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► **To cite this version:**

François Desbouvries. Entropy Computation in Partially Observed Markov Chains. Bayesian Inference and Maximum Entropy Methods In Science and Engineering, MaxEnt, Jul 2006, Paris, France. pp.355-357, 10.1063/1.2423294 . hal-04415402

**HAL Id: hal-04415402**

**<https://hal.science/hal-04415402>**

Submitted on 24 Jan 2024

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# ENTROPY COMPUTATION IN PARTIALLY OBSERVED MARKOV CHAINS

François Desbouvries

*Institut National des Télécommunications, Evry, France*

**Abstract.** Let  $X = \{X_n\}_{n \in \mathbb{N}}$  be a hidden process and  $Y = \{Y_n\}_{n \in \mathbb{N}}$  be an observed process. We assume that  $(X, Y)$  is a (pairwise) Markov Chain (PMC). PMC are more general than Hidden Markov Chains (HMC) and yet enable the development of efficient parameter estimation and Bayesian restoration algorithms. In this paper we propose a fast (i.e.,  $O(N)$ ) algorithm for computing the entropy of  $\{X_n\}_{n=0}^N$  given an observation sequence  $\{y_n\}_{n=0}^N$ .

**Keywords:** Entropy, Hidden Markov Models, Partially observed Markov Chains

**PACS:** 05.70.-a, 65.40.Gr, 02.50.-r, 02.50.Ga.

## INTRODUCTION

Let  $(X, Y) = \{X_n, Y_n\}_{n \in \mathbb{N}}$  be a joint process in which  $X$  is unobserved and  $Y$  is observed. We assume that  $X$  and  $Y$  are both discrete with  $X_n \in \{1, \dots, K\}$  and  $Y_n \in \{1, \dots, M\}$  for all  $n \in \mathbb{N}$ . Let  $X_{i:j} = \{X_n\}_{i \leq n \leq j}$ ,  $Y_{i:j} = \{Y_n\}_{i \leq n \leq j}$ ,  $x_{i:j} = \{x_n\}_{i \leq n \leq j}$  and  $y_{i:j} = \{y_n\}_{i \leq n \leq j}$  (upper case letters denote random variables (r.v.) and lower case letters their realizations). Let also  $p(x_{i:j}|y_{i:j})$ , say, denote the conditional probability that  $X_{i:j} = x_{i:j}$  given  $Y_{i:j} = y_{i:j}$ . In some applications it is relevant to compute the entropy of  $X_{0:N} = \{X_n\}_{n=0}^N$  given an observation  $y_{0:N} = \{y_n\}_{n=0}^N$ , i.e. we want to compute

$$H(X_{0:N}|y_{0:N}) = - \sum_{x_{0:N}} p(x_{0:N}|y_{0:N}) \log p(x_{0:N}|y_{0:N}). \quad (1)$$

The brute force computation of (1) requires  $O(K^N)$  elementary operations. However, a fast (i.e.,  $O(K^2N)$ ) algorithm for computing (1) has been proposed recently [1] in the framework of Hidden Markov Chains (HMC) (see e. g. the recent tutorials [2] [3]), i.e. of processes  $(X, Y)$  satisfying

$$p(x_{n+1}|x_{0:n}) = p(x_{n+1}|x_n); \quad (2)$$

$$p(y_{0:N}|x_{0:N}) = \prod_{n=0}^N p(y_n|x_{0:N}); \quad (3)$$

$$p(y_n|x_{0:N}) = p(y_n|x_n) \text{ for all } n, 0 \leq n \leq N. \quad (4)$$

Now, HMC have been generalized recently to Pairwise Markov Chains (PMC) [4], i.e. to joint processes  $(X, Y)$  which satisfy

$$p(x_n, y_n|x_{0:n-1}, y_{0:n-1}) = p(x_n, y_n|x_{n-1}, y_{n-1}). \quad (5)$$

As we see from the definition, a PMC can be seen as a (vector) Markov chain in which one component is observed and the other one is hidden. Now, (2)-(4) imply (5), so any HMC is a PMC. The converse is not true, as can be seen at the local level, since in a PMC the transition probability reads

$$p(x_n, y_n | x_{n-1}, y_{n-1}) = p(x_n | x_{n-1}, y_{n-1}) p(y_n | x_n, x_{n-1}, y_{n-1}); \quad (6)$$

so an HMC is indeed a PMC in which  $p(x_n | x_{n-1}, y_{n-1})$  reduces to  $p(x_n | x_{n-1})$  and  $p(y_n | x_n, x_{n-1}, y_{n-1})$  reduces to  $p(y_n | x_n)$ . In other words, making use of PMC enables to model rather complex physical situations, since at time  $n$ , conditionally on the previous state  $x_{n-1}$ , the probability of the current state  $x_n$  may still depend on the previous observation  $y_{n-1}$ ; and conditionally on  $x_n$ , the probability of observation  $y_n$  may still depend on the previous state  $x_{n-1}$  and on the previous observation  $y_{n-1}$ .

It happens that it is possible to extend from HMC to PMC [4] the existing efficient Bayesian restoration or parameter estimation algorithms. As we shall see in this paper, it is also possible in the context of PMC to compute  $H(X_{0:N} | y_{0:N})$  efficiently. More precisely, our aim here is to extend to PMC the algorithm of [1]; the algorithm we obtain remains  $O(K^2N)$ .

