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Analysis of models used by student primary teachers when addressing a geometric estimation task

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We present a qualitative research study aimed at exploring the models produced by student primary teachers when addressing an estimation task in which the three spatial dimensions play a major role and the objects whose quantity is to be estimated are deformable. Under a grounded theory approach, we analyse the written productions of 80 student primary teachers who were asked to quantify the amount of bunches of grapes that fit in a box of given dimensions. We highlight the diversity of models (linearisation, base unit, density, direct estimate, counting; also found in studies with other kind of tasks) and of quantities (volume, length, area, weight) on which their models were based. Geometric considerations (e. g., using a geometric body to model a bunch of grapes) were mainly in the service of metric interests (calculating volumes) and rarely to spatial ones (arranging bunches). Our findings extend and add value to those observed in related studies.

Keywords: Modelling, estimation task, geometry, analysis of models, student primary teachers.

Introduction

Mathematical modelling, as part of mathematics education, is recognised as a powerful mean to develop mathematical skills as well as a fundamental end in itself for both pre-university and university levels (Ärlebäck & Albarracín, 2019; Stillman et al., 2015). In particular, it has been shown to be of great relevance in the training of primary school teachers: it promotes their reflection on the significance of scholar mathematical contents and processes (Huinchahue Arcos et al., 2018; Kaiser et al., 2006), inspires them to integrate modelling in their prospective teaching through useful teaching examples (Doerr, 2007), and offers opportunities to strength their mathematical reasoning and abilities to make assumptions, communicate mathematical ideas, and question and interpret results obtained in context-based situations. However, formal and informal inclusion of mathematical modelling in primary teacher training programs and in the daily classroom activity is currently not widespread. From the mathematics education perspective, one of the most valuable information sources to plan and design this inclusion is precisely that obtained from the analysis of the productions and performances of student primary teachers when addressing tasks that require modelling. This allows teacher trainers, and mathematics educators in general, to anticipate solving strategies, identify aspects in the modelling tasks that influence the solving strategies, be aware of frequent mistakes and misconceptions to prevent them and treat them as learning opportunities, set learning expectations, graduate the challenge level in the proposed problems, etc. From the research in mathematics education, it is then of research interest characterising the models that student primary teachers use to address mathematical modelling tasks. In this paper we present a research study whose main objective is to analyse the models used by student primary teachers to estimate the amount of content (objects of a kind) that fit in a container of given dimensions.

Background

A large family of estimation tasks that have proven to be effective in promoting modelling are the so-called *Fermi problems*. These are “open, non-standard problems requiring the students to make assumptions about the problem situation and estimate relevant quantities before engaging in, often, simple calculations” (Ärlebäck 2009, p. 331). Detailed descriptions of the emergent solving strategies have been provided in the case of Fermi problems that request estimating large quantities of objects that fit on a delimited flat surface. Segura (2022) uses the following four categories to classify solving plans followed by student primary teachers when faced with series of this type of problems. *Counting* proposes a direct and exhaustive procedure to obtain an estimation. *Linearisation* reduces a two-dimensional (2D) question to a one-dimensional (1D) one to give a final estimate (placing the objects whose quantity is to be estimated on a grid of rows and columns is the most usual example in this category). *Base unit* consists of determining the number of times that the surface occupied by an object on the total surface fits in the total surface, calculating the respective areas and operating correspondingly. *Density* is based on quantifying the number of objects that fit in a unit area chosen by the solver, and then multiplying it by the times that this area fits in the total surface area. These categories are linked to the strategies found in previous studies with secondary school students (Albarracín & Gorgorió, 2014) and primary school students (Ferrando & Albarracín, 2021). It should be noted that the Fermi problems proposed in these studies have certain features in common. On one hand, they could be reformulated and interpreted as 2D problems, meaning that not considering the three dimensions of the problem situation does not lead to less valid models. On the other hand, they are “large number problems” in which the total surface area of the situation is much larger than that occupied by the type of object assumed to be on it and for which an estimate is requested. Consequently, the complete shape of the objects whose quantity is to be estimated might not play a significant role and inaccuracy in it could have a low impact on the resulting estimate. In this scenario, it is natural to wonder about the impact that these characteristics of the problems could have in the emerging strategies observed in those studies or how these strategies could vary if the characteristics were others. However, the emerging solving strategies in the case of estimation problems set in delimited 3D spaces that do not admit sensible 2D modelling have not been explored as much in the literature so far. Sevinc and Ferrando (2022) investigated the models used by student primary teachers to estimate the amount of paper rolls that can be stored in a given wardrobe. They found 3D models involving geometric properties such as the orientation of the rolls or their arrangement, which were not considered in the categories discussed by Segura (2022). Nevertheless, those 3D models were based on reductions to 2D estimations, maybe due to the geometry of the objects whose quantity is to be estimated (cylinders), which can be stored in tiers. It is therefore reasonable to question whether considering objects with more irregular geometry, or deformable ones, would lead to new models not based on reductions to 2D estimations. The mathematical treatment of such objects could also promote the development of models involving further geometric properties. These possibilities, up to the best of our knowledge, are yet to be explored, which motivated the research goals of the present study: (1) Identify and characterise the models used by student primary teachers when addressing an estimation task in which the three spatial dimensions play a significant role in the contextual situation and the objects whose quantity is to be estimated are deformable (see Figure 1);

