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# Stability Of The $\mathbf{P}_{nc}^1 \times (P^0 + P^1)$ Element

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**Abstract** We solve the Stokes problem numerically. We analyse the  $\mathbf{P}_{nc}^1 \times (P^0 + P^1)$  mixed finite element method which exhibits interesting numerical features. However, only an incomplete proof of the inf-sup condition is available. We prove here this condition and the stability of the method.

## 1 Stokes Problem

A popular low-order method for solving the Stokes problem is the nonconforming Crouzeix-Raviart mixed finite element method [1, Example 4] (cf. [2, 3] for higher orders). The discrete velocity is piecewise affine, continuous in the barycentre of the interfaces between elements, but not globally continuous,  $\mathbf{P}_{nc}^1$ ; and the discrete pressure is piecewise constant,  $P^0$ . This method introduces some consistency error on the discrete velocity, which may lead to an unphysical solution when solving the Navier-Stokes equations, especially when the source field is a strong gradient [4, 5]. To recover consistency, the discrete pressure space can be generated by  $P^1$ -continuous plus  $P^0$  basis functions, cf. the TrioCFD code [6, 7]. This leads to the  $\mathbf{P}_{nc}^1 \times (P^0 + P^1)$  mixed finite element method, which exhibits interesting numerical features. However, only an incomplete proof of the stability is available in [8, 9]. Below, we prove the inf-sup condition. Consider the Stokes problem set in an open,

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bounded, connected subset  $\Omega$  of  $\mathbb{R}^d$ ,  $d = 2, 3$ , with a Lipschitz boundary  $\partial\Omega$ :

$$\text{Find } (\mathbf{u}, p) \text{ such that } -\nu \Delta \mathbf{u} + \nabla p = \mathbf{f} \text{ and } \operatorname{div} \mathbf{u} = 0 \text{ in } \Omega, \quad (1)$$

with homogeneous boundary conditions  $\mathbf{u} = 0$  on  $\partial\Omega$  and a normalisation condition:  $\int_{\Omega} p = 0$ . The case of inhomogeneous Dirichlet boundary conditions, that is  $\mathbf{u} = \mathbf{g}$  on  $\partial\Omega$  with  $\mathbf{g} \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$  such that  $\int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n}|_{\partial\Omega} = 0$  can be studied similarly. Given  $D$  an open subset of  $\mathbb{R}^d$ , let us set  $\mathbf{L}^2(D) = (L^2(D))^d$ , and  $L^2_{zmv}(D) = \{q \in L^2(D) \mid \int_D q = 0\}$ . We denote by  $(\cdot, \cdot)_D$  the  $L^2(D)$  or  $\mathbf{L}^2(D)$  inner product, and  $\|\cdot\|_D$  the associated norm. Due to Poincaré-Steklov inequality [10, Lemma 3.24], the seminorm in  $H^1_0(\Omega)$  is equivalent to the natural norm, so that  $(v, w)_{H^1_0(\Omega)} = (\nabla v, \nabla w)_{\Omega}$  and  $\|v\|_{H^1_0(\Omega)} = \|\nabla v\|_{\Omega}$ . Let us set  $\mathbf{V} = \{\mathbf{v} \in \mathbf{H}^1_0(\Omega) \mid \operatorname{div} \mathbf{v} = 0\}$  which is a closed subset of  $\mathbf{H}^1_0(\Omega)$ . We denote by  $\mathbf{V}^{\perp}$  the orthogonal of  $\mathbf{V}$  in  $\mathbf{H}^1_0(\Omega)$ .

**Proposition 1** [13, Theorem 3.1 (e)]. *The operator  $\operatorname{div} : \mathbf{H}^1_0(\Omega) \rightarrow L^2(\Omega)$  is an isomorphism of  $\mathbf{V}^{\perp}$  onto  $L^2_{zmv}(\Omega)$ . We call  $C_{\operatorname{div}} > 1$  the constant such that:  $\forall p \in L^2_{zmv}(\Omega), \exists! \mathbf{v}_p \in \mathbf{V}^{\perp} \mid \operatorname{div} \mathbf{v}_p = p$  and  $\|\mathbf{v}_p\|_{\mathbf{H}^1_0(\Omega)} \leq C_{\operatorname{div}} \|p\|_{\Omega}$ .*

Let  $\mathbf{f} \in (\mathbf{H}^1_0(\Omega))'$ . The variational formulation of Problem (1) reads:  
Find  $(\mathbf{u}, p) \in \mathbf{H}^1_0(\Omega) \times L^2_{zmv}(\Omega)$  such that  $\forall (\mathbf{v}, q) \in \mathbf{H}^1_0(\Omega) \times L^2_{zmv}(\Omega)$ :

$$\nu(\mathbf{u}, \mathbf{v})_{\mathbf{H}^1_0(\Omega)} - (p, \operatorname{div} \mathbf{v})_{\Omega} - (q, \operatorname{div} \mathbf{u})_{\Omega} = \langle \mathbf{f}, \mathbf{v} \rangle_{\mathbf{H}^1_0(\Omega)}. \quad (2)$$

One can prove that Problem (2) is well-posed with the T-coercivity theory [14, 15, 16], using Poincaré-Steklov inequality [10, Lemma 3.24] and Proposition 1.

## 2 Discretization And Stability Of The $\mathbf{P}^1_{nc} \times (P^0 + P^1)$ Scheme

Consider  $(\mathcal{T}_h)_h$  a family of simplicial triangulations of  $\Omega$ . For a given  $\mathcal{T}_h := \{K\}$ , we call  $\mathcal{F}_h := \{F\}$  its set of  $(d-1)$ -simplices, called facets (faces if  $d = 3$ , edges if  $d = 2$ ), and  $\mathcal{V}_h := \{S\}$  its set of vertices. In the bounds, we will use  $\lesssim$  and  $\gtrsim$  for (hidden) constants independent of  $\mathcal{T}_h$ .

- For  $K \in \mathcal{T}_h$ , we call  $h_K$  and  $\rho_K$  the diameters of  $K$  and its inscribed sphere respectively, and we let:  $\sigma_K = \frac{h_K}{\rho_K}$ . When the  $(\mathcal{T}_h)_h$  is a shape-regular triangulation family (see e.g. [10, def. 11.2]), there exists a constant  $\sigma > 1$ , called the shape regularity parameter, such that for all  $h$ , for all  $K \in \mathcal{T}_h$ ,  $\sigma_K \leq \sigma$ .
- Let  $\mathcal{F}_h = \mathcal{F}_h^i \cup \mathcal{F}_h^b$ , where  $\mathcal{F}_h^b = \{F \in \mathcal{F}_h \mid F \subset \partial\Omega\}$ ,  $\mathcal{F}_h^i = \mathcal{F}_h \setminus \mathcal{F}_h^b$ . For  $K \in \mathcal{T}_h$ , we set  $\mathcal{F}_K^i = \{F \in \mathcal{F}_h^i \mid F \subset \partial K\}$ .