## EFFICIENT ENTROPY COMPUTATION IN PMC

From now on we assume that  $(X, Y)$  is a PMC, i.e. that (5) holds. Let us first recall [5] the following basic properties of entropy :

$$h(U, V | w) = h(U | w) + h(V | U, w), \quad (7)$$

$$h(V | U, w) = \sum_u h(V | u, w) p(u | w). \quad (8)$$

Let us now address the computation of  $H(X_{0:N} | y_{0:N})$ . Let  $0 \leq n \leq N$ . From (7), (8) we get

$$\begin{aligned} H(X_{0:n} | y_{0:n}) &= H(X_n | y_{0:n}) + H(X_{0:n-1} | X_n, y_{0:n}) \\ &= H(X_n | y_{0:n}) + \sum_{x_n} H(X_{0:n-1} | x_n, y_{0:n}) p(x_n | y_{0:n}). \end{aligned} \quad (9)$$

On the other hand, from (5) we get

$$p(x_{0:n-2} | x_{n-1}, x_n, y_{0:n}) = p(x_{0:n-2} | x_{n-1}, y_{0:n-1}), \quad (10)$$

so  $H(X_{0:n-1} | x_n, y_{0:n})$  in (9) can be computed recursively by

$$\begin{aligned} H(X_{0:n-1} | x_n, y_{0:n}) &= H(X_{n-1} | x_n, y_{0:n}) + H(X_{0:n-2} | X_{n-1}, x_n, y_{0:n}) \\ &= H(X_{n-1} | x_n, y_{0:n}) + \sum_{x_{n-1}} H(X_{0:n-2} | x_{n-1}, x_n, y_{0:n}) p(x_{n-1} | x_n, y_{0:n}) \\ &\stackrel{(10)}{=} H(X_{n-1} | x_n, y_{0:n}) + \sum_{x_{n-1}} H(X_{0:n-2} | x_{n-1}, y_{0:n-1}) p(x_{n-1} | x_n, y_{0:n}) \quad (11) \end{aligned}$$

It remains to compute  $p(x_n|y_{0:n})$  and  $p(x_{n-1}|x_n, y_{0:n})$  efficiently. This can be performed by an algorithm which extends to PMC [4] the (forward pass of) the Forward-Backward algorithm [6] [7] [8] [9], and which we now recall

$$\begin{aligned}
p(x_{n-1}, x_n | y_{0:n}) &= \frac{p(x_{n-1}, x_n, y_{0:n})}{\sum_{x_{n-1}, x_n} p(x_{n-1}, x_n, y_{0:n})} \\
&\stackrel{(5)}{=} \frac{p(x_n, y_n | x_{n-1}, y_{n-1}) p(x_{n-1} | y_{0:n-1}) p(y_{0:n-1})}{\sum_{x_{n-1}, x_n} p(x_n, y_n | x_{n-1}, y_{n-1}) p(x_{n-1} | y_{0:n-1}) p(y_{0:n-1})} \\
&= \frac{p(x_n, y_n | x_{n-1}, y_{n-1}) p(x_{n-1} | y_{0:n-1})}{\sum_{x_{n-1}, x_n} p(x_n, y_n | x_{n-1}, y_{n-1}) p(x_{n-1} | y_{0:n-1})}, \tag{12}
\end{aligned}$$

$$p(x_n | y_{0:n}) = \sum_{x_{n-1}} p(x_{n-1}, x_n | y_{0:n}), \tag{13}$$

$$p(x_{n-1} | x_n, y_{0:n}) = \frac{p(x_{n-1}, x_n | y_{0:n})}{p(x_n | y_{0:n})}. \tag{14}$$

Let us summarize the discussion. We got the following algorithm :

**Fast algorithm for computing  $H(X_{0:N}|y_{0:N})$ .**

- At time  $n - 1$  :
  - assume that we have  $\{H(X_{0:n-2}|x_{n-1}, y_{0:n-1})\}_{x_{n-1}=1}^K, \{p(x_{n-1}|y_{0:n-1})\}_{x_{n-1}=1}^K$ .
- Iteration  $n - 1 \rightarrow n$  :
  - compute  $\{p(x_n|y_{0:n})\}_{x_n=1}^K$  and  $\{p(x_{n-1}|x_n, y_{0:n})\}_{x_{n-1}, x_n=1}^K$  via (12), (13) and (14);
  - compute  $\{H(X_{n-1}|x_n, y_{0:n}) = -\sum_{x_{n-1}} p(x_{n-1}|x_n, y_{0:n}) \log p(x_{n-1}|x_n, y_{0:n})\}_{x_n=1}^K$ ;
  - compute  $\{H(X_{0:n-1}|x_n, y_{0:n})\}_{x_n=1}^K$  via (11);
  - compute  $H(X_n|y_{0:n}) = -\sum_{x_n} p(x_n|y_{0:n}) \log p(x_n|y_{0:n})$ ;
  - compute  $H(X_{0:n}|y_{0:n})$  via (9).

Note that the algorithm is  $O(K^2N)$ , as was the original algorithm of [1]. Finally, we assumed that  $Y_n$  is a discrete r.v., but the extension to continuous emission probability densities is straightforward.

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