $\mathcal{C}_0$  by  $\Lambda_{\{2\},\{3\}}$  is infinite, and there are no projective transformations between  $\mathcal{C}_k, \mathcal{C}_{k'}$  for any  $k \neq k'$ .*

Indeed, for  $\mathcal{C}_0 = \mathcal{C}_0(z_o)$ , one just have to take  $z_o \in \mathbb{P}^1$  such that the sequence  $F_\Lambda^{on}(z_o)$  tends to  $\infty$ .

#### 2.4. The coordinates of the iterations of unassuming arrangements.

In this section, in order to understand the dynamics of the operator  $\Lambda_{\{2\},\{3\}}$  on unassuming arrangements, not up to projective automorphism of  $\mathbb{P}^2$  but in the plane, we would like to know what are the equations of the six lines of the successive images  $\mathcal{C}_k, k = 1, 2, \dots$  of an unassuming arrangement  $\mathcal{C}_0$  by  $\Lambda_{\{2\},\{3\}}$ . That will be used in Section 3.3, where we study periodic and pre-periodic unassuming arrangements for the action of  $\Lambda_{\{2\},\{3\}}$ .

Consider the arrangement of lines  $\mathcal{C}(a, b, c)$  whose normals are the columns of the following matrix

$$(2.7) \quad \begin{pmatrix} 0 & 0 & b & -b & c & -c \\ a & -a & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

For  $a, b, c$  generic, it is projectively equivalent to  $\mathcal{C}_0(\frac{ac}{b})$ , and therefore it is an unassuming arrangement. One computes that the image of  $\mathcal{C}(a, b, c)$  by  $\Lambda_{\{2\},\{3\}}$  is the unassuming arrangement  $\mathcal{C}(\frac{b}{c}, ac, \frac{b}{a})$ .

Rather than computing the coordinates of the lines of the sequence associated to  $\mathcal{C}_0$ , let us compute those associated to the arrangement  $\tilde{\mathcal{C}}_0$ , where

$$(2.8) \quad \tilde{\mathcal{C}}_0 = \mathcal{C}(1, 1, z_o),$$

since that choice greatly simplify the formulas and computations.

Let us define the integer sequence  $(\nu_k)_{k \geq 0}$  by  $\nu_k = \frac{1}{3}(2^k - (-1)^k)$  for  $k \in \mathbb{N}$ . That sequence may also be inductively defined by

$$\nu_0 = 0, \nu_{k+1} = 2\nu_k + (-1)^{k+1}$$

It is the OEIS sequence A001045 and it begins with 0, 1, 1, 3, 5, 11, 21, 43...

Let us also define the sequences  $(a_k)_{k \geq 0}, (b_k)_{k \geq 0}, (c_k)_{k \geq 0}$  by

$$a_k = z_o^{\nu_k(-1)^k}, \quad b_k = z_o^{\nu_k(-1)^{k+1}} = \frac{1}{a_k}, \quad c_k = z_o^{\nu_k(-1)^{k+1}} = z_o a_k.$$

Let  $(\tilde{\mathcal{C}}_k)$  be the sequence defined by  $\tilde{\mathcal{C}}_0 = \mathcal{C}(1, 1, z_o)$  and  $\tilde{\mathcal{C}}_{k+1} = \Lambda_{\{2\},\{3\}}(\tilde{\mathcal{C}}_k)$ .

**Theorem 8.** *We have  $\tilde{\mathcal{C}}_k = \mathcal{C}(a_k, b_k, c_k)$ .*

*Proof.* This is true when  $k = 0$ . Let  $k \in \mathbb{N}$  be an integer, and suppose that this is true for  $k$ . One has

$$\begin{aligned} \frac{b_k}{c_k} &= \frac{z_o^{\nu_k(-1)^{k+1}}}{z_o^{\nu_k(-1)^k}} = z_o^{(-1)^{k+1}(2\nu_k + (-1)^k)} = a_{k+1}, \\ a_k c_k &= z_o^{2\nu_k(-1)^{k+1}} = z_o^{(-1)^k \nu_{k+1}} = b_{k+1}, \\ \frac{b_k}{a_k} &= \frac{z_o^{\nu_k(-1)^{k+1}}}{z_o^{\nu_k(-1)^k}} = z_o^{(-1)^{k+1}(2\nu_k + (-1)^k - (-1)^k)} = z_o^{(-1)^{k+1} \nu_{k+1} + 1} = c_{k+1}, \end{aligned}$$

thus the result.  $\square$

*Remark 9.* When  $|z_o| \neq 1$ , the sequence of unassuming arrangements  $(\tilde{\mathcal{C}}_k)$  tends to  $x^2 y^2 z^2 = 0$ . The sequence of the 15 double points of  $(\tilde{\mathcal{C}}_k)$  tends to the set of three points  $\{(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1)\}$ .

Working modulo projective automorphism, these 15 points tend to a set  $P_9$  of 9 points (given by the columns of the  $3 \times 15$  matrix in Section (2.2), with  $z_o$  replaced by 0). The dual arrangement  $\mathcal{D}(P_9)$  is the simplicial line arrangement  $A(9, 48)$  (see [1] for the notations for simplicial arrangements); it is fixed by  $\Lambda_{\{3\},\{2\}}$ . The line arrangement  $\mathcal{L}_{\{2\}}(P_9)$  is the simplicial arrangement  $A(13, 96)_3$ .