For  $F \in \mathcal{F}_h$ ,  $M_F$  denotes the barycentre of  $F$ , and  $\mathbf{n}_F$  its unit normal (outward oriented if  $F \subset \partial\Omega$ ). For  $K \in \mathcal{T}_h$  and  $F \subset \partial K$ , we denote by  $\mathbf{n}_{F,K}$  the normal vector of  $F$  outgoing from  $K$ . Let  $\hat{K}$  be the reference simplex. For  $K \in \mathcal{T}_h$ , we denote by  $T_K : \hat{K} \rightarrow K$  the geometric mapping such that  $\forall \hat{\mathbf{x}} \in \hat{K}$ ,  $\mathbf{x}|_K = T_K(\hat{\mathbf{x}}) = \mathbb{B}_K \hat{\mathbf{x}} + \mathbf{b}_K$ ,

where  $\mathbb{B}_K \in \mathbb{R}^{d \times d}$  and  $\mathbf{b}_K \in \mathbb{R}^d$ . Let us set  $J_K = \det(\mathbb{B}_K)$ . Recall that [10, Lemma 11.1]:  $|J_K| = |K|/|\hat{K}|$ ,  $\|\mathbb{B}_K\| \leq h_K/\rho_{\hat{K}}$ ,  $\|\mathbb{B}_K^{-1}\| \leq h_{\hat{K}}/\rho_K$ . For  $v \in L^2(\Omega)$ , we set:  $v_K = v|_K$ ,  $\hat{v}_K = v \circ T_K$ . By changing the variable, we get:  $\|v\|_K^2 = |J_K| \|\hat{v}_K\|_{\hat{K}}^2$ . Let  $v \in H^1(K)$ , then  $\nabla v|_K = (\mathbb{B}_K^{-1})^T \nabla_{\hat{x}} \hat{v}$ , and  $\|\nabla v\|_K^2 \leq \|\mathbb{B}_K^{-1}\|^2 |J_K| \|\nabla_{\hat{x}} \hat{v}\|_{\hat{K}}^2$ ,  $\|\nabla_{\hat{x}} \hat{v}\|_{\hat{K}}^2 \leq \|\mathbb{B}_K\|^2 |J_K|^{-1} \|\nabla v\|_K^2$ . Let  $K \in \mathcal{T}_h$  and  $F \subset \partial K$ , we denote by  $T_{F,K}$  the geometric mapping such that  $\forall \mathbf{x} \in F$ ,  $T_{F,K}(T_K^{-1}(\mathbf{x})) = \mathbf{x}$ . We let  $\hat{F}_{F,K} = T_K^{-1}(F)$ . According to [1, Eq. (3.17)]:

$$\forall K \in \mathcal{T}_h, \forall F \subset \partial K, \quad |F| |K|^{-1} \lesssim \rho_K^{-1}. \quad (3)$$

We set  $\mathcal{P}_h H^1 = \{v \in L^2(\Omega) \mid \forall K \in \mathcal{T}_h, v|_K \in H^1(K)\}$ , endowed with:  $(v, w)_h := \sum_{K \in \mathcal{T}_h} (\nabla v, \nabla w)_K$ ,  $\|v\|_h^2 = \sum_{K \in \mathcal{T}_h} \|\nabla v\|_K^2$ . Let  $F \in \mathcal{F}_h^i$  such that  $F = \partial K_L \cap \partial K_R$  and  $\mathbf{n}_F = \mathbf{n}_{F,K_L}$ . The jump of  $v \in \mathcal{P}_h H^1$  across  $F$  is defined by:  $[v]_F := v|_{K_L} - v|_{K_R}$ . For  $F \in \mathcal{F}_h^b$ , we set:  $[v]_F := v|_F$ .

For a subset  $D$  of  $\mathbb{R}^d$ , for  $k \in \mathbb{N}$ , we call  $P^k(D)$  the set of degree  $k$  polynomials on  $D$ ,  $\mathbf{P}^k(D) = (P^k(D))^d$ , and  $\mathcal{P}_{disc}^k(\mathcal{T}_h) = \{q \in L^2(\Omega) \mid \forall K \in \mathcal{T}_h, q|_K \in P^k(K)\}$ . For  $k > 0$  we define  $\mathcal{P}^k(\mathcal{T}_h) = C^0(\overline{\Omega}) \cap \mathcal{P}_{disc}^k(\mathcal{T}_h)$ . We call  $\pi_0 : L^2(\Omega) \mapsto \mathcal{P}_{disc}^0(\mathcal{T}_h)$  the  $L^2$ -orthogonal projection onto  $\mathcal{P}_{disc}^0(\mathcal{T}_h)$ , such that  $\forall q \in L^2(\Omega)$ ,  $\pi_0(q) = \sum_{K \in \mathcal{T}_h} q_K \mathbb{1}_K$  where  $\forall K \in \mathcal{T}_h$ ,  $q_K = \int_K q / |K|$ .

For  $K \in \mathcal{T}_h$  and a vertex  $S$  of  $K$ ,  $\lambda_{S,K}$  denotes the barycentric coordinate related to  $S$ . We can endow  $\mathcal{P}^1(\mathcal{T}_h)$  with the basis  $(\phi_S)_{S \in \mathcal{V}_h}$  such that:  $\forall K \in \mathcal{T}_h$ ,  $\phi_S|_K = \lambda_{S,K}$  if  $S \in K$ , zero otherwise, so that  $\mathcal{P}^1(\mathcal{T}_h) = \text{vect}((\phi_S)_{S \in \mathcal{V}_h})$ .

