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An analysis of mathematical modelling in university textbooks to address Klein's second discontinuity in teacher education

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In Honduras, mathematics teachers receive formal mathematics training at university, which does not prepare them to integrate extra-mathematical modelling in their practice, as it demanded in high school curriculum. This corresponds to Klein's second discontinuity. To address this issue, we propose analysing the dimension of extra-mathematical modelling (university vs high school). We report here the analysis of a university differential equations textbook. We found that usefulness of Laplace transform in solving differential equations is unrelated to a mathematical (or an extra-mathematical) modelling activity. A complementary control theory textbook analysis showed the crucial role of the Laplace transform in solving mathematical modelling tasks in engineering, e.g., controlling systems represented by circuits, which are taught in high school, being an extra-mathematical modelling activity can be integrated into math courses to pre-service mathematics teachers.

Keywords: Differential equations, Klein's second discontinuity, Laplace transform, mathematical modelling, mathematics teachers.

Introduction

Mathematical modelling ability is one of the goals to be taught in the high school curriculum of many countries (Frejd, 2013; Jessen & Kjeldsen, 2021; Rodríguez, 2010). The Organisation for Economic Co-operation and Development (OECD), in its international assessments (e.g., PISA) has promoted the integration of modelling in elementary mathematics teaching to develop citizens with the ability to understand, question and modify their world, which has led to changes in the curriculum. However, not only teachers do not prioritise its development in the classroom for epistemological and systemic reasons (Frejd, 2013) or because of “insufficient previous experience with modelling tasks” (Sen Zeytun et al., 2023, p. 12). But also, because their formal university mathematics training does not provide them with the tools to integrate modelling into their teaching practice, and even less so in non-mathematical contexts (Ibidem). This corresponds to Klein's second discontinuity, identified more than a hundred years ago (Kilpatrick, 2019). Klein (2016/1933) argues that when an undergraduate student becomes a teacher, he is expected to “teach the traditional elementary mathematics according to school practice; and, since he will be scarcely able, unaided, to discern any connection between this task and his university mathematics” (p. 1). This issue has been taken up again, showing the validity of its essence and the particularities imposed by current conditions (e.g., Barquero & Winsløw, 2022; Eichler & Isaev, 2023). Winsløw and Grønbaek (2014) consider Klein's second discontinuity from an institutional approach –the Anthropological Theory of the Didactic (ATD)– analysed in three dimensions, namely, “[t]he *institutional context* (of university vs. school); [t]he difference in the *subject's role* within the institution (a student in university or school, vs. a teacher of school mathematics); [t]he difference in *mathematical contents* (elementary vs. advanced)”

(p. 64), through relations between subjects with praxeologies or knowledge in a certain institution. This model allows us to propose a fourth dimension: *extra-mathematical modelling in context* (university vs. high school). Klein argues that students should learn to contemplate mathematical phenomena from their environment instead of applying mathematics to artificial contexts (Mattheis, 2019). With the aim of contributing to the study of this fourth dimension, we analyse characteristics of the teaching of mathematical modelling at university through a differential equations (DE) textbook and a control theory textbook and the way in which this analysis can be considered to generate a mathematical modelling proposal for a module of a DE course. And thus, to provide pre-service teachers with tools to integrate mathematical modelling in non-mathematical contexts into the high school.

Elements of the anthropological theory of the didactic

Institutions are stable social organisations that make possible human activities, e.g., mathematical activities and mathematical modelling, which are similar in that in TAD “doing mathematics mostly consists in producing, transforming, interpreting and developing mathematical models” (Barquero et al., 2019, p. 321). Human activities can be analysed through a unique model: praxeology. Its four elements are task type (T , what is to be done), technique (τ , how it is to be done), technology (θ , is a discourse that produces, justifies and explains the τ) and theory (Θ , more generally what produces, justifies and explains the θ). Praxeologies can be created, taught and used in any academic or non-academic institution and can circulate between institutions (Chevallard, 1999), undergoing, in effect, transpositional processes, i.e. transformations. Turning a praxeology from an academic (e.g., mathematics) or non-academic (e.g., a workplace) institution into a teaching object implies a process of *didactic transposition* involving several institutions: academia and others (workplace), educational system, classroom and community of study, as illustrated by Gascón and Nicolás (2022, p. 16):

