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► **To cite this version:**

Erell Jamelot, Patrick Ciarlet, Stefan Sauter. Stability of the $\mathbf{P}_{nc^1 - (P^0 + P^1)}$ element. ENUMATH 2023 - The European Conference on Numerical Mathematics and Advanced Applications, Sep 2023, Lisbonne, Portugal. hal-04414894v3

HAL Id: hal-04414894

<https://hal.science/hal-04414894v3>

Submitted on 3 Sep 2024 (v3), last revised 15 Oct 2024 (v4)

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Stability Of The $\mathbf{P}_{nc}^1 - (P^0 + P^1)$ Element

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Abstract We solve the Stokes problem numerically. We analyse the $\mathbf{P}_{nc}^1 - (P^0 + P^1)$ mixed finite element method which exhibits interesting numerical features. However, only an incomplete proof of the inf-sup condition is available. We prove here this condition and the stability of the method.

1 Stokes Problem

A popular low-order method for solving the Stokes problem is the nonconforming Crouzeix-Raviart mixed finite element method [1, Example 4] (cf. [2, 3] for higher orders). The discrete velocity is piecewise affine, continuous in the barycentre of the interfaces between elements, but not globally continuous, \mathbf{P}_{nc}^1 ; and the discrete pressure is piecewise constant, P^0 . This method introduces some consistency error on the discrete velocity, which may lead to an unphysical solution when solving the Navier-Stokes equations, especially when the source field is a strong gradient [4, 5]. To recover consistency, the discrete pressure space can be generated by P^1 -continuous plus P^0 basis functions, cf. the TrioCFD code [6, 7]. This leads to the $\mathbf{P}_{nc}^1 - (P^0 + P^1)$ mixed finite element method, which exhibits interesting numerical features. However, only an incomplete proof of the stability is available in [8, 9]. Below, we prove the inf-sup condition. Consider the Stokes problem set in an open,

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bounded, connected subset Ω of \mathbb{R}^d , $d = 2, 3$, with a Lipschitz boundary $\partial\Omega$:

$$\text{Find } (\mathbf{u}, p) \text{ such that } -\nu \Delta \mathbf{u} + \nabla p = \mathbf{f} \text{ and } \operatorname{div} \mathbf{u} = 0 \text{ in } \Omega, \quad (1)$$

with homogeneous boundary conditions $\mathbf{u} = 0$ on $\partial\Omega$ and a normalisation condition: $\int_{\Omega} p = 0$. The case of inhomogeneous Dirichlet boundary conditions, that is $\mathbf{u} = \mathbf{g}$ on $\partial\Omega$ with $\mathbf{g} \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$ such that $\int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n}|_{\partial\Omega} = 0$ can be studied similarly. Given D an open subset of \mathbb{R}^d , let us set $\mathbf{L}^2(D) = (L^2(D))^d$, and $L^2_{zmv}(D) = \{q \in L^2(D) \mid \int_D q = 0\}$. We denote by $(\cdot, \cdot)_D$ the $L^2(D)$ or $\mathbf{L}^2(D)$ inner product, and $\|\cdot\|_D$ the associated norm. Due to Poincaré-Steklov inequality [10, Lemma 3.24], the seminorm in $H^1_0(\Omega)$ is equivalent to the natural norm, so that $(v, w)_{H^1_0(\Omega)} = (\nabla v, \nabla w)_{\Omega}$ and $\|v\|_{H^1_0(\Omega)} = \|\nabla v\|_{\Omega}$. Let us set $\mathbf{V} = \{\mathbf{v} \in \mathbf{H}^1_0(\Omega) \mid \operatorname{div} \mathbf{v} = 0\}$ which is a closed subset of $\mathbf{H}^1_0(\Omega)$. We denote by \mathbf{V}^{\perp} the orthogonal of \mathbf{V} in $\mathbf{H}^1_0(\Omega)$.