Using [12, Thm. 5] and Eq. (3), one can prove that:

$$\forall w \in P^k(K), \forall F \subset \partial K, \quad \|w\|_F \lesssim (|F| |K|^{-1})^{\frac{1}{2}} \|w\|_K \lesssim (\rho_K)^{-\frac{1}{2}} \|w\|_K. \quad (4)$$

Using Eq. (4) with  $w = v - \underline{v}_K$  and Poincaré-Steklov inequality [10, Lemma 12.11] in  $K$ , it holds (with below,  $\underline{v}_F = \int_F v / |F|$ ):

$$\forall v \in P^k(K), \forall F \subset \partial K, \quad \|v - \underline{v}_F\|_F \lesssim \|v - \underline{v}_K\|_F \lesssim (\sigma_K h_K)^{1/2} \|\nabla v\|_K. \quad (5)$$

Using the proof of [17, Lemma 2] and [12, Thm. 5], we can prove that:

$$\forall v \in P^1(K), \quad \|\nabla v\|_K \lesssim (\rho_K)^{-1} \|v - \underline{v}_K\|_K. \quad (6)$$

Let  $\mathbf{X}_{0,h}$  be the space of the discrete velocity of the nonconforming Crouzeix-Raviart mixed finite element method, defined by:

$$\mathbf{X}_h = (X_h)^d, \quad X_h = \{v_h \in \mathcal{P}_{disc}^1(\mathcal{T}_h) \mid \forall F \in \mathcal{F}_h^i, \int_F [v_h] = 0\}; \quad (7)$$

$$\mathbf{X}_{0,h} = (X_{0,h})^d, \quad X_{0,h} = \{v_h \in X_h \mid \forall F \in \mathcal{F}_h^b, \int_F v_h = 0\}. \quad (8)$$

The condition on the jumps of  $v_h$  on the inner facets is often called the patch-test condition. Due to this condition, one can prove that :

**Proposition 2** [1, Lemma 2]. *The broken norm  $v_h \rightarrow \|v_h\|_h$  is a norm over  $X_{0,h}$ .*

For  $K \in \mathcal{T}_h$  and  $F \subset \partial K$ ,  $\lambda_{F,K}$  denotes the barycentric coordinate related to the vertex  $S_{F,K}$  opposite to facet  $F$  in  $K$ . We can endow  $X_h$  with the basis  $(\psi_F)_{F \in \mathcal{F}_h}$  such that for all  $K \in \mathcal{T}_h$ ,  $\psi_F|_K = 1 - d\lambda_{F,K}$  if  $F \subset \partial K$  and zero otherwise. We then have  $\psi_F|_F = 1$ , so that  $[\psi_F]_F = 0$  if  $F \in \mathcal{F}_h^i$ , and for all  $F' \neq F$ ,  $\int_{F'} \psi_F = 0$ . We have:  $X_h = \text{vect}((\psi_F)_{F \in \mathcal{F}_h})$  and  $X_{0,h} = \text{vect}((\psi_F)_{F \in \mathcal{F}_h^i})$ .

Let  $\Pi_h : \mathbf{H}^1(\Omega) \mapsto \mathbf{X}_h$  be the Crouzeix-Raviart interpolation operator, such that for all  $\mathbf{v} \in \mathbf{H}^1(\Omega)$ ,  $\Pi_h(\mathbf{v}) = \sum_{F \in \mathcal{F}_h} \underline{\mathbf{v}}_F \psi_F$ ,  $\underline{\mathbf{v}}_F = \int_F \mathbf{v}/|F|$ .

**Proposition 3** [17, Lemma 2]. *The interpolation operator  $\Pi_h$  is such that for all  $\mathbf{v}_h \in \mathbf{H}_0^1(\Omega)$ ,  $\|\Pi_h(\mathbf{v}_h)\|_h \leq \|\mathbf{v}_h\|_{\mathbf{H}_0^1(\Omega)}$ .*

However, due to the consistency error on the discrete velocity, it is well-known [18, 19] that choosing the space of the discrete pressures equal to  $\mathcal{P}_{disc}^0(\mathcal{T}_h) \cap L_{zmv}^2(\Omega)$  yields inaccurate results when solving Navier-Stokes equations. Starting from [20], a new element is proposed in [8]. The space of the discrete pressures is then:  $Q_h = Q_{0,h} + Q_{1,h}$ , where  $Q_{0,h} := \mathcal{P}_{disc}^0(\mathcal{T}_h) \cap L_{zmv}^2$  and for  $k > 0$ ,  $Q_{k,h} := \mathcal{P}^k(\mathcal{T}_h) \cap L_{zmv}^2(\Omega)$ . One can build an  $L^2$ -orthogonal sum letting  $\tilde{Q}_{1,h} = \{q_h \in Q_h \mid \forall K \in \mathcal{T}_h, \int_K q_h = 0\}$ .

**Proposition 4** *We have:  $Q_h = Q_{0,h} \overset{\perp}{\oplus} \tilde{Q}_{1,h}$ . It holds for all  $q_h \in Q_h$ :  $\|q_h\|_{\Omega}^2 = \|\pi_0(q_h)\|_{\Omega}^2 + \|q_h - \pi_0(q_h)\|_{\Omega}^2$ .*

To prove the discrete inf-sup condition of  $\mathbf{X}_{0,h} \times Q_h$ , we need some Lemmas.

**Lemma 1** *Let  $k > 0$ ,  $(\mathbf{v}, q) \in \mathbf{X}_{0,h} \times Q_{k,h}$ . It holds:*

$$|\sum_{K \in \mathcal{T}_h} \sum_{F \subset \partial K} (\mathbf{v} \cdot \mathbf{n}_{F,K}, q)_F| \lesssim \sigma \|\mathbf{v}\|_h \|q - \pi_0(q)\|_{\Omega}. \quad (9)$$

*Proof.* We start with the proof of [20, Lemma 3.1]. Using the patch-test condition, one obtains:

$$\sum_{K \in \mathcal{T}_h} \sum_{F \subset \partial K} (\mathbf{v} \cdot \mathbf{n}_{F,K}, q)_F = \sum_{K \in \mathcal{T}_h} \sum_{F \subset \partial K} \left( (\mathbf{v}_K - \underline{\mathbf{v}}_F) \cdot \mathbf{n}_{F,K}, q_K - \underline{q}_K \right)_F.$$

We have, using Eq. (4) with  $w = q_K - \underline{q}_K$  and Eq. (5) for the components of  $\mathbf{v}$ :

$$\left| \left( (\mathbf{v}_K - \underline{\mathbf{v}}_F) \cdot \mathbf{n}_{F,K}, q_K - \underline{q}_K \right)_F \right| \lesssim \sigma_K \|\underline{\nabla} \mathbf{v}_K\|_K \|q_K - \underline{q}_K\|_K.$$

Summing and using the discrete Cauchy-Schwarz inequality, we obtain (9).  $\square$

Introducing the element-wise divergence operator  $\text{div}_h$ , we prove next that one can build an explicit right inverse of  $\text{div}_h$  from  $Q_h$  to  $\mathbf{X}_{0,h}$ .