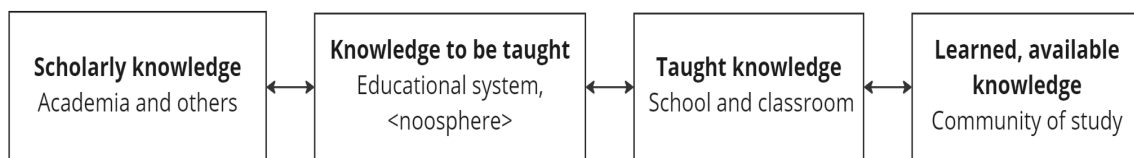


Figure 1. Institutions and phases involved in didactic transposition

According to Barquero et al. (2019) the process of transposing a praxeology from academia or other institutions to the classroom corresponds to the *unit of analysis* in this theory, which involves studying academia or other institutions (mathematics discipline, workplace, everyday life), the educational system (curriculum, educational model) and the classroom (mathematics courses).

Methodology

This paper presents part of the unit of analysis of this research, which corresponds to the curriculum of pre-service mathematics teachers and university courses, DE and control theory with the aim of recognising the place given to mathematical modelling in non-mathematical contexts.

Stage 1. The pre-service mathematics teacher's programme

The pre-service mathematics teachers programme at the Universidad Pedagógica Nacional Francisco Morazán in Honduras was analysed in this phase. Three curricular areas were identified: (a) General

and pedagogical foundations; (b) Didactics of mathematics; and (c) Specialisation in mathematics. The mathematical knowledge to be taught includes, in the third year, a DE course in which the techniques for solving first and second-order DE and their applications in the real world are taught. Educators design their didactic proposals.

Stage 2. Analysis of mathematics taught in textbooks: DE and control theory

To recognise the role given to mathematical modelling in the DE course, two textbooks were analysed. The first one is Zill (2013), as it is the basis of the DE course for pre-service teachers, while the second one (Ogata, 2010) is used in university education for teaching extra-mathematical tasks. This analysis was guided by the question Q_0 : How does it teach a task of type T : solving a DE using the Laplace transform (LT)? A process of modelling with DE, suggested by Zill (2013), was identified. Subsequently, four analytical questions were answered, representing the elements of a praxeology. First, the question *What is done?* was asked in order to identify the type of task, focus on the proposed activities (e.g., exercises, problems). Then, to identify the technique taught, the question was asked: *what procedures were presented to solve the tasks?* These procedures were described through the steps to perform the tasks. Finally, to identify the technology, the focus was on the definitions, theorems and properties that justify each of the steps of the technique, i.e., what justifications are presented to support each procedure, while the theory was identified by answering: *what mathematical theory is presented to support these justifications?* Zill's (2013) textbook analyses a whole chapter on teaching the LT as a method for solving DE, while Ogata (2010) only one task in context was analysed. This made it possible to propose a local praxeology of teaching the LT in Zill (2013) and a local praxeology of modelling electrical systems in Ogata (2010).