**Definition 10.** Let  $\mathcal{C}$  be an unassuming arrangement. Then the line arrangement  $\mathcal{C} \cup \mathcal{C}'$  (where  $\mathcal{C}' = \Lambda_{\{2\},\{3\}}(\mathcal{C})$ ) has exactly three 4-points  $p_1, p_2, p_3$ . By induction, these points are common to all iterates of  $\mathcal{C}$  under  $\Lambda_{\{2\},\{3\}}$ ; we call these three points the *base points* of  $\mathcal{C}$ , and we call the lines dual to these points *the triangle of  $\mathcal{C}$* . The triangle of  $\mathcal{C}$  is contained in the dual line arrangement  $\tilde{\mathcal{C}} = \mathcal{L}_2(\mathcal{D}(\mathcal{C}))$ .

An unassuming arrangement with base points  $\mathcal{T}_3 = \{(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1)\}$  is projectively equivalent under a projective map  $g$  to some arrangement  $\tilde{\mathcal{C}}_0 = \mathcal{C}(1, 1, t)$ , which has the same base points. The maps  $g$  must preserve the 3 base points and therefore is the product of a diagonal map  $(x : y : z) \rightarrow (ux : vy : wz)$  (for some  $u, v, w$ ) and a permutation of the coordinates (which permutation has no effect on the unassuming arrangement). Thus we obtain that

**Lemma 11.** *The unassuming arrangements  $\mathcal{C}(a, b, c)$  in (2.7) are all unassuming arrangements with base points in  $\mathcal{T}_3$ .*

2.4.1. *Cross ratio.* The triangle (see Definition 10) of the unassuming line arrangements  $\mathcal{C}(1, 1, z_0)$  is the union of the lines  $L_1 : x = 0$ ,  $L_2 : y = 0$ ,  $L_3 : z = 0$ . We define a parametrization of these lines by  $a \rightarrow (0 : a : 1)$ ,  $b \rightarrow (b : 0 : 1)$ ,  $c \rightarrow (c : 1 : 0)$ . We remark that if one defines sequences  $A_k, B_k, C_k$  of pairs of points in  $L_1, L_2, L_3$  by respectively  $A_k = \{a_k, -a_k\}$ ,  $B_k = \{b_k, -b_k\}$  and  $C_k = \{c_k, -c_k\}$ , where we recall that

$$a_k = t^{\nu_k(-1)^k}, \quad b_k = t^{\nu_k(-1)^{k+1}} = \frac{1}{a_k}, \quad c_k = t^{\nu_k(-1)^{k+1}} = ta_k,$$

we obtain the relations

(2.9)

$$\text{Cross}(A_{k-1}; A_k) = \text{Cross}(B_{k-1}; B_k) = \text{Cross}(C_{k-1}; C_k) = \left\{ \left( \frac{t^{2^k} - 1}{t^{2^k} + 1} \right)^{\pm 2} \right\}.$$

the points in  $A_k, B_k, C_k$  seen as elements in  $L_1, L_2, L_3$  defines the normals of the lines of  $\tilde{\mathcal{C}}_k$ .

### 3. DYNAMICS AND APPLICATIONS

3.1. **On the dynamics of  $\chi(z) = -\frac{1}{z^2}$  and  $F_\Lambda = 2z^2 - 1$ .** For basic notions on dynamic we refer to the books [3] and [15].

3.1.1. *The semi-conjugacy between  $\chi$  and  $F_\Lambda$ .* Let us also recall that the rational function  $F_\Lambda : z \rightarrow 2z^2 - 1$  gives the action of  $\Lambda_{\{2\},\{3\}}$  on the moduli space  $\mathfrak{B}_6/S_6 = \mathbb{P}^1$  of unassuming arrangements. We obtained  $F_\Lambda$  by making the following diagram to commute

$$\begin{array}{ccc} \mathcal{C}_0(t) \in \mathfrak{B}_6 & \rightarrow & P(\mathcal{C}_1(t)) \in \mathfrak{B}_6 \\ \downarrow & & \downarrow \\ \mathfrak{B}_6/S_6 & \xrightarrow{F_\Lambda} & \mathfrak{B}_6/S_6 \end{array}$$

Here  $P(\mathcal{C}_1(z_o))$  the image of the line arrangement  $\mathcal{C}_1(z_o) = \Lambda_{\{2\},\{3\}}(\mathcal{C}_0(z_o))$  by the projective transformation sending the first four lines to the canonical basis, so that it is in  $\mathfrak{B}_6$ . The horizontal map is given in the period map (see (2.6))  $\mathcal{Y} : z_o \rightarrow \frac{1}{2}(z_o^2 + z_o^{-2})$  and the above diagram is

$$\begin{array}{ccc} \mathbb{P}^1 & \xrightarrow{\chi} & \mathbb{P}^1 \\ \mathcal{Y} \downarrow & & \mathcal{Y} \downarrow \\ \mathbb{P}^1 & \xrightarrow{F_\Lambda} & \mathbb{P}^1. \end{array}$$

It is commutative so that one has

$$(3.1) \quad F_\Lambda \circ \mathcal{Y}(z_o) = \mathcal{Y} \circ \chi(z_o),$$

in other words: the map  $\mathcal{Y}$  is a semi-conjugacy between  $\chi$  and  $F_\Lambda$ . A consequence is that if  $z_o$  is a periodic point of  $\chi$ , then it is a periodic point of  $F_\Lambda$  (we give below an example for which the period of  $F_\Lambda(z_o)$  is 2 for  $F_\Lambda$ , but the period of  $z_o$  for  $\chi$  is 4).