**Lemma 2** *For all  $q_h \in Q_h$ , there exists  $\mathbf{v}_{0,h} \in \mathbf{X}_{0,h}$  such that:*

$$-(\text{div}_h \mathbf{v}_{0,h}, \pi_0(q_h))_{\Omega} = \nu^{-1} \|\pi_0(q_h)\|_{\Omega}^2, \quad (10)$$

$$\|\mathbf{v}_{0,h}\|_h \leq C_{\text{div}} \nu^{-1} \|\pi_0(q_h)\|_{\Omega}. \quad (11)$$

*Proof.* Let  $q_h \in Q_h$ . From Proposition 1,  $\exists \mathbf{v}_{\pi_0(q_h)} \in \mathbf{H}_0^1(\Omega) \mid \operatorname{div} \mathbf{v}_{\pi_0(q_h)} = \pi_0(q_h)$  and  $\|\mathbf{v}_{\pi_0(q_h)}\|_{\mathbf{H}_0^1(\Omega)} \leq C_{\operatorname{div}} \|\pi_0(q_h)\|_{\Omega}$ . Let  $\mathbf{v}_{0,h} = -\nu^{-1} \Pi_h(\mathbf{v}_{\pi_0(q_h)})$ . Integrating by part twice, we get:  $\forall K \in \mathcal{T}_h$ ,  $\int_K \operatorname{div} \mathbf{v}_{0,h} = -\nu^{-1} \int_K \operatorname{div} \mathbf{v}_{\pi_0(q_h)}$ , hence (10). Using Proposition 3, we get (11).  $\square$

We can also build an explicit right inverse of the discrete divergence operator from  $Q_{1,h}$  to  $\mathbf{X}_{T,h} = \{\mathbf{v}_h \in \mathbf{X}_{0,h} \mid \forall F \in \mathcal{F}_h^i, \mathbf{v}_h \cdot \mathbf{n}_F = 0\}$ , if [20, Hyp. 4.1] holds:

**Hypothesis 1.** We suppose that  $\mathcal{T}_h$  is such that each element  $K \in \mathcal{T}_h$  has at most  $d - 1$  facets that lie on  $\partial\Omega$ .

**Lemma 3** Assuming Hypothesis 1,  $\forall q_{1,h} \in Q_{1,h}$ ,  $\exists \mathbf{v}_{1,h} \in \mathbf{X}_{T,h}$  such that:

$$(\nabla q_{1,h}, \mathbf{v}_{1,h})_{\Omega} \gtrsim \nu^{-1} \sigma^{-2(d-2)} \|q_{1,h} - \pi_0(q_{1,h})\|_{\Omega}^2, \quad (12)$$

$$(\operatorname{div} \mathbf{v}_{1,h}, q_{0,h})_K = 0 \quad \forall q_{0,h} \in Q_{0,h}, \quad (13)$$

$$\|\mathbf{v}_{1,h}\|_h \lesssim \nu^{-1} \sigma^2 \|q_{1,h} - \pi_0(q_{1,h})\|_{\Omega}. \quad (14)$$

*Proof.* Let us consider  $q_{1,h} \in Q_{1,h}$ . For all  $K \in \mathcal{T}_h$ , we let  $q_K := q_{1,h}|_K$  and  $\hat{q}_K = q_K \circ T_K$ . Since  $q_{1,h} \in H^1(\Omega)$  we have (see e.g. [21, Prop. 2.2.10]):  $\forall F \in \mathcal{F}_h^i \mid F = K \cap K', \nabla q_K \times \mathbf{n}_F = \nabla q_{K'} \times \mathbf{n}_F$ . We construct  $\mathbf{v}_{1,h} \in \mathbf{X}_{T,h} \mid \forall F \in \mathcal{F}_h^i$ ,  $\mathbf{v}_{1,h}(M_F) = \nu^{-1} h_F^2 \mathbf{n}_F \times (\nabla q_{1,h} \times \mathbf{n}_F)$ , where  $h_F$  is the diameter of  $F$ . We have:

$$\forall K \in \mathcal{T}_h \quad (\nabla q_{1,h}, \mathbf{v}_{1,h})_K = \nu^{-1} (d+1)^{-1} |K| \sum_{F \in \mathcal{F}_K^i} h_F^2 |\nabla q_K \times \mathbf{n}_F|^2. \quad (15)$$

Let  $\mathbf{n}_{\hat{F}_{F,K}}$  be the unit normal vector of the facet  $\hat{F}_{F,K}$  outgoing from  $\hat{K}$ . We have:  $\mathbf{n}_{F,K} = |K| (\mathbb{B}_K^{-1})^T |\hat{F}_{F,K}| |F|^{-1} \mathbf{n}_{\hat{F}_{F,K}}$ .

Consider  $d = 2$ . Changing the variable, assuming Hypothesis 1, we get:

$$\sum_{F \in \mathcal{F}_K^i} h_F^2 |\nabla q_K \times \mathbf{n}_F|^2 = \sum_{F \in \mathcal{F}_K^i} |\hat{F}_{F,K}|^2 |\nabla_{\hat{\mathbf{x}}} \hat{q}_K \times \mathbf{n}_{\hat{F}_{F,K}}|^2 \gtrsim |\nabla_{\hat{\mathbf{x}}} \hat{q}_K|^2. \quad (16)$$

Consider  $d = 3$ . Changing the variable, one now finds that  $|\nabla q_K \times \mathbf{n}_F| = |\hat{F}_{F,K}| |F|^{-1} |\mathbb{B}_K \nabla_{\hat{\mathbf{x}}} \hat{q}_K \times \mathbf{n}_{\hat{F}_{F,K}}|$ . Hence:  $|\nabla q_K \times \mathbf{n}_F| \gtrsim |F|^{-1} \|\mathbb{B}_K^{-1}\|^{-1} |\nabla_{\hat{\mathbf{x}}} \hat{q}_K \times \mathbf{n}_{\hat{F}_{F,K}}|$ . Notice that  $h_F |F|^{-1} \|\mathbb{B}_K^{-1}\|^{-1} \gtrsim \sigma_K^{-1}$ . Assuming Hypothesis 1, we then have:

$$\sum_{F \in \mathcal{F}_K^i} h_F^2 |\nabla q_K \times \mathbf{n}_F|^2 \gtrsim \sigma_K^{-2} |\nabla_{\hat{\mathbf{x}}} \hat{q}_K|^2. \quad (17)$$

Using Poincaré-Steklov inequality [10, Lemma 12.11] in  $\hat{K}$ , we have:  $|\hat{K}| |\nabla_{\hat{\mathbf{x}}} \hat{q}_K|^2 = \|\nabla_{\hat{\mathbf{x}}} \hat{q}_K\|_{\hat{K}}^2 \gtrsim \|\hat{q}_K - \underline{q}_K\|_{\hat{K}}^2 = |K|^{-1} \|q_K - \underline{q}_K\|_K^2$  with  $\underline{q}_K = \int_K q_K / |K|$ . Using (16) and (17) in (15), it holds:

$$(\nabla q_{1,h}, \mathbf{v}_{1,h})_K \gtrsim \nu^{-1} \sigma_K^{-2(d-2)} |K| |\nabla_{\hat{\mathbf{x}}} \hat{q}_K|^2 \gtrsim \nu^{-1} \sigma_K^{-2(d-2)} \|q_K - \underline{q}_K\|_K^2. \quad (18)$$

We obtain (12) by summation. Remark that  $\nabla \psi_{F|K} = |K|^{-1} |F| \mathbf{n}_{F,K}$ .