Results

Analysis of the LT teaching in a DE textbook

In Zill's textbook (2013), explicit expressions allude to mathematical modelling (e.g., real life, real world, model, mathematical model). In introducing DE as mathematical models, the author points out their importance: "It is often desirable to describe the behavior of some real-life system or phenomenon, whether physical, sociological, or even economic, in mathematical terms" (p. 20). The analysis focused on solving DE and DE systems and their applications presented in chapter 7 of Zill's (2013) textbook. Three types of tasks were identified, T : solving a DE of the form $a_2y'' + a_1y' + a_0y = f(t)$; T' : solving a system of DE; and T'' : solving a DE in context of the form $a_2y'' + a_1y' + a_0y = f(t)$. A general technique for solving tasks of these three types (T, T', T'') is presented, which consists of four steps and is based on LT. In his textbook, Zill (2013) highlights the potential of this technique by pointing out: "Yes, there is a lot of algebra inherent in the use of the Laplace transform, but observe that we do not have to use variation of parameters or worry about the cases and algebra in the method of undetermined coefficients" (p. 287). The local praxeology¹ is schematised in Figure 2:

¹ It has only one technology that produces several techniques to solve different types of tasks. In this case, the technology is a Laplace transform.

Task types (T)	Technique (τ)	Technology (θ)	Theory (θ)
(T) Solving a DE	Step 1: Apply the LT to the DE. Step 2: Solve the equation for $Y(s)$. Step 3: Partial Fraction Decomposition. Step 4: Find $y(t)$ using the inverse of LT.	Laplace transform (LT)	Differential equations (DE)
(T') Solving a system of DE			
(T'') Solving a DE in context			

Figure 2: Praxeology for solving a DE using the LT

Were identified four tasks of type T (t_1, t_2, t_3, t_4), one of type T' and one of type T'' (Figure 3). T and T'' differ from each other by the function $f(t)$ of the DE given (e.g., algebraic function, trigonometric function, e -based exponential, product of two such functions, unit step, Dirac delta). The definition is not used to find the LT of these functions. Still, LT theorems organised in tables or previously stated (e.g., first and second-translation theorems) are used to justify the technique. In T'' , $f(t)$ is an integrodifferential equation, its transform is found using the convolution theorem; finally, T' consists of solving a system of DE using the steps of the general technique (step 1) and then solving the resulting system of algebraic equations.

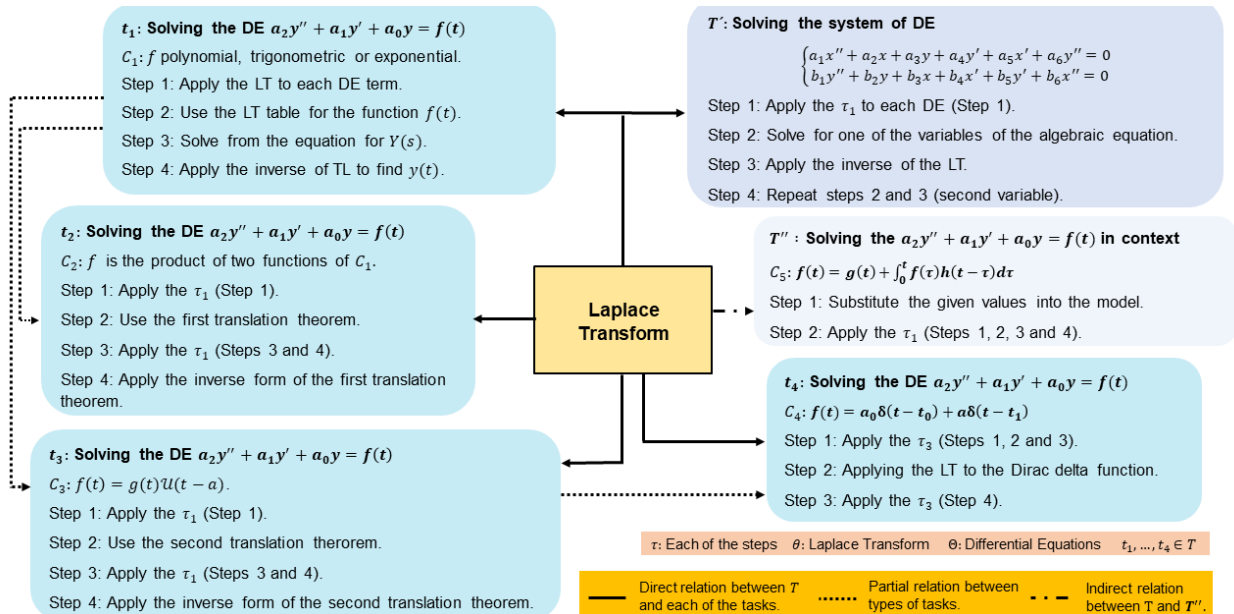
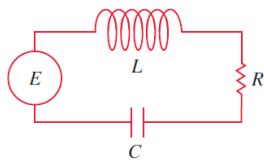