(2) Analyse the similarities and differences of these models with those reported in other related studies in which other characteristics were considered in the design of the estimation task.

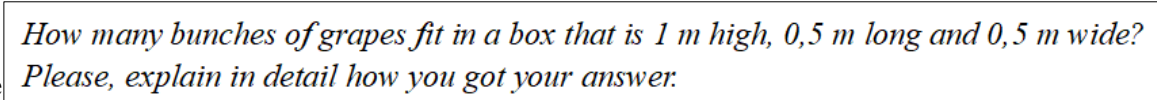
Methodology

This qualitative research work is framed in a larger project aimed at exploring the models produced by students from different academic levels when addressing estimation tasks in which the characteristics of the objects whose quantity is to be estimated and the information provided on them are variable.

Participants and design of the task

Eighty student primary teachers of mixed abilities in mathematics from the same public university in Spain participated in the study. This group was intentionally selected based on the course on mathematics education in which they were enrolled and their collaborative attitude. Participants were duly informed about the research study in advance and gave their consent to participate in it. At the time of the study, they had not been instructed in mathematical modelling as part of their teacher training.

According to our research goals, we designed the estimation task shown in Figure 1 to be presented to the participants. The variables considered in the design were: (i) estimate type, choosing the one based on estimating the content (number of objects of a kind) that fit in a specific container; (ii) deformability of the content, considering a deformable object; (iii) number of relevant spatial dimensions of both the container and the content, selecting three dimensions in both cases.

Figure 1: 

1:

English version of the paper-based task presented to the participants

Data collection and analysis

The data collection was performed in one session of the second semester of the two of a Spanish academic year, during regular class time of the participants. They were strongly encouraged to individually address the task shown in Figure 1 making use of their own resources (meaning knowledge and abilities) and to write their responses in pen on the sheet provided to each one. During their work, the participants did not receive any feedback about the suitability of their approaches or the reasonableness of their answers. Upon completion, participants' worksheets were delivered to the research team and constitute the data for this qualitative research.

The data analysis procedure was inspired by the principles of Grounded Theory (Corbin & Strauss, 1990) and it was structured in three phases, all three carried out jointly by the research team. In the first phase, a preliminary review of the 80 productions was conducted. During this review we paid attention to all the aspects considered relevant in the productions and that were repeatedly subject of discussion, namely: (i) *drawings* (pictorial representations), (ii) *quantities* (measurable qualities involved), (iii) *geometric treatment* (geometric considerations about the content and/or the container of the estimation task), (iv) *data and variables* (variables from the task considered in the production

and data provided by the solver), (v) *approach* (mathematical ideas and procedure to get a final answer to the question of the task), (vi) *sensibility of the final answer* (reasonableness of it).

The second phase consisted of another review of the 80 productions, focused on analysing and codifying all these aspects for each one, assigning corresponding values and including comments. Participants' anonymity was ensured by assigning each a label: S_j , $j \in \mathbb{N} \cap ([1, 32] \cup [76, 92] \cup [107, 119] \cup [144, 161])$ (the peculiarity of this number set comes from the choice of a sub-sample of the larger sample of the project). By way of example, the following could illustrate the information associated to the production of a random participant S_k as the result of the work in this phase: (i) it includes the drawing of a prism representing the box, (ii) it considers the volume of the box and of a bunch of grapes, (iii) it proposes a cone to geometrically model a bunch of grapes, (iv) it deals with the given measures of the box and it assumes specific lengths for the radius and the height of the proposed cone, (v) it calculates the volume of the prism and of the cone (through their respective formulas) and it divides the first by the second, (vi) it gives 325 bunches of grapes as final answer.