Let us prove (13). Recall the orthogonality  $\nabla \psi_{F|K}(M_F) \perp \mathbf{v}_{1,h}(M_F)$ , so that  $(\operatorname{div} \mathbf{v}_{1,h}, q_{0,h})_K = |K| q_{0,h}(K) \sum_{F \in \mathcal{F}_K^i} \nabla \psi_{F|K}(M_F) \cdot \mathbf{v}_{1,h}(M_F) = 0$ .

Let us prove (14). For  $K \in \mathcal{T}_h$ ,  $\underline{\mathbf{v}}_{1,h|K} = \sum_{F \in \mathcal{F}_K^i} \mathbf{v}_{1,h}(M_F) \otimes \nabla \psi_{F|K}$ .

Hence:  $\|\underline{\nabla} \mathbf{v}_{1,h}\|_K^2 \lesssim \nu^{-2} \sum_{F \in \mathcal{F}_K^i} |F|^2 |K|^{-1} h_F^4 |\nabla q_K \times \mathbf{n}_F|^2$ . Using (3), and assuming Hypothesis 1, we get:  $\|\underline{\nabla} \mathbf{v}_{1,h}\|_K^2 \lesssim \nu^{-2} \sigma_K^2 h_K^2 |K| \sum_{F \in \mathcal{F}_K^i} |\nabla q_K \times \mathbf{n}_F|^2 \lesssim \nu^{-2} \sigma_K^2 h_K^2 \|\nabla q_K\|_K^2$ . From Eq. (6):  $\|\underline{\nabla} \mathbf{v}_{1,h}\|_K \lesssim \nu^{-1} \sigma_K^2 \|q_K - \underline{q}_K\|_K$ .  $\square$

Let  $b_h : \mathbf{X}_h \times Q_h \rightarrow \mathbb{R}$  be defined by  $b_h(\mathbf{v}_h, q_h) = -(\operatorname{div}_h \mathbf{v}_h, q_{0,h})_\Omega + (\mathbf{v}_h, \nabla q_{1,h})_\Omega$  with  $q_h = q_{0,h} + q_{1,h}$  such that  $(q_{0,h}, q_{1,h}) \in Q_{0,h} \times Q_{1,h}$ .

**Theorem 1 (Stability)** *The following continuity property holds:*

$$\forall (\mathbf{v}_h, q_h) \in \mathbf{X}_{0,h} \times Q_h, \quad |b_h(\mathbf{v}_h, q_h)| \lesssim \sigma \|\mathbf{v}_h\|_h \|q_h\|_\Omega. \quad (19)$$

Assuming Hypothesis 1, the following inf-sup condition holds:

$$\forall q_h \in Q_h, \exists \mathbf{v}_h \in \mathbf{X}_{0,h}, \quad b_h(\mathbf{v}_h, q_h) \gtrsim \frac{1}{\sqrt{2}} C_{\operatorname{div}}^{-2} \sigma^{-2d} \|\mathbf{v}_h\|_h \|q_h\|_\Omega. \quad (20)$$

*Proof.* Let us prove (19). Let  $(\mathbf{v}_h, q_h) \in \mathbf{X}_{0,h} \times Q_h$ . Integrating by parts, we have:

$$b_h(\mathbf{v}_h, q_h) = -(\operatorname{div}_h \mathbf{v}_h, \pi_0(q_h))_\Omega + \sum_{K \in \mathcal{T}_h} \sum_{F \subset \partial K} (\mathbf{v}_h \cdot \mathbf{n}_{F,K}, q_{1,h})_F. \quad (21)$$

Using Cauchy-Schwarz, we have:  $|(\operatorname{div}_h \mathbf{v}_h, \pi_0(q_h))_\Omega| \leq \sqrt{d} \|\mathbf{v}_h\|_h \|\pi_0(q_h)\|_\Omega$ . Using (9) in (21), we obtain (19) from Proposition 4. Let us prove (20), starting from the proof of [20, Lemma 4.2]. Let  $q_h \in Q_h \setminus \{0\}$ , where  $q_h = q_{0,h} + q_{1,h}$  is such that  $(q_{0,h}, q_{1,h}) \in Q_{0,h} \times Q_{1,h}$ . Let  $\mathbf{v}_h := \mathbf{v}_{0,h}$  be like in Lemma 2. Using (21) with  $\mathbf{v}_h = \mathbf{v}_{0,h}$ , (9) and (10), letting  $\tilde{q}_{1,h} = q_h - \pi_0(q_h)$ , we have:

$$b_h(\mathbf{v}_{0,h}, q_h) \gtrsim \nu^{-1} \|\pi_0(q_h)\|_\Omega^2 - \sigma \|\mathbf{v}_{0,h}\|_h \|\tilde{q}_{1,h}\|_\Omega. \quad (22)$$

Using (11) and Young inequality, we have for all  $\varepsilon > 0$ :

$$-\|\mathbf{v}_{0,h}\|_h \|\tilde{q}_{1,h}\|_\Omega \geq -\frac{1}{2} C_{\operatorname{div}} \nu^{-1} (\varepsilon \|\pi_0(q_h)\|_\Omega^2 + \varepsilon^{-1} \|\tilde{q}_{1,h}\|_\Omega^2). \quad (23)$$

We now insert the bound (23) in (22) to get:

$$b_h(\mathbf{v}_{0,h}, q_h) \gtrsim \nu^{-1} \left( \left(1 - \frac{\varepsilon}{2} C_{\operatorname{div}} \sigma\right) \|\pi_0(q_h)\|_\Omega^2 - \frac{C_{\operatorname{div}}}{2\varepsilon} \sigma \|\tilde{q}_{1,h}\|_\Omega^2 \right). \quad (24)$$

Let  $\mathbf{v}_{1,h}$  be like in Lemma 3. Using (12) and (13), we get:

$$b_h(\mathbf{v}_{1,h}, q_h) = (\nabla q_{1,h}, \mathbf{v}_{1,h})_\Omega \gtrsim \nu^{-1} \sigma^{-2(d-2)} \|\tilde{q}_{1,h}\|_\Omega^2 \quad (25)$$