3.1.2. *Julia sets of  $\chi : z \rightarrow -\frac{1}{z^2}$  and  $F_\Lambda : z \rightarrow 2z^2 - 1$ .* The orbit of 0 under  $\chi$  is  $\{0, \infty\}$ . Consider  $z = \rho e^{i\theta} \in \mathbb{C}$ ,  $z \neq 0$  with  $\rho \in \mathbb{R}^+$ ,  $\theta \in \mathbb{R}$ . If  $\rho \neq 1$ , the sequence  $\chi^{2n}(z)$  (resp.  $\chi^{2n+1}(z)$ ) converges to 0 (resp.  $\infty$ ) if  $\rho < 1$  and to  $\infty$  (resp. 0) if  $\rho > 1$ . The Julia set of  $\chi$  is the unit circle.

The polynomial  $F_\Lambda$  is the degree two Tchebychev polynomial, one has:

$$F_\Lambda(\cos(\theta)) = \cos(2\theta).$$

The Julia set of  $F_\Lambda$  is the interval  $[-1, 1]$  (see e.g., [3]).

The map  $\chi$  restricted to the unit ball (identified with  $\mathbb{R}/2\pi\mathbb{Z}$ ) is the map  $\theta \rightarrow -2\theta \in \mathbb{R}/2\pi\mathbb{Z}$ . The map  $F_\Lambda$  restricted to  $[-1, 1]$  maps  $\cos(\theta)$  to  $\cos^2(\theta) - 1 = \cos(2\theta)$ . The quotient map  $\mathcal{Y} : z \rightarrow \frac{1}{2}(z^2 + z_o^{-2})$  restricted to the unit ball is the map

$$\theta \in \mathbb{R}/2\pi\mathbb{Z} \rightarrow \cos(2\theta).$$

The image of the Julia set of  $\chi$  by  $\mathcal{Y}$  is the Julia set of  $F_\Lambda$ .

3.1.3. *Periodic and pre-periodic sequences  $(\mathcal{C}_k)_{k \in \mathbb{N}}$ .* Let us study when the sequence  $\tilde{\mathcal{C}}_k$  with first term  $\tilde{\mathcal{C}}_0(z_o) = \mathcal{C}(1, 1, z_o)$  is periodic. We recall that one has  $\tilde{\mathcal{C}}_k = \mathcal{C}(a_k, b_k, c_k)$  where

$$a_k = z_o^{\nu_k(-1)^k}, \quad b_k = z_o^{\nu_k(-1)^{k+1}} = \frac{1}{a_k}, \quad c_k = z_o^{\nu_k(-1)^{k+1}} = z_o a_k,$$

and  $\nu_k = \frac{1}{3}(2^k + (-1)^{k+1})$ . The sequence  $(\tilde{\mathcal{C}}_k)_k$  is therefore periodic if and only if  $(a_k)_k$  is periodic. If the sequence  $(a_k)_k$  is periodic with period  $n$  then

$$z_o^{\nu_{n-1}(-1)^{n-1}} = 1,$$

therefore  $z_o$  is an  $\nu_{n-1}$  root of unity. Conversely, if  $z_o$  is an  $m$ -root of unity, with  $m$  dividing  $\nu_{n-1}$  and not dividing  $\nu_k$  for  $k \leq n-2$ , we obtain an  $n$ -periodic sequence of unassuming arrangements.

3.1.4. *Invertibility of the operator  $\Lambda_{\{2\},\{3\}}$ .* Using the formula in (2.7), one may search for the antecedents of the configuration  $\mathcal{C}(u, v, w)$ .

**Theorem 12.** *The operator  $\Lambda_{\{2\},\{3\}}$  has an inverse on the space of real unassuming arrangements, and has degree two on the space of complex unassuming arrangements.*

*Proof.* An arrangement  $\mathcal{C}_{-1}(a, b, c)$  is such that  $\Lambda_{\{2\},\{3\}}(\mathcal{C}_{-1}) = \mathcal{C}(u, v, w)$  if and only if the parameters  $a, b, c$  are such that

$$\frac{b}{a} = \pm u, \quad ac = \pm v, \quad \frac{b}{c} = \pm w.$$

Over the complex field, the solutions  $(a, b, c)$  are among the triplets

$$(3.2) \quad \left(\pm\rho, \pm w\rho, \pm \frac{w}{u}\rho\right),$$

where  $\rho = \sqrt{\pm \frac{uv}{w}}$ . The choice of the three signs in Equation (3.2) does not matter for the line arrangement  $\mathcal{C}_{-1}$ , since that choice only permutes the marking of the arrangement. Thus we can suppose that  $(a, b, c)$  is equal to  $(\rho, w\rho, \frac{w}{u}\rho)$ , and there are two possibilities for  $\rho$ : either  $\rho_1 = \sqrt{\frac{uv}{w}}$  or  $\rho_2 = \sqrt{-\frac{uv}{w}}$ . Over the complex numbers, one has two antecedents. Suppose that the unassuming arrangement  $\mathcal{C}(u, v, w)$  is real i.e.,  $u, v, w \in \mathbb{R}$ . Then, only one of the two numbers  $\rho_1, \rho_2$  is real, and therefore there is a unique antecedent for a real unassuming arrangement.  $\square$

**3.2. Singularities of the union of the iterates of  $\Lambda_{\{2\},\{3\}}$ .** Let  $(\mathcal{C}_k)_{k \geq 0}$  be the sequence of line arrangements inductively defined by  $\mathcal{C}_0 = \mathcal{C}(1, 1, z_o)$  for  $z_o \in \mathbb{P}^1$  and  $\mathcal{C}_{k+1} = \Lambda_{\{2\},\{3\}}(\mathcal{C}_k)$ .