During the third phase we reviewed our data coding of the productions and identified the most significant aspects, that is, those that allowed us to better group and differentiate them. These turned out to be approach, quantities and geometric treatment. We grouped the productions according to their similarities respect to these aspects and named each group, giving rise to the following system of categories of models. Names of the categories were motivated by the nature of the models they house and the existing ones in related research studies (e. g., Albarracín & Gorgorió, 2014; Ferrando & Albarracín, 2021; Segura, 2022).

Categories of models identified in our analysis

- *Linearisation*. The 3D-question is transformed in three 1D-questions. Length, height and width measurements of the content are assumed and the numbers of times that these respectively fit into the length, height and width measurements of the container (expressed in the same units) are obtained. The product of these numbers is offered as the final answer.
- *Base unit*. The measure of a specific quantity of the container and the measure of the same quantity of a content object (the "unit") are estimated. Then the first is divided by the second and the quotient of this division is given as final answer.
- *Density*. A part of the container is considered and the number of content objects that fit in it is found out. Then this number is multiplied by the number of times that the considered part fits in the container, giving this product as final answer.
- *Direct estimate*. An estimate of the content that fits in the container is directly given without any intermediary calculation. Some considerations can be included but how (if so) they are taken into account in the given estimate is not made explicit.
- *Counting*. It refers to counting the number of content objects needed to fill the container.

Results and discussion

Regarding participation rate, all participants evidenced in their written productions that they had addressed the task, although not all of them gave a final response to it. Use of models was identified

in 44 of the 80 productions. The remaining 36 were reduced to statements of inability to answer due to lack of data, annotations of the data provided in the task and isolated calculations with them, or they included so sparse and incomplete information that a model could not be identified.

Table 1 summarises the characteristics of the models identified in the participants' productions according to their approach, quantity and geometric treatment considered (pictures of the participants' productions in which these models were identified are shown in López Centella and Montejo-Gómez (2023)).

Table 1: Models identified in the participants' productions

Model	Approach	Quantity	Geometric treatment	
			Geometric body to model content	Arrangement of content in container
Linearisation	Multiply the quotients obtained by dividing the length of each of the dimensions of the container by the corresponding (assumed) of the content	Length	Cube / Cuboid (implicit or explicitly)	Not explicit / Stacking of cuboids
Base unit	Divide the volume of the container by the volume of the content	Volume	Cube / Cuboid (implicit or explicitly) / Pyramid / Cone / Sphere	Content treated as a fluid / non-fluid
	Divide the area of a prefixed subsurface of the container by the area occupied by the content when placed on it	Area	Rectangle	NA
	Divide the "weight that fits in the container" by the weight of the content (under an implicit/explicit use of the non-general equivalence $1\text{kg} = 1\text{l} = 1\text{dm}^3$).	Weight		Content treated as a fluid
Density	Quantify the content that fits in a part of the container and multiply this quantity by the times that the part fits in the container	Volume	Cuboid	Stacking of cuboids
Direct estimate	Provide an estimate of the content that fits in the container without specifying calculations or how further considerations are taken into account			
Counting	Count the number of content objects needed to fill the container			

As specified in Table 1, several versions of the base unit model were used. It was based on three different quantities: volume (S76, S107), area (S10) and weight (S82). In the case of volume, we highlight two aspects concerning geometric treatment: the particular geometric body to model the content, and the arrangement of the content in the container. Regarding the first, cuboids (S76, S107), cones (S160) and pyramids (S150) were used to geometrically model a bunch of grapes, and a sphere to do so with a grape (S12). Regarding the second, some participants who use cuboids to model a bunch of grapes represented the stacking of the cuboids in the box (S153). Remarkably, most of the participants who used a base unit model involving volume and a specific geometric body to model a bunch of grapes or a grape, merely considered this geometric modelling with the aim of calculating the volume of the content object. Once obtained, they usually proceeded to di-

vide the volume of the box by the volume of the geometric body in question, treating the grapes as a fluid (S76, S107, S150, S160) and paying no attention to their arrangement in the box. Nevertheless, there were some exceptions to this: some participants deliberately deducted some grapes of the obtained quotient claiming that “taking into account that they [grapes] do not have a shape that completely adapts to the interior of the box, it is necessary to subtract some of them, since the whole space will not be filled by grapes” (S12).