Last, we set  $\mathbf{v}_h^* = \mu \mathbf{v}_{0,h} + \mathbf{v}_{1,h}$ , where  $\mu > 0$ . Using (24) and (25), we have:

$$\begin{aligned} b_h(\mathbf{v}_h^*, q_h) &= \mu b_h(\mathbf{v}_{0,h}, q_h) + b_h(\mathbf{v}_{1,h}, q_h), \\ &\gtrsim \nu^{-1} \left( \mu \left(1 - \frac{\varepsilon}{2} C_{\operatorname{div}} \sigma\right) \|\pi_0(q_h)\|_\Omega^2 + \left(\sigma^{-2(d-2)} - \frac{\mu C_{\operatorname{div}}}{2\varepsilon} \sigma\right) \|\tilde{q}_{1,h}\|_\Omega^2 \right). \end{aligned}$$

Let us choose  $\varepsilon = (C_{\operatorname{div}} \sigma)^{-1} < 1$  and  $\mu = \sigma^{-2(d-2)} \varepsilon^2$ . We obtain that:

$$b_h(\mathbf{v}_h^*, q_h) \gtrsim C_{\min} \nu^{-1} \|q_h\|_{\Omega}^2 \text{ with } C_{\min} = \varepsilon^2 \sigma^{-2(d-2)} = C_{\text{div}}^{-2} \sigma^{-2(d-1)}. \quad (26)$$

Let us bound  $\|\mathbf{v}_h^*\|_h$  by  $\|q_h\|_{\Omega}$ . We have:  $\|\mathbf{v}_h^*\|_h^2 \lesssim (\mu^2 \|\mathbf{v}_{0,h}\|_h^2 + \|\mathbf{v}_{1,h}\|_h^2)$ . Using (11) and (14)  $\|\mathbf{v}_h^*\|_h^2 \lesssim \nu^{-2} (\mu^2 C_{\text{div}}^2 \|\pi_0(q_h)\|_{\Omega}^2 + \sigma^4 \|\tilde{q}_{1,h}\|_{\Omega}^2)$ , hence:

$$\|\mathbf{v}_h^*\|_h^2 \leq C_{\max}^2 \nu^{-2} \|q_h\|_{\Omega}^2 \text{ with } C_{\max} = \sqrt{2} \sigma^2. \quad (27)$$

Using (27) in (26), we get (20) since  $C_{\min} C_{\max}^{-1} = \frac{1}{\sqrt{2}} C_{\text{div}}^{-2} \sigma^{-2d}$ .  $\square$

Let  $\mathbf{V}_h = \{\mathbf{v}_h \in \mathbf{X}_{0,h} \mid \nabla q_h \in Q_h, b_h(\mathbf{v}_h, q_h) = 0\}$ . We know from [22, Lemma 3.1] that  $\mathbf{V}_h \subset \{\mathbf{v}_h \in \mathbf{X}_{0,h} \mid \forall K \in \mathcal{T}_h, \text{div } \mathbf{v}_h|_K = 0\}$ . Using  $P^0 + P^1$  discrete pressures improves consistency. Indeed, integrating by parts, we have that for all  $(\mathbf{v}_h, q_{1,h}) \in \mathbf{V}_h \times Q_{1,h}$ ,  $0 = b_h(\mathbf{v}_h, q_{1,h}) = -(\text{div}_h \mathbf{v}_h, \pi_0(q_{1,h}))_{\Omega} + \sum_{F \in \mathcal{F}_h} \int_F q_{1,h} [\mathbf{v}_h] \cdot \mathbf{n}_F = 0 + \sum_{F \in \mathcal{F}_h} \int_F q_{1,h} [\mathbf{v}_h] \cdot \mathbf{n}_F$ . Hence:

**Lemma 4** For all  $(\mathbf{v}_h, q_{1,h}) \in \mathbf{V}_h \times Q_{1,h}$ , it holds:  $\sum_{F \in \mathcal{F}_h} \int_F q_{1,h} [\mathbf{v}_h] \cdot \mathbf{n}_F = 0$ .

Let  $\ell_{\mathbf{f}} \in \mathcal{L}(\mathbf{X}_h, \mathbb{R})$  be such that  $\forall \mathbf{v}_h \in \mathbf{X}_h$ ,  $\ell_{\mathbf{f}}(\mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h)_{\Omega}$  if  $\mathbf{f} \in \mathbf{L}^2(\Omega)$ ,  $\ell_{\mathbf{f}}(\mathbf{v}_h) = \langle \mathbf{f}, \mathcal{I}_h(\mathbf{v}_h) \rangle_{\mathbf{H}_0^1(\Omega)}$  if  $\mathbf{f} \notin \mathbf{L}^2(\Omega)$ , where  $\mathcal{I}_h : \mathbf{X}_{0,h} \rightarrow \mathbf{Y}_{0,h}$  is for instance an averaging operator [10, §22.4.1], with  $\mathbf{Y}_{0,h} = \{\mathbf{v}_h \in \mathbf{H}_0^1(\Omega) \mid \forall K \in \mathcal{T}_h, \mathbf{v}_h|_K \in \mathbf{P}^k(K)\}$ . The discretization of Problem (1) with the  $\mathbf{P}_{nc}^1 \times (P^0 + P^1)$  element reads: Find  $(\mathbf{u}_h, p_h) \in \mathbf{X}_{0,h} \times Q_h$  such that  $\forall (\mathbf{u}_h, p_h) \in \mathbf{X}_{0,h} \times Q_h$

$$v(\mathbf{u}_h, \mathbf{v}_h)_h + b_h(\mathbf{v}_h, p_h) + b_h(\mathbf{u}_h, q_h) = \ell_{\mathbf{f}}(\mathbf{v}_h). \quad (28)$$

Due to Proposition 2 and Theorem 1, Problem (28) is well posed.

### 3 Convergence Of The $\mathbf{P}_{nc}^1 \times (P^0 + P^1)$ Element

Following the proofs of [1, Theorems 3, 4, 6], we can prove

**Theorem 2 (Convergence)** Suppose that Hypothesis 1 holds and that the solution of Problem (1) is such that  $(\mathbf{u}, p) \in (\mathbf{V} \cap \mathbf{H}^2(\Omega)) \times (H^1(\Omega) \cap L_{zmv}^2(\Omega))$ . It holds:

$$\|\mathbf{u} - \mathbf{u}_h\|_h \lesssim \sigma h \nu^{-1} \|\mathbf{f}\|_{\Omega}. \quad (29)$$

$$\text{If } \Omega \text{ is convex } \|\mathbf{u} - \mathbf{u}_h\|_{\Omega} \lesssim \sigma^2 h^2 \nu^{-1} \|\mathbf{f}\|_{\Omega}. \quad (30)$$

$$\|(p - p_h) - \pi_0(p - p_h)\|_{\Omega} \lesssim \nu \sigma^{(2d-1)} h \|\mathbf{f}\|_{\Omega}. \quad (31)$$

$$\|\pi_0(p - p_h)\|_{\Omega} \lesssim \nu C_{\text{div}} \sigma^{(2d+1)} h \|\mathbf{f}\|_{\Omega}. \quad (32)$$

In the case of inhomogeneous Dirichlet boundary conditions, one can also recover (29) and (30), following [1, §7], where a lifting of  $\mathbf{g}$  is used.