Proposition 1 [13, Theorem 3.1 (e)]. *The operator $\operatorname{div} : \mathbf{H}^1_0(\Omega) \rightarrow L^2(\Omega)$ is an isomorphism of \mathbf{V}^{\perp} onto $L^2_{zmv}(\Omega)$. We call $C_{\operatorname{div}} > 1$ the constant such that: $\forall p \in L^2_{zmv}(\Omega), \exists! \mathbf{v}_p \in \mathbf{V}^{\perp} \mid \operatorname{div} \mathbf{v}_p = p$ and $\|\mathbf{v}_p\|_{\mathbf{H}^1_0(\Omega)} \leq C_{\operatorname{div}} \|p\|_{\Omega}$.*

Let $\mathbf{f} \in (\mathbf{H}^1_0(\Omega))'$. The variational formulation of Problem (1) reads:
Find $(\mathbf{u}, p) \in \mathbf{H}^1_0(\Omega) \times L^2_{zmv}(\Omega)$ such that $\forall (\mathbf{v}, q) \in \mathbf{H}^1_0(\Omega) \times L^2_{zmv}(\Omega)$:

$$\nu(\mathbf{u}, \mathbf{v})_{\mathbf{H}^1_0(\Omega)} - (p, \operatorname{div} \mathbf{v})_{\Omega} - (q, \operatorname{div} \mathbf{u})_{\Omega} = \langle \mathbf{f}, \mathbf{v} \rangle_{\mathbf{H}^1_0(\Omega)}. \quad (2)$$

One can prove that Problem (2) is well-posed with the T-coercivity theory [14, 15, 16], using Poincaré-Steklov inequality [10, Lemma 3.24] and Proposition 1.

2 Discretization And Stability Of The $\mathbf{P}^1_{nc} - (\mathbf{P}^0 + \mathbf{P}^1)$ Scheme

Consider $(\mathcal{T}_h)_h$ a family of simplicial triangulations of Ω . For a given $\mathcal{T}_h := \{K\}$, we call $\mathcal{F}_h := \{F\}$ its set of $(d-1)$ -simplices, called facets (faces if $d = 3$, edges if $d = 2$), and $\mathcal{V}_h := \{S\}$ its set of vertices. In the bounds, we will use \lesssim and \gtrsim for (hidden) constants independent of \mathcal{T}_h .

- For $K \in \mathcal{T}_h$, we call h_K and ρ_K the diameters of K and its inscribed sphere respectively, and we let: $\sigma_K = \frac{h_K}{\rho_K}$. When the $(\mathcal{T}_h)_h$ is a shape-regular triangulation family (see e.g. [10, def. 11.2]), there exists a constant $\sigma > 1$, called the shape regularity parameter, such that for all h , for all $K \in \mathcal{T}_h$, $\sigma_K \leq \sigma$.
- Let $\mathcal{F}_h = \mathcal{F}_h^i \cup \mathcal{F}_h^b$, where $\mathcal{F}_h^b = \{F \in \mathcal{F}_h \mid F \subset \partial\Omega\}$, $\mathcal{F}_h^i = \mathcal{F}_h \setminus \mathcal{F}_h^b$. For $K \in \mathcal{T}_h$, we set $\mathcal{F}_K^i = \{F \in \mathcal{F}_h^i \mid F \subset \partial K\}$.

For $F \in \mathcal{F}_h$, M_F denotes the barycentre of F , and \mathbf{n}_F its unit normal (outward oriented if $F \subset \partial\Omega$). For $K \in \mathcal{T}_h$ and $F \subset \partial K$, we denote by $\mathbf{n}_{F,K}$ the normal vector of F outgoing from K . Let \hat{K} be the reference simplex. For $K \in \mathcal{T}_h$, we denote by $T_K : \hat{K} \rightarrow K$ the geometric mapping such that $\forall \hat{\mathbf{x}} \in \hat{K}$, $\mathbf{x}|_K = T_K(\hat{\mathbf{x}}) = \mathbb{B}_K \hat{\mathbf{x}} + \mathbf{b}_K$,