**Proposition 13.** *For  $z_o$  in a dense open set, the line arrangement  $\mathcal{A}_k = \cup_{0 \leq j \leq k} \mathcal{C}_j$  has singularities*

$$t_2(\mathcal{A}_k) = 12(k^2 - k + 1), \quad t_3(\mathcal{A}_k) = 12k, \quad t_{2k+2}(\mathcal{A}_k) = 3,$$

and  $t_r = 0$  for  $r \notin \{2, 3, 2k + 2\}$ .

*Proof.* For  $z_o$  in a dense open set of  $\mathbb{P}^1$ , the sequence  $(z_o^{\nu_k})_{k \geq 2}$  has only distinct elements, and the line arrangement  $\mathcal{A}_k$  is the union of  $6(k + 1)$  distinct lines (see Theorem 8). We proceed by induction. All the lines in  $\mathcal{A}_k$  pass through one of the three base points  $p, q, r$  (see Definition 10). That implies that apart in these three points, the singularities of  $\mathcal{A}_k$  can only be double or triple points. The line arrangement  $\mathcal{C}_k \cup \mathcal{C}_{k+1}$  has singularities

$$t_2 = t_3 = 12, \quad t_4 = 3.$$

The 4-points of  $\mathcal{C}_k \cup \mathcal{C}_{k+1}$  are at  $p, q, r$ , these points are the three  $2k + 2$ -points of  $\mathcal{A}_k$ , thus  $\mathcal{A}_{k+1}$  has three  $2k + 4$ -points. Any line of  $\mathcal{C}_{k+1}$  passes through three of the 15 double points of  $\mathcal{C}_k$ . The triple points of  $\mathcal{C}_k \cup \mathcal{C}_{k+1}$  are obtained as intersection points of two lines of  $\mathcal{C}_k$  and one line in  $\mathcal{C}_{k+1}$ . The 12 nodes of  $\mathcal{C}_k \cup \mathcal{C}_{k+1}$  are the 12 nodes of  $\mathcal{C}_{k+1}$  distinct from  $p, q, r$ . The

Point $z_o$	$\mathcal{C}_0$ fixed by	$z_o$ fixed by	$\Upsilon z_o$ fixed by	Configuration
$\zeta_3$	$\Lambda_{\{2\},\{3\}}^3$	$\chi$	$F_\Lambda$	Ceva(6)
$\zeta_5$	$\Lambda_{\{2\},\{3\}}^4$	$\chi^4$	$F_\Lambda^2$	$t_2 = t_3 = 48, t_8 = 3$ $\bar{\mathcal{C}} = \text{Ceva}(10)$
$\zeta_{11}$	$\Lambda_{\{2\},\{3\}}^5$	$\chi^5$	$F_\Lambda^5$	$t_2 = 2t_3 = 120, t_{10} = 3$ $\bar{\mathcal{C}} = \text{Ceva}(22)$
$\zeta_7$	$\Lambda_{\{2\},\{3\}}^6$	$\chi^6$	$F_\Lambda^3$	$t_2 = 72, t_3 = 120, t_{12} = 3$ $\bar{\mathcal{C}} = \text{Ceva}(14)$
$\zeta_{21}$	$\Lambda_{\{2\},\{3\}}^6$	$\chi^6$	$F_\Lambda^6$	$t_2 = 216, t_3 = 72, t_{12} = 3$ $\bar{\mathcal{C}} = \text{Ceva}(42)$
$\zeta_{43}$	$\Lambda_{\{2\},\{3\}}^7$	$\chi^7$	$F_\Lambda^7$	$t_2 = 336, t_3 = 84, t_{14} = 3$ $\bar{\mathcal{C}} = \text{Ceva}(86)$
$\zeta_{17}$	$\Lambda_{\{2\},\{3\}}^8$	$\chi^8$	$F_\Lambda^4$	$t_2 = 480, t_3 = 96, t_{16} = 3$ $\bar{\mathcal{C}} = \text{Ceva}(34)$
$\zeta_9$	$\Lambda_{\{2\},\{3\}}^9$	$\chi^3$	$F_\Lambda^3$	Ceva(18)
$\zeta_{19}$	$\Lambda_{\{2\},\{3\}}^9$	$\chi^9$	$F_\Lambda^9$	$t_2 = 432, t_3 = 180, t_{18} = 3$ $\bar{\mathcal{C}} = \text{Ceva}(38)$
$\zeta_{31}$	$\Lambda_{\{2\},\{3\}}^{10}$	$\chi^{10}$	$F_\Lambda^5$	$t_2 = 840, t_3 = 120, t_{20} = 3$ $\bar{\mathcal{C}} = \text{Ceva}(62)$
$\zeta_{15}$	$\Lambda_{\{2\},\{3\}}^{12}$	$\chi^4$	$F_\Lambda^4$	$t_2 = t_3 = 432, t_{24} = 3$ $\bar{\mathcal{C}} = \text{Ceva}(30)$
$\zeta_{13}$	$\Lambda_{\{2\},\{3\}}^{12}$	$\chi^{12}$	$F_\Lambda^6$	$t_2 = 144, t_3 = 528, t_{24} = 3$ $\bar{\mathcal{C}} = \text{Ceva}(26)$
$\zeta_{33}$	$\Lambda_{\{2\},\{3\}}^{15}$	$\chi^5$	$F_\Lambda^5$	$t_2 = 2t_3 = 1080, t_{30} = 3$ $\bar{\mathcal{C}} = \text{Ceva}(66)$
$\zeta_{23}$	$\Lambda_{\{2\},\{3\}}^{22}$	$\chi^{22}$	$F_\Lambda^{11}$	$t_2 = 264, t_3 = 1848, t_{44} = 3$ $\bar{\mathcal{C}} = \text{Ceva}(46)$
$\zeta_{29}$	$\Lambda_{\{2\},\{3\}}^{28}$	$\chi^{28}$	$F_\Lambda^{14}$	$t_2 = 336, t_3 = 3024, t_{56} = 3$ $\bar{\mathcal{C}} = \text{Ceva}(58)$