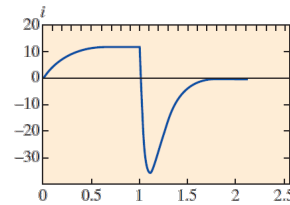
Figure 3: Praxeology of the DE using the LT

An example of a task of type T'' is to determine the current $i(t)$ in a single-loop LRC-series circuit with L , R and C known, with $i(0) = 0$ and $E(t)$ given. Zill (2013) explains that an LRC circuit is a physical system with an inductor L , a resistor R and a capacitor C (Figure 4a), with the switch closed where L , R and C are generally constant. Thus, the mathematical model $L \frac{di}{dt} + iR + \frac{1}{C} \int_0^t i(\tau) d\tau = E(t)$ is constructed by means of Kirchhoff's second law. This is a first-order DE with constant coefficients, with initial conditions all zero, similar to T -type of task. In the context of the LRC circuit, the function $f(t)$ has an integrodifferential equation. The author suggests, as an initial step, substituting the given data in the mathematical model above, i.e., DE becomes $0.1 \frac{di}{dt} + 2i + 10 \int_0^t i(\tau) d\tau = 120t - 120tU(t-1)$; subsequently, the general technique (Figure 2) is shown

applying the LT to each term, by using partial fractions and applying the inverse LT to $I(s)$, the current $i(t)$ is obtained as a solution, the graph of which is Figure 4b:



(a)



(b)

Figure 4: (a) LRC-series circuit. (b) Graph of current $i(t)$ (Zill, 2013, pp. 305–306)

According to Zill (2013), mathematicians have adopted some engineering words as their own that they consider appropriate when describing a phenomenon. Examples are the *input* or *driving* function and *output* or *response* function, the *transient term*, and the *steady state* of the solution. In this same problem, the author presents the graph of the solution (Figure 4b) and comments, very superficially, that the output $i(t)$ is a continuous function. However, the input function $E(t)$ is discontinuous. In another circuit example, in interpreting the solution $i(t) = E_0/R + ce^{-(R/L)t}$, he mentions that the transient term $ce^{-(R/L)t}$ is an often-important concept in applied problems. In this case, $i(t) \rightarrow E_0/R$, when $t \rightarrow \infty$. This allows relationships such as Ohm's law ($E = iR$) to be established for large values of time. Therefore, the term E_0/R is the steady state current. According to the technique of the task of type T'' exemplified, its solution consists of substituting the data in the given mathematical model (in real life, this is not always the case) and then solving the first-order DE without any need to reflect on the meaning of the parameters R , C and L or of the current i and the voltage E . This is a task that fulfils only steps 3 and 4 of the modelling process with DE proposed by Zill (2013): "(1) Assumptions and hypotheses, (2) mathematical formulation, (3) Solve the EDs, (4) Display predictions of model" (p. 21). This seems to limit the teaching of mathematical modelling in non-mathematical contexts.