where $\mathbb{B}_K \in \mathbb{R}^{d \times d}$ and $\mathbf{b}_K \in \mathbb{R}^d$. Let us set $J_K = \det(\mathbb{B}_K)$. Recall that [10, Lemma 11.1]: $|J_K| = |K|/|\hat{K}|$, $\|\mathbb{B}_K\| \leq h_K/\rho_{\hat{K}}$, $\|\mathbb{B}_K^{-1}\| \leq h_{\hat{K}}/\rho_K$. For $v \in L^2(\Omega)$, we set: $v_K = v|_K$, $\hat{v}_K = v \circ T_K$. By changing the variable, we get: $\|v\|_K^2 = |J_K| \|\hat{v}_K\|_{\hat{K}}^2$. Let $v \in H^1(K)$, then $\nabla v|_K = (\mathbb{B}_K^{-1})^T \nabla_{\hat{x}} \hat{v}$, and $\|\nabla v\|_K^2 \leq \|\mathbb{B}_K^{-1}\|^2 |J_K| \|\nabla_{\hat{x}} \hat{v}\|_{\hat{K}}^2$, $\|\nabla_{\hat{x}} \hat{v}\|_{\hat{K}}^2 \leq \|\mathbb{B}_K\|^2 |J_K|^{-1} \|\nabla v\|_K^2$. Let $K \in \mathcal{T}_h$ and $F \subset \partial K$, we denote by $T_{F,K}$ the geometric mapping such that $\forall \mathbf{x} \in F$, $T_{F,K}(T_K^{-1}(\mathbf{x})) = \mathbf{x}$. We let $\hat{F}_{F,K} = T_K^{-1}(F)$. According to [1, Eq. (3.17)]:

$$\forall K \in \mathcal{T}_h, \forall F \subset \partial K, \quad |F| |K|^{-1} \lesssim \rho_K^{-1}. \quad (3)$$

We set $\mathcal{P}_h H^1 = \{v \in L^2(\Omega) \mid \forall K \in \mathcal{T}_h, v|_K \in H^1(K)\}$, endowed with: $(v, w)_h := \sum_{K \in \mathcal{T}_h} (\nabla v, \nabla w)_K$, $\|v\|_h^2 = \sum_{K \in \mathcal{T}_h} \|\nabla v\|_K^2$. Let $F \in \mathcal{F}_h^i$ such that $F = \partial K_L \cap \partial K_R$ and $\mathbf{n}_F = \mathbf{n}_{F,K_L}$. The jump of $v \in \mathcal{P}_h H^1$ across F is defined by: $[v]_F := v|_{K_L} - v|_{K_R}$. For $F \in \mathcal{F}_h^b$, we set: $[v]_F := v|_F$.

For a subset D of \mathbb{R}^d , for $k \in \mathbb{N}$, we call $P^k(D)$ the set of degree k polynomials on D , $\mathbf{P}^k(D) = (P^k(D))^d$, and $\mathcal{P}_{disc}^k(\mathcal{T}_h) = \{q \in L^2(\Omega) \mid \forall K \in \mathcal{T}_h, q|_K \in P^k(K)\}$. For $k > 0$ we define $\mathcal{P}^k(\mathcal{T}_h) = C^0(\overline{\Omega}) \cap \mathcal{P}_{disc}^k(\mathcal{T}_h)$. We call $\pi_0 : L^2(\Omega) \mapsto \mathcal{P}_{disc}^0(\mathcal{T}_h)$ the L^2 -orthogonal projection onto $\mathcal{P}_{disc}^0(\mathcal{T}_h)$, such that $\forall q \in L^2(\Omega)$, $\pi_0(q) = \sum_{K \in \mathcal{T}_h} q_K \mathbb{1}_K$ where $\forall K \in \mathcal{T}_h$, $q_K = \int_K q / |K|$.