TABLE 2. First periodic configurations

configuration  $\mathcal{C}_{k+1}$  meets the  $6k$  lines of  $\mathcal{A}_{k-1}$  in  $6 \cdot 6k - 6 \cdot 2k = 24k$  double points. Thus the number of double points of  $\mathcal{A}_{k+1}$  is

$$12(k^2 - k + 1) + 24k = 12(k^2 + k + 1) = 12((k + 1)^2 - (k + 1) + 1),$$

and the number of triple points of  $\mathcal{A}_{k+1}$  is  $12(k + 1)$ .  $\square$

**3.3. Periodic points of  $\Lambda_{\{2\},\{3\}}$ ,  $\chi$  and  $F_\Lambda$ .** In Table 2, we study the line arrangement  $\mathcal{C}_0(z_o)$  and its orbit by  $\Lambda_{\{2\},\{3\}}$  for various roots of unity  $z_o$ , so that the operator  $\Lambda_{\{2\},\{3\}}$  is periodic. In the first column is the value of  $z_o$ , the columns 2, 3, 4, are the period under the action of  $\Lambda_{\{2\},\{3\}}$ ,  $\chi$  and  $F_\Lambda$  respectively. The last column gives in the first line the invariants of the arrangement  $\mathcal{C} = \cup_{k \geq 0} \mathcal{C}_k(z_o)$ , and when possible or needed, in the second line it gives invariants or identification of the line arrangement  $\bar{\mathcal{C}}$  which is obtained by taking the union of the orbits  $\mathcal{O}(\mathcal{C}_k(z'_o))$  when  $z'_o$  varies among the conjugates of  $z_o$ . We denote by  $\zeta_k$  a primitive  $k$ -root of unity.

For the values of  $\zeta_k$  in Table 2, the union of the periodic line arrangements  $\mathcal{C}_k$  is contained in some Ceva( $m(k)$ ) line arrangement, for some  $m(k) \in \mathbb{N}$ . It would be interesting to know if this is true for any  $k$ . By the following remarks, necessarily one has  $\forall k, m(k) \notin \{3, 5, 7, 9\}$ :

*Remark 14.* Let be  $m \in \{3, 5, 7, 9\}$ . By doing a systematic search among the sub-arrangements of six lines of Ceva( $m$ ), one obtains that Ceva( $m$ ) does not contain an unassuming arrangement.

Let  $N(k)$  be the order of  $-2$  in the multiplicative group  $(\mathbb{Z}/k\mathbb{Z})^*$ . Since  $\chi$  is the map  $z_o \rightarrow -\frac{1}{z_o^2}$ , a primitive  $k$ -root of unity  $\zeta_k$  with  $k$  odd is periodic of period  $N(k)$  for  $\chi$ .

**3.4. Antecedents of 0 by  $\chi$ .** The configuration  $\mathcal{C}_0 = \mathcal{C}_0(i)$  (for  $i^2 = -1$ ) is such that  $\Upsilon(i) = 1$ . The arrangement  $\mathcal{C}_1 = \Lambda_{\{2\},\{3\}}(\mathcal{C}_0)$  is the complete quadrilateral and  $\mathcal{C}_2 = \Lambda_{\{2\},\{3\}}(\mathcal{C}_1)$  is equal to  $\emptyset$ , moreover

**Proposition 15.** *The arrangement  $\mathcal{A}_{12} = \mathcal{C}_0 \cup \mathcal{C}_1$  has 12 lines with singularities*

$$t_3 = 16, t_4 = 3, t_r = 0 \text{ for } r \neq 3, 4.$$

*The lines and the triple points form a  $(16_3, 12_4)$ -configuration, which is the Ceva(4) arrangement.*

For  $n \leq 8$  and  $\beta \in (\chi^n)^{-1}(0)$ , we computed that the sequence  $(\mathcal{C}_k(\beta))_k$  has length  $n + 1$ :  $\mathcal{C}_0(\beta), \dots, \mathcal{C}_{n-1}(\beta)$  are arrangements of 6 lines in  $\mathfrak{B}_6$ ,  $\mathcal{C}_n$  is the complete quadrilateral, and  $\mathcal{C}_k = \emptyset$  for  $k \geq n$ . By taking the union of arrangements  $\mathcal{C}_k(\beta)$  with  $\beta \in (\chi^n)^{-1}(0)$ , one obtains the configurations Ceva( $2^n$ ).