Analysis of the LT teaching in a control theory textbook

In Ogata's (2010) control theory textbook, a task was analysed in Chapter 2, which is structured in sections. The first section introduces the notions of control theory. In the second section, the transfer function and impulse response are studied. LT is presented as a powerful tool for solving control system problems (although the teaching of LT is cursory, its review in a DE textbook is suggested), whose behaviour is modelled by a transfer function (defined as the ratio of the LT of the output function and the LT of the input function of a system). The following sections study automatic control systems, block diagrams and the *Matlab* software for calculating the transfer function. This function is a key concept in a system since its determination allows us to know its stability. Ogata (2010) mentions that if the transfer function of a system is known, the output or response to various forms of input is studied to understand the nature of the system. In school, the first step in analysing and controlling a system is to determine its mathematical model. However, in the real world, the input signal of a system is usually not known in advance, and the instantaneous input cannot be expressed analytically. However, it can be known or determined in certain cases from typical tests using input functions (e.g., unit step, ramp, impulse, parabola). According to Ogata (2010), the response of a control system consists of two parts, transient and steady-state (this is briefly noted in Zill, 2013). Designing a control involves predicting the system's behaviour from the knowledge of its components. The most important characteristic is the absolute stability of the system: stable or unstable, but it has other behaviours, such as relative stability and steady-state error. A similar task

to T'' was analysed and called T_*'' : modelling electrical system, which includes three tasks: t_1^* , finding the transfer function $G(s)$ of an electrical system; t_2^* , finding the output signal $c(t)$; and t_3^* , analysing the absolute stability of a system. The techniques to solve these tasks are based on the LT described in Figure 5. There is a partial relation between t_1^* , t_2^* and t_3^* because t_2^* and t_3^* require the τ_1^* in their steps (Figure 5). The transfer function allows to analyse the absolute and relative stability of the system, which involves an energy exchange. The output of the system at a given input function exhibits a transient response before reaching a steady state. In t_3^* , some of the steps of the τ_1^* and τ_2^* are necessary to examine and quantitatively describe the system and the method of zeros and poles can be used. To find the poles, one can use *Matlab* software and plot these points on the complex plane and thus conclude whether the system is stable or unstable.

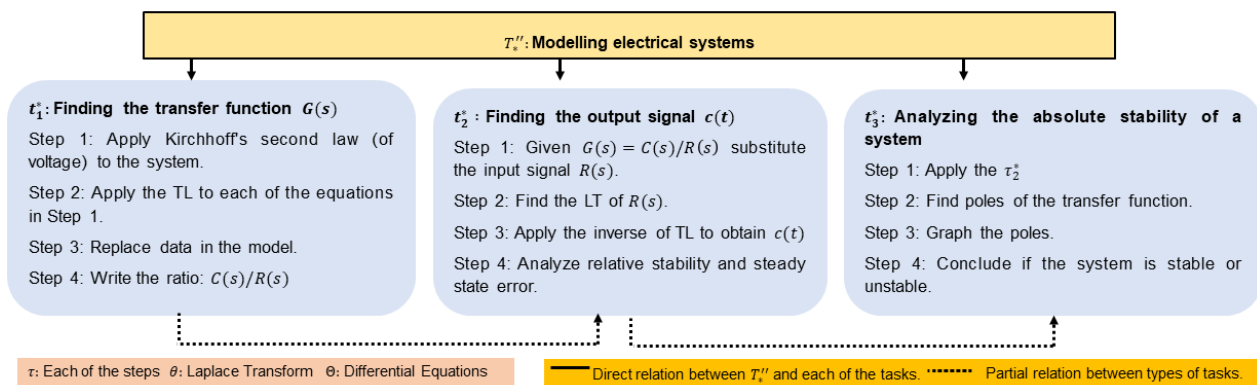


Figure 5: Praxeology of the LT in control theory

An example of t_2^* : determine the output function of a closed-loop first-order system (Figure 6a) given the unit step input function $R(s) = 1/s$, whose transfer function is: $C(s)/R(s) = 1/(Ts + 1)$. Which consists of calculating the LT of each term, decomposing it into partial fractions and applying the inverse LT, and the output $c(t) = 1 - e^{-t/T}$.