When  $\Omega \subset \mathbb{R}^2$ , we have moreover the two Lemmas below.

**Lemma 5** For all  $(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_{2,h}$ ,  $(\nabla q_h, \mathbf{v}_h)_{\Omega} = 0$ .



*Proof.* Let  $(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_{2,h}$ . Integrating by parts and using Lemma 4, we get that  $\forall q_{1,h} \in Q_{1,h}$ :  $(\nabla q_h, \mathbf{v}_h)_\Omega = -\sum_{F \in \mathcal{F}_h} \int_F q_h [\mathbf{v}_h] \cdot \mathbf{n}_F = -\sum_{F \in \mathcal{F}_h} \int_F (q_h - q_{1,h}) [\mathbf{v}_h] \cdot \mathbf{n}_F = \sum_{F \in \mathcal{F}_h} \int_F (q_h - q_{1,h}) [\mathbf{v}_h - \mathbf{v}_h(M_F)] \cdot \mathbf{n}_F$ . Hence, we obtain:  $(\nabla q_h, \mathbf{v}_h)_\Omega = -\sum_{K \in \mathcal{T}_h} \sum_{F \subset \partial K} \int_F (q_h - q_{1,h}) (\mathbf{v}_h - \mathbf{v}_h(M_F)) \cdot \mathbf{n}_{F,K}$ . Choose  $q_{1,h}$  such that  $\forall S \in \mathcal{V}_h$ ,  $q_{1,h}(S) = q_{2,h}(S)$ . Then,  $\forall F \in \mathcal{F}_h$ , the degree 3 polynomial  $(q_h - q_{1,h})(\mathbf{v}_h - \mathbf{v}_h(M_F))$  vanishes at the quadrature points of Simpson's rule. Since it is exact for degree 3 polynomials,  $\forall F \in \mathcal{F}_h$ ,  $\int_F (q_h - q_{1,h})(\mathbf{v}_h - \mathbf{v}_h(M_F)) \cdot \mathbf{n}_{F,K} = 0$  and  $(\nabla q_h, \mathbf{v}_h)_\Omega = 0$ .  $\square$

**Lemma 6** Let  $\phi \in H^3(\Omega) \cap L^2_{zmv}(\Omega)$  such that  $|\phi|_{H^3(\Omega)} \neq 0$ .

Then,  $\forall \mathbf{v}_h \in \mathbf{V}_h$ ,  $(\nabla \phi, \mathbf{v}_h)_\Omega \lesssim \sigma h^2 |\phi|_{H^3(\Omega)} \|\mathbf{v}_h\|_\Omega$ .

*Proof.* Let  $\mathbf{v}_h \in \mathbf{V}_h$ . Using Lemma 5 and Cauchy-Schwarz, it holds:  $|(\nabla \phi, \mathbf{v}_h)_\Omega| = |(\nabla(\phi - q_h), \mathbf{v}_h)_\Omega|$  for all  $q_h \in Q_{2,h}$ . We then use [10, Lemma 11.9] to conclude.  $\square$

**Theorem 3 (Pressure Robustness)** Suppose that  $\Omega$  is convex. Let  $p \in H^3(\Omega) \cap L^2_{zmv}(\Omega)$  such that  $|p|_{H^3(\Omega)} \neq 0$ . Consider Problem (1) with  $\mathbf{f} = \nabla p$ . Then  $(\mathbf{u}_h, p_h)$ , the solution of Problem (28) is such that (where  $\delta p = p - p_h$ ):

$$\nu \|\mathbf{u}_h\|_h \lesssim \sigma^2 h^3 |p|_{H^3(\Omega)}, \quad \nu \|\mathbf{u}_h\|_\Omega \lesssim \sigma^3 h^4 |p|_{H^3(\Omega)}. \quad (33)$$

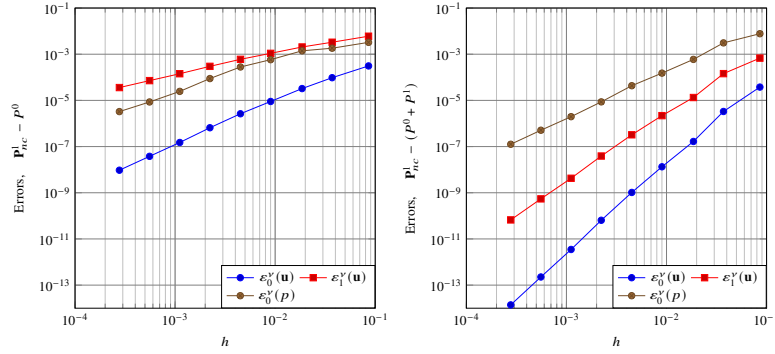
$$\|\delta p - \pi_0(\delta p)\|_\Omega \lesssim \sigma^4 h^2 |p|_{H^2(\Omega)}, \quad \|\pi_0(\delta p)\|_\Omega \lesssim C_{\text{div}} \sigma^6 h^2 |p|_{H^2(\Omega)}. \quad (34)$$

*Proof.* Setting  $\mathbf{v}_h = \mathbf{u}_h$  as test-function in Problem (28) and using Lemma 6, it holds:  $\nu \|\mathbf{u}_h\|_h^2 = (\nabla \phi, \mathbf{u}_h)_\Omega \lesssim \sigma h^2 |\phi|_{H^3(\Omega)} \|\mathbf{u}_h\|_\Omega$ . From [1, Theorem 4], we have:  $\|\mathbf{u}_h\|_\Omega \lesssim \sigma h \|\mathbf{u}_h\|_h$ . We deduce that  $\nu \|\mathbf{u}_h\|_h \lesssim \sigma^2 h^3 |\phi|_{H^3(\Omega)}$ . Using this estimate, we get that  $\nu \|\mathbf{u}_h\|_\Omega \lesssim \sigma^3 h^4 |\phi|_{H^3(\Omega)}$ . The proof of (34) is similar to that of Eqs (31)-(32), using (33) and noticing that  $\forall K \in \mathcal{T}_h$ ,  $\|\nabla(\phi - \pi_1 \phi)\|_K \lesssim \sigma_K h_K |\phi|_{H^2(K)}$ .  $\square$