**3.5. Assembling the Hesse configuration.** Let  $\mathcal{C}_0$  be the configuration of 6 lines such that their normals are the columns of following matrix

$$(3.3) \quad \begin{pmatrix} 1 & 0 & 0 & 1 & -j & \frac{1}{3}(1-j) \\ 0 & 1 & 0 & 1 & j+1 & \frac{1}{3}(j+2) \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix},$$

where  $j^2 + j + 1 = 0$ . This is an unassuming arrangement: the configuration  $\Lambda_2(\mathcal{D}(\mathcal{C}_0))$  has singularities  $t_2 = 27, t_3 = 6, t_5 = 6$  and the 5-points are not contained in a conic. A direct computation shows that:

**Proposition 16.** *The associated  $\Lambda_{\{2\},\{3\}}$ -sequence  $(\mathcal{C}_k)_{k \in \mathbb{N}}$  is periodic with period 2 and  $\mathcal{H} = \mathcal{C}_0 \cup \mathcal{C}_1$  is the Hesse configuration of 12 lines with singularities  $t_2 = 12, t_4 = 9$ .*

*The arrangements  $\mathcal{C}_0$  and  $\mathcal{C}_1$  corresponds to the two points of the labelled moduli space of realization of the second matroid in Section 2.1.*

*Remark 17.* The moduli space in Section 2.1 has two points, but this is the moduli of labelled line arrangements, forgetting the marking gives projectively equivalent arrangements.

One can check moreover that the Hesse configuration contains exactly 6 arrangements  $\mathcal{C}$  of six lines having only nodal singularities. Each of these line arrangement  $\mathcal{C}$  is an unassuming arrangement. These six arrangements are projectively equivalent: they form an orbit of an order 6 projective automorphism of  $\mathbb{P}^2$ .

We remark that the dual Hesse arrangement is also known as the Ceva(3) arrangement.

Finally, consider  $\check{\mathcal{C}}_0 = \mathcal{L}_2(\mathcal{D}(\mathcal{C}_0))$  and  $\check{\mathcal{C}}_1 = \Lambda_{\{3\},\{2\}}(\check{\mathcal{C}}_0)$ . One has

$$\Lambda_{\{3\},\{2\}}(\check{\mathcal{C}}_1) = \check{\mathcal{C}}_0.$$

The union  $\check{\mathcal{C}}_0 \cup \check{\mathcal{C}}_1$  has 21 lines and singularities  $t_2 = 36, t_4 = 9, t_5 = 12$ ; it is the reflexion line arrangement associated to the complex reflexion group  $G_{26}$  in Shephard-Todd classification.

**3.6. Cubic surfaces and unassuming arrangements.** The aim of this section is to prove the following result:

**Proposition 18.** *Let  $\mathcal{C}_0$  be an unassuming arrangement. Then  $\mathcal{C}_0$  is in the family  $\mathfrak{B}_6/S_6$  or in the second moduli space described in Section 2.1.*

That result on matroids has been checked by L. Kühne (personal communication) using combinatorial tool. That proves Theorem 1 without assumption on the six dual points not being on a conic. We give below a geometric proof.

Rather than studying  $\mathcal{C}_0$ , we will be using its dual  $\check{\mathcal{C}}_0 = \mathcal{L}_2(\mathcal{D}(\mathcal{C}_0))$ . We say that a set  $P_6$  of six points  $p_1, \dots, p_6$  in the plane is unassuming if these points are dual to an unassuming arrangement, which means that the points are not contained in a conic and  $\check{\mathcal{C}}_0 = \mathcal{L}_2(P_6)$  is a line arrangement of 15 lines with singularities

$$t_2 = 27, t_3 = t_5 = 6.$$

For classifying these arrangements, we will use the classification of cubic surfaces and their automorphism groups.

Our reference for cubic surfaces is [8, Section V.4]. Let  $P_6 = \{p_1, \dots, p_6\}$  be an arrangement of points in the plane such that no three are on a line and not all are contained in a conic. Let  $\pi : X \rightarrow \mathbb{P}^2$  be the blow-up at the points in  $P_6$ . By taking the pull-back to  $X$  of the linear system of cubics with base points  $p_1, \dots, p_6$ , one obtains a morphism  $X \rightarrow \mathbb{P}^3$ ; it is an embedding and we identify  $X$  with its image, which is a smooth cubic surface.

The cubic surface  $X$  contains exactly 27 lines, which are the strict transform  $F_{ij}$  of the 15 lines through points  $p_i, p_j$ , ( $1 \leq i < j \leq 6$ ), the six exceptional curves  $E_1, \dots, E_6$  above the points  $p_1, \dots, p_6$ , the strict transform  $G_1, \dots, G_6$  of the six conics  $C_1, \dots, C_6$  containing five of the six points  $p_1, \dots, p_6$  (we denote by  $G_k$  the strict transform of the conic which does not contain  $p_k$ ). Conversely, if  $E'_1, \dots, E'_6$  are six disjoint lines on a smooth cubic surface, there exists a blow-down map  $\pi' : X \rightarrow \mathbb{P}^2$  contracting the six lines

$E'_j$  to points  $p'_1, \dots, p'_6$  and such that the image of the 21 remaining lines by  $\pi'$  are 15 lines through the points  $p'_j$  and six conics as above.

A point on a smooth cubic surface  $X$  which is intersection of three lines is called an Eckardt point. The image of an Eckardt point by the blow-down map  $\pi : X \rightarrow \mathbb{P}^2$  can be

- a) either a triple point of the line arrangement  $\mathcal{L}_2(P_6)$ , or
- b) a 5-point  $p_j$  of the line arrangement  $\mathcal{L}_2(P_6)$ , in that case, one conic  $C_k$  among the 5 conics containing  $p_j$  is tangent to one of the lines  $L_{ij}$  of  $\mathcal{L}_2(P_6)$  at  $p_j$ , so that the exceptional curve  $E_j$ , and the lines  $F_{ij}$  and  $G_k$  meet at one point.