For $K \in \mathcal{T}_h$ and a vertex S of K , $\lambda_{S,K}$ denotes the barycentric coordinate related to S . We can endow $\mathcal{P}^1(\mathcal{T}_h)$ with the basis $(\phi_S)_{S \in \mathcal{V}_h}$ such that: $\forall K \in \mathcal{T}_h$, $\phi_S|_K = \lambda_{S,K}$ if $S \in K$, zero otherwise, so that $\mathcal{P}^1(\mathcal{T}_h) = \text{vect}((\phi_S)_{S \in \mathcal{V}_h})$.

Using [12, Thm. 5] and Eq. (3), one can prove that:

$$\forall w \in P^k(K), \forall F \subset \partial K, \quad \|w\|_F \lesssim (|F| |K|^{-1})^{\frac{1}{2}} \|w\|_K \lesssim (\rho_K)^{-\frac{1}{2}} \|w\|_K. \quad (4)$$

Using Eq. (4) with $w = v - \underline{v}_K$ and Poincaré-Steklov inequality [10, Lemma 12.11] in K , it holds (with below, $\underline{v}_F = \int_F v / |F|$):

$$\forall v \in P^k(K), \forall F \subset \partial K, \quad \|v - \underline{v}_F\|_F \lesssim \|v - \underline{v}_K\|_F \lesssim (\sigma_K h_K)^{1/2} \|\nabla v\|_K. \quad (5)$$

Using the proof of [17, Lemma 2] and [12, Thm. 5], we can prove that:

$$\forall v \in P^1(K), \quad \|\nabla v\|_K \lesssim (\rho_K)^{-1} \|v - \underline{v}_K\|_K. \quad (6)$$

Let $\mathbf{X}_{0,h}$ be the space of the discrete velocity of the nonconforming Crouzeix-Raviart mixed finite element method, defined by:

$$\mathbf{X}_h = (X_h)^d, \quad X_h = \{v_h \in \mathcal{P}_{disc}^1(\mathcal{T}_h) \mid \forall F \in \mathcal{F}_h^i, \int_F [v_h] = 0\}; \quad (7)$$

$$\mathbf{X}_{0,h} = (X_{0,h})^d, \quad X_{0,h} = \{v_h \in X_h \mid \forall F \in \mathcal{F}_h^b, \int_F v_h = 0\}. \quad (8)$$

The condition on the jumps of v_h on the inner facets is often called the patch-test condition. Due to this condition, one can prove that :

Proposition 2 [1, Lemma 2]. *The broken norm $v_h \rightarrow \|v_h\|_h$ is a norm over $X_{0,h}$.*

For $K \in \mathcal{T}_h$ and $F \subset \partial K$, $\lambda_{F,K}$ denotes the barycentric coordinate related to the vertex $S_{F,K}$ opposite to facet F in K . We can endow X_h with the basis $(\psi_F)_{F \in \mathcal{F}_h}$ such that for all $K \in \mathcal{T}_h$, $\psi_F|_K = 1 - d\lambda_{F,K}$ if $F \subset \partial K$ and zero otherwise. We then have $\psi_F|_F = 1$, so that $[\psi_F]_F = 0$ if $F \in \mathcal{F}_h^i$, and for all $F' \neq F$, $\int_{F'} \psi_F = 0$. We have: $X_h = \text{vect}((\psi_F)_{F \in \mathcal{F}_h})$ and $X_{0,h} = \text{vect}((\psi_F)_{F \in \mathcal{F}_h^i})$.

Let $\Pi_h : \mathbf{H}^1(\Omega) \mapsto \mathbf{X}_h$ be the Crouzeix-Raviart interpolation operator, such that for all $\mathbf{v} \in \mathbf{H}^1(\Omega)$, $\Pi_h(\mathbf{v}) = \sum_{F \in \mathcal{F}_h} \underline{\mathbf{v}}_F \psi_F$, $\underline{\mathbf{v}}_F = \int_F \mathbf{v}/|F|$.