## 4 Numerical Results

To get  $p_h \in L^2_{zmv}(\Omega)$ , we eliminate a degree of freedom in  $\mathcal{P}^0_{disc}(\mathcal{T}_h)$  and in  $\mathcal{P}^1(\mathcal{T}_h)$  and post-process by subtracting the mean-value. Let us set:  $\|(\mathbf{u}, p)\|_\nu^2 := \|\nabla \mathbf{u}\|_\Omega^2 + \nu^{-2} \|p\|_\Omega^2$ . Let  $\delta_h \mathbf{u} = \Pi_h \mathbf{u} - \mathbf{u}_h$ ,  $\delta p = p - p_h$  and  $\pi^1_{disc}$  be the  $L^2$ -orthogonal projection onto  $\mathcal{P}^1_{disc}(\mathcal{T}_h)$ . We compute the discrete errors values:  $\varepsilon_0^\nu(\mathbf{u}_h) := \|\delta_h \mathbf{u}\|_\Omega / \|(\mathbf{u}, p)\|_\nu$ ,  $\varepsilon_1^\nu(\mathbf{u}_h) := \|\delta_h \mathbf{u}\|_h / \|(\mathbf{u}, p)\|_\nu$  and  $\varepsilon_0^\nu(p_h) := \nu^{-1} (\|\pi_0(\delta p)\|_\Omega^2 + \|\pi^1_{disc}(\delta p) - \pi_0(\delta p)\|_\Omega^2)^{\frac{1}{2}} / \|(\mathbf{u}, p)\|_\nu$ . Let  $\tau_{0,\mathbf{u}}$ ,  $\tau_{1,\mathbf{u}}$  and  $\tau_p$  be the averaged convergence rates of  $\varepsilon_0^\nu(\mathbf{u})$ ,  $\varepsilon_1^\nu(\mathbf{u})$  and  $\varepsilon_0^\nu(p)$ .

We first consider a 2D case illustrating Thm. 3, with the prescribed solution  $\mathbf{u} = 0$ ,  $\mathbf{f} = \nabla \psi$  with  $\psi(x, y) = \exp(-10(1 - x + 2y))$ . Fig. 1 shows  $\varepsilon_0^\nu(\mathbf{u}_h)$ ,  $\varepsilon_1^\nu(\mathbf{u}_h)$  and  $\varepsilon_0^\nu(p_h)$  against  $h$ , for  $\nu = 10^{-2}$ , with  $P^0$  (left) or  $P^0 + P^1$  (right) discrete pressures, comparing the  $\mathbf{P}_{nc}^1 \times P^0$  [1, Example 4] and  $\mathbf{P}_{nc}^1 \times (P^0 + P^1)$  schemes. Tab. 1 reports observed convergence rates, which are better for the  $\mathbf{P}_{nc}^1 \times (P^0 + P^1)$  scheme.

**Fig. 1** 2D case. Plots of  $\varepsilon_0^v(\mathbf{u}_h)$ ,  $\varepsilon_1^v(\mathbf{u}_h)$  and  $\varepsilon_0^v(p_h)$  for  $\nu = 10^{-2}$ .**Table 1** 2D case, averaged convergence rates on the last five meshes.

	$P^0$	$P^0 + P^1$		$P^0$	$P^0 + P^1$		$P^0$	$P^0 + P^1$		
$\tau_{0,\mathbf{u}}$	2.02	4.03		$\tau_{1,\mathbf{u}}$	1.01	3.07		$\tau_p$	1.60	2.10

Let us consider stationary Navier-Stokes equations set in  $\Omega = (0, 1)^3$ : Find  $(\mathbf{u}, p)$  such that  $-\nu \Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mathbf{f}$  and  $\text{div } \mathbf{u} = 0$  in  $\Omega$ , with inhomogeneous Dirichlet boundary conditions. Computations are done with the TrioCFD code [6, 7] with the  $\mathbf{P}_{nc}^1 \times (P^0 + P^1)$  scheme. The convection term is discretized using the MUSCL scheme [6]. We study the prescribed solution  $\mathbf{u} = (y - z, z - x, x - y)^T$ ,  $p = \frac{1}{2}(x^2 + y^2 + z^2) - xy - xz - yz - 1/4$  so that  $\mathbf{f} = 0$ . Fig. 2 shows  $\varepsilon_0^v(\mathbf{u}_h)$  (left) and  $\varepsilon_0^v(p_h)$  (right) against  $h$ , for  $\nu = 10^{-1}$ ;  $10^{-2}$ ;  $10^{-3}$ .

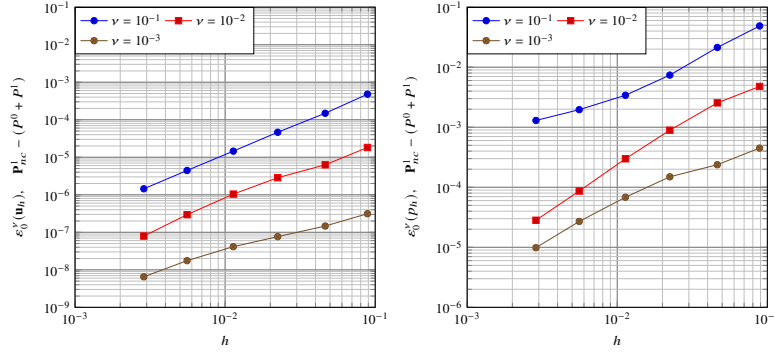
**Fig. 2** 3D case. Plots of  $\varepsilon_0^v(\mathbf{u}_h)$  and  $\varepsilon_0^v(p_h)$  for  $\nu = 10^{-1}$ ,  $\nu = 10^{-2}$  or  $\nu = 10^{-3}$ .

Table 2 reports the convergence rates  $\tau_{0,\mathbf{u}}$  (left), and  $\tau_p$  (right). More numerical results are available in [8, 9, 6, 23]. The  $\mathbf{P}_{nc}^1 \times (P^0 + P^1)$  scheme shows good approximation properties and should therefore be suitable for solving the Navier-Stokes

**Table 2** 3D case, averaged convergence rates on the last three meshes.

$\nu$	$10^{-1}$	$10^{-2}$	$10^{-3}$	$\nu$	$10^{-1}$	$10^{-2}$	$10^{-3}$
$\tau_{0,\mathbf{u}}$	1.68	1.88	1.41	$\tau_p$	0.70	1.72	1.35

equations in the context of industrial simulation. More generally, it seems interesting to design and study discrete pressure enrichment for higher order nonconforming Crouzeix-Raviart mixed FEM [2, 3].

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