Conversely, a configuration described in a) or b) corresponds to an Eckardt point on the surface  $X$ . Eckardt points on a cubic surface are of special interest, since to each Eckardt point there corresponds an involution of  $X$  fixing the Eckardt point and an hyperplane. The automorphism groups of cubic surfaces are classified in [7]. In characteristic 0, cubic surfaces with at least six Eckardt points are classified as follows

$ \mathcal{E} $	$G$	Strata	dim
6	$S_4$	4B	1
9	$H_3(3) \rtimes \mathbb{Z}/2\mathbb{Z}$	3A	1
9	$H_3(3) \rtimes \mathbb{Z}/4\mathbb{Z}$	12A	$\{pt\}$
10	$S_5$	5A	Clebsch
18	$(\mathbb{Z}/3\mathbb{Z})^3 \rtimes S_4$	3C	Fermat

Here  $H_3(3)$  denotes the Heisenberg group. The column  $|\mathcal{E}|$  indicates the number of Eckardt points, the next column is for the automorphism group of the cubic surface; the surfaces are classified according to their strata (which is irreducible). When the strata has a unique point and when it exists, we give the name of the corresponding surface, otherwise we give the dimension of the strata.

Let us proceed according to the strata. By definition of the strata, the abstract configuration of lines and Eckardt points remains the same in each strata, so that it is enough to know that configuration in one example. One can find the equations of the cubic surfaces according to the strata in [7], or [9].

Case 4B. The configurations  $\check{C}_0(z_o)$  obtained from the family  $\mathfrak{B}_6$  give all the cubic surfaces in the strata 4B. Using Magma, one compute that such a cubic has equation

$$X_1X_2X_3 - \frac{1}{2}X_2^2X_3 + \frac{z_o^2+1}{z_o^2-1}X_2X_3^2 - X_1^2X_4 + \frac{1}{8}(z_o^2+3)X_2^2X_4 - \frac{3z_o^2+1}{z_o^2-1}X_2X_3X_4 + X_3^2X_4 + 2X_1X_4^2 + \frac{1}{8}\frac{z_o^4+6z_o^2+9}{z_o^2-1}X_2X_4^2 - 2X_3X_4^2 = 0$$

Cases 3A and 12A. Using Magma, one computes the 27 lines and 9 Eckardt points for some surfaces in strata 3A and 12A. Then we check that each of the 27 lines contains a unique Eckardt point. Therefore the image of these 9 points through any blow-down map  $\pi' : X \rightarrow \mathbb{P}^2$  will be the six 5-points and 3 triple points of the associated 15 line arrangement.

Case 3C. Consider the six points from (3.3), obtained from the Hesse configuration. By blowing-up these points, one obtains the Fermat cubic surface. Each line of the Fermat cubic contains two Eckardt points, thus blowing down any set of six disjoint lines gives a 15 lines arrangement with six triple points. It correspond to the rigid unassuming arrangements.

Case 5A. The Clebsch cubic is obtained by blowing-up the set  $P_6 = \{c_1, \dots, c_6\}$  of six points which are the vertices of a regular pentagon and the center of the pentagon. The line arrangement  $\mathcal{L}_2(P_6)$ , which has 15 lines, is know under the name of the Icosahedral line arrangement. It is also the simplicial arrangement denoted by  $\mathcal{A}(15)_{120}$  in [1] and it has singularities

$$t_2 = 15, t_3 = 10, t_5 = 6,$$

thus the 5-points  $c_1, \dots, c_6$  are not an unassuming arrangement. The Clebsch cubic surface  $X$  possesses 10 Eckardt points, the inverse image of the ten triple points under the blow-up map. The 15 lines  $F_{ij}$  which are the strict transform of the lines through  $c_i, c_j$  form a  $(10_3, 15_2)$ -configuration. The 12 remaining lines  $E_j, G_k$  of  $X$  do not contain Eckardt points.

Let us prove that using the Clebsch cubic  $X$ , it is not possible to construct a set of six disjoint lines such that the image of the lines on  $X$  by the map which contracts these six lines gives an arrangement of 15 lines with  $t_2 = 27, t_3 = t_5 = 6$ . If one wants to construct such a set of lines, one must take disjoint lines  $F_{st}$  (each containing two Eckardt points), to be mapped to two 5-points, thus we must choose two lines  $F_{ij}, F_{i'j'}$  (which are disjoint) for some  $i, j \neq i', j'$ . The lines that are disjoint to  $F_{ij}$  and  $F_{i'j'}$  are the 10 lines  $F_{ik}, E_k, G_k$  for  $k \in \{1, \dots, 6\} \setminus \{i, j, i', j'\}$  and the line  $F_{j,j'}$ . One can not take another line of the form  $F_{rs}$ , moreover it is easy to check that among the remaining 6 lines, one cannot choose 4 lines disjoint from  $F_{i,j}, F_{i',j'}$ . Thus one cannot obtain a new unassuming arrangement by using the Clebsch cubic.

That finishes the proof of Proposition 18.