Concerning the geometric treatment, participants (as a whole) seemed to have tacitly taken into account the deformability of the bunches of grapes according to the length measurements assumed for the geometric bodies through which they modelled them: the cones were given a height generally greater than the length given to the side of the cubes or any side of the cuboids.

Interestingly enough, to respond this estimation question formulated in terms of capacity, some participants worked with weights (S12). It seemed that they had more references available to estimate the weight of a bunch of grapes than its volume and it came more naturally to them. This fact may be related to their level of acquaintization with these quantities and the phenomena associated to the objects involved in the question in their daily activity (weigh some grapes in a market or deal with fruit weights). Then, in their attempts to maintain consistency under the base unit model that they were pursuing to apply, they converted the measure of capacity of the box into a measure of its weight by using the following: $1 \text{ dm}^3 = 1 \text{ l} = 1 \text{ kg}$. Apparently this was assumed to be a universal equivalence between measurement units —no further written considerations were found regarding it when it was used. The fact or the need of interpreting the estimation question by means of a quantity different than volume/capacity not only was observed in the base unit model. The linearisation model made the work of many participants easier turning a 3D-question to 1D questions based on length.

Linearisation and density are the only models in which considerations about the arrangement of the bunches of grapes in the box were found. These considerations were typically reflected through pictorial representations showing the stacking of cuboids in the box and descriptions of the number of bunches of grapes fitting in a “tier” of the box and total number of tiers. On the other hand, far of the apparent rudimentary nature or low sophistication level of the counting model, this is invoked by some participants (S116) as the only one that allows a right answer: “the only way to check how many [bunches of grapes] fit is to put one by one in the box until not one more fits and to count how many”.

Conclusions

On the models identified in our analysis

This work describes models used by student primary teachers when addressing a geometric estimation task consisting on quantifying the number of content units (bunches of grapes) that fit in a given container (dimensioned box). According to the approach to face the task, we identified up to five categories of models in the student primary teachers written productions: linearisation, base unit, density, direct estimate and counting. Subcategories of the first three were described under the consideration of two characteristics: the main quantity involved and the geometric treatment carried out in the models, namely the geometric body to model the content object and the arrangement of the

content in the container. Regarding the first, we highlight that several models and many productions were based on a different quantity to volume. The most frequently used models fall into the categories of linearisation, base unit based on volume, and density. We underline the unexpected consideration of the quantity weight in the base unit model, based on the estimation of the weight of a content unit and an attempt to convert the measure of capacity of the container into “the measure of weight that it can hold”. Regarding the second characteristic, density focused more on the arrangement of the content (stacking in the box) than on its shape, more often considered in base unit models.

In another order of things, the impact of the geometric features taken into account in the design of the estimation task (namely, the singular shape and deformability of the content and the relevance of the three spatial dimensions) was different in the distinct categories of models. In linearisation, direct estimate and counting they seem to have had less impact. In contrast, base unit models attempted to better geometrically modelling the content object according to its shape through the use of cones and pyramids for a bunch of grapes, and of spheres for a grape itself.

On the similarities and differences of the models with those identified in other related studies

Among the main research interests motivating the presented study was to explore if considering deformable or irregularly shaped objects in estimation tasks would lead to new models not based on reductions to 2D estimations as the described in a series of works (e.g., Albarracín & Gorgorió, 2014; Ferrando & Albarracín, 2021; Segura, 2022). The impact of the aforementioned features of our proposed task seem to have been primarily on the quantities involved in the models and the geometric treatment of the content. Remarkably, the description of the models observed in our study can be seen as a description that extends and expands that reported by Segura (2022).

Impact, reach and use of the results

From the research perspective, our findings contribute to expanding the literature on models considered by student primary teachers when addressing estimation tasks, extending existing results for other types of estimation tasks, and pointing out three-dimensionality, shape and deformability as aspects of modelling tasks that can influence these models. From the point of view of teaching and of mathematics education, they anticipate student primary teachers solving approaches, reveal some common mistakes and misconceptions (that can be taken as learning opportunities) and help to set learning expectations.

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