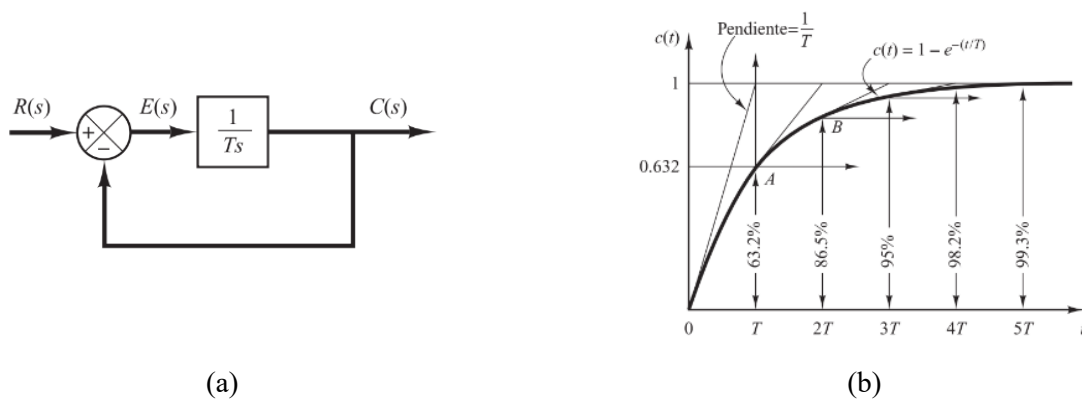


Figure 6: a) Block diagram of a first-order system. b) Unit step response. (Ogata, 2010, pp. 161–162)

The output function $c(t)$ behaves like the input function (the unit step function) when $t \rightarrow \infty$, since the slope decreases as the value of t increases, i.e., the steady state is mathematically reached at infinite time. However, in the real world, a reasonable estimate is the value of $t = 4T$ (Figure 6b), to ensure that the system's behaviour is stationary. Hence, the response is kept within 2% of the final value. This analysis shows the crucial role of differential equations in modelling and controlling the behaviour of physical systems. The LT is fundamental to converting DE into algebraic equations and

moving from the temporal study of the phenomenon to the algebraic one. It is an authentic extra-mathematical context that can be used in a didactic modelling proposal for the DE course.

Conclusion

The analyses suggest that the main objective of Zill (2013) is to solve a DE using LT where technical (intra-mathematical) expertise is privileged. That is, the tasks are focused on the application of LT as a technique to solve certain types of DE, but no reflection is made on whether they allow modelling real phenomena and whether or not this is a limited view of reality (Frejd, 2013; Jessen & Kjeldsen, 2021). In Ogata (2010), a notion of control theory is appreciated: the transfer function, which allows the study and control of physical systems modelled with DE. That is, to determine the stability of the system. LT is fundamental in defining the transfer function, a mathematical function that models the system's output for each possible input, and in the resolution of extra-mathematical modelling tasks. Integrating tasks in non-mathematical contexts to mathematics courses is associated with great complexity since different types of mathematical and non-mathematical knowledge are involved, as well as different objectives in the use of modelling techniques and different disciplinary logic underlying mathematical modelling. Because of this, the real world often does not seem to play a role in modelling problems in mathematics education (Jessen & Kjeldsen, 2021). It is, therefore, necessary to generate specific proposals for teaching to integrate mathematical modelling in non-mathematical contexts into teacher education. To this end, it is necessary to generate an analysis of the teaching of mathematical modelling in high school, of the extra-mathematical contexts used, e.g., physical (electrical circuits), which contributes elements to the fourth dimension of Klein's second discontinuity studied here and how it can be approached in pre-service mathematics teacher education. Also, looking ahead for this research, we need to pay attention to whether teacher education is producing modelling skills or competencies or capabilities, which are relevant and linked to those of high school and in what ways education in these institutions is imitating real-world practices (engineering, data science, physics, etc.) (Rodríguez, 2010). Pre-service teachers must have experience with modelling task design throughout their careers. These experiences and research could provide them with helpful tools for supporting the implementation of such tasks in real classrooms, considering the conditions and constraints that may impact their viability.

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