### 3.7. Concluding Remarks.

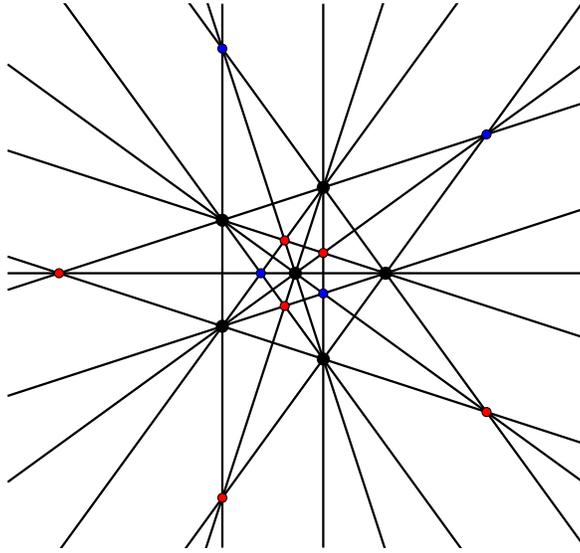
3.7.1. *Special values of the parameter  $z_o$ .* Let us define  $\mathcal{S} = \{\pm 2 \pm \sqrt{5}\}$ ; these are the values  $z_o$  such that the line arrangement  $\mathcal{C}_0(z_o)$  defined in 2.1 is not an unassuming arrangement:  $\mathcal{C}_0(z_o)$  has 15 nodes but the line arrangement  $\mathcal{L}_2(\mathcal{D}(\mathcal{C}_0)) = \mathcal{D}(\mathcal{P}_2(\mathcal{C}_0))$  has 15 lines and singularities

$$t_2 = 15, t_3 = 10, t_5 = 6.$$

That line arrangement is the simplicial arrangement  $\mathcal{A}(15)_{120}$  (see [6]).

The image  $\mathcal{C}'_1$  by  $\Lambda_{\{2\},\{3\}}$  of  $\mathcal{C}_0 = \mathcal{C}_0(z_o)$  is a line arrangement of 10 lines with 45 nodes. The line arrangement  $\mathcal{C}_0 \cup \mathcal{C}'_1$  has 16 lines with singularities  $t_2 = 30, t_4 = 15$ . We conjecture that the number of lines of the iterates of  $\mathcal{C}_0$  under  $\Lambda_{\{2\},\{3\}}$  diverges toward infinity.

In the degenerate cases  $z_o \in \{0, \pm 1, \infty\}$ , one has the relations  $\mathcal{Y}(1) = \mathcal{Y}(-1) = 1, \mathcal{Y}(0) = \mathcal{Y}(\infty) = \infty$  and the peculiarity of these cases is reflected by the map  $F_\Lambda$  since  $F_\Lambda(0) = -1, F_\Lambda(-1) = 1$  and the points 1 and  $\infty$  are

FIGURE 3.1. The simplicial arrangement  $\mathcal{A}(15)_{120}$ 

fixed points of  $F_\Lambda$ . The image by the period map  $\mathcal{Y}$  of  $z_o \in \mathcal{S}$  equals to 161 and we remark that the peculiar dynamical behavior of  $\mathcal{C}_0(z_o)$  under the map  $\Lambda_{\{2\},\{3\}}$  is unseen by the map  $F_\Lambda$ .

Finally, we notice that the line arrangement  $\mathcal{C}'_1$  contains exactly 5 subsets of 6 lines which are unassuming arrangements, all are such that their image by  $\mathcal{Y}$  equals  $51841 = 2 \cdot 161^2 - 1$ . Each of the 15 double points of  $\mathcal{C}_0$  is the vertex three base points (see Definition 10) of unassuming arrangements.

Figure 3.1 is a picture of  $\mathcal{A}(15)_{120}$ ; the six 5-points of  $\check{\mathcal{C}}_0(z_o)$  (for fixed  $z_o \in \mathcal{S}$ ) are in black, the ten triple points are in red or blue, the six points in red are dual to an unassuming arrangement, the 4 other sets of six points with that property are obtained by taking the image by the order 5 rotation through the center.

**3.7.2. Free arrangements.** In [12], we study other arrangements of six lines  $\mathcal{F}_0$ , which we call flashing arrangements. They have the property that the image  $\mathcal{F}_1 = \Lambda_{\{2\},\{3\}}(\mathcal{F}_0)$  is also a flashing arrangement and  $\Lambda_{\{2\},\{3\}}(\mathcal{F}_1) = \mathcal{F}_0$ . The arrangement  $\mathcal{L}_2(\mathcal{D}(\mathcal{F}_0))$  has 13 lines.

Lukas Kühne pointed out to us the paper [2], where is studied, in view of Terao's Conjecture, two line arrangements  $L_{13}$  and  $L_{15}$  of 13 lines and 15 lines, with one dimensional moduli. Both arrangements have the property that they are free but non recursively free arrangements. It turns out that the arrangements of 15 lines  $\mathcal{L}_2(\mathcal{D}(\mathcal{C}))$  associated to the unassuming arrangements  $\mathcal{C}$  in  $\mathfrak{B}_6$  are the arrangements  $L_{15}$ . The arrangements  $\mathcal{L}_2(\mathcal{D}(\mathcal{F}))$  associated to the flashing arrangements are the arrangements  $L_{13}$ .

We found our first example of unassuming arrangement (and therefore the nonbases  $\text{NB}_1$  (see Section (2.1)) as a sub-arrangement of  $\Lambda_{\{2\},\{3\}}(\mathcal{A}(15)_{120})$ ;

the flashing arrangements were also derived from  $\mathcal{A}(15)_{120}$ . We were then searching by random for a line arrangement  $\mathcal{C}_0$  with the smallest number of lines possible such that the number of lines of the sequence defined by  $\mathcal{C}_{k+1} = \Lambda_{\{2\},\{3\}}(\mathcal{C}_k)$  diverges to infinity.

We found the second nonbases  $\text{NB}_2$  (see Equation (2.4)) with a rigid realization space by searching unassuming line arrangements contained in the Ceva line arrangements.

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