Proposition 3 [17, Lemma 2]. *The interpolation operator Π_h is such that for all $\mathbf{v}_h \in \mathbf{H}_0^1(\Omega)$, $\|\Pi_h(\mathbf{v}_h)\|_h \leq \|\mathbf{v}_h\|_{\mathbf{H}_0^1(\Omega)}$.*

However, due to the consistency error on the discrete velocity, it is well-known [18, 19] that choosing the space of the discrete pressures equal to $\mathcal{P}_{disc}^0(\mathcal{T}_h) \cap L_{zmv}^2(\Omega)$ yields inaccurate results when solving Navier-Stokes equations. Starting from [20], a new element is proposed in [8]. The space of the discrete pressures is then: $Q_h = Q_{0,h} + Q_{1,h}$, where $Q_{0,h} := \mathcal{P}_{disc}^0(\mathcal{T}_h) \cap L_{zmv}^2$ and for $k > 0$, $Q_{k,h} := \mathcal{P}^k(\mathcal{T}_h) \cap L_{zmv}^2(\Omega)$. One can build an L^2 -orthogonal sum letting $\tilde{Q}_{1,h} = \{q_h \in Q_h \mid \forall K \in \mathcal{T}_h, \int_K q_h = 0\}$.

Proposition 4 *We have: $Q_h = Q_{0,h} \overset{\perp}{\oplus} \tilde{Q}_{1,h}$. It holds for all $q_h \in Q_h$: $\|q_h\|_{\Omega}^2 = \|\pi_0(q_h)\|_{\Omega}^2 + \|q_h - \pi_0(q_h)\|_{\Omega}^2$.*

To prove the discrete inf-sup condition of $\mathbf{X}_{0,h} \times Q_h$, we need some Lemmas.

Lemma 1 *Let $k > 0$, $(\mathbf{v}, q) \in \mathbf{X}_{0,h} \times Q_{k,h}$. It holds:*

$$|\sum_{K \in \mathcal{T}_h} \sum_{F \subset \partial K} (\mathbf{v} \cdot \mathbf{n}_{F,K}, q)_F| \lesssim \sigma \|\mathbf{v}\|_h \|q - \pi_0(q)\|_{\Omega}. \quad (9)$$

Proof. We start with the proof of [20, Lemma 3.1]. Using the patch-test condition, one obtains:

$$\sum_{K \in \mathcal{T}_h} \sum_{F \subset \partial K} (\mathbf{v} \cdot \mathbf{n}_{F,K}, q)_F = \sum_{K \in \mathcal{T}_h} \sum_{F \subset \partial K} \left((\mathbf{v}_K - \underline{\mathbf{v}}_F) \cdot \mathbf{n}_{F,K}, q_K - \underline{q}_K \right)_F.$$

We have, using Eq. (4) with $w = q_K - \underline{q}_K$ and Eq. (5) for the components of \mathbf{v} :

$$\left| \left((\mathbf{v}_K - \underline{\mathbf{v}}_F) \cdot \mathbf{n}_{F,K}, q_K - \underline{q}_K \right)_F \right| \lesssim \sigma_K \|\underline{\nabla} \mathbf{v}_K\|_K \|q_K - \underline{q}_K\|_K.$$

Summing and using the discrete Cauchy-Schwarz inequality, we obtain (9). \square

Introducing the element-wise divergence operator div_h , we prove next that one can build an explicit right inverse of div_h from Q_h to $\mathbf{X}_{0,h}$.

Lemma 2 *For all $q_h \in Q_h$, there exists $\mathbf{v}_{0,h} \in \mathbf{X}_{0,h}$ such that:*

$$-(\text{div}_h \mathbf{v}_{0,h}, \pi_0(q_h))_{\Omega} = \nu^{-1} \|\pi_0(q_h)\|_{\Omega}^2, \quad (10)$$

$$\|\mathbf{v}_{0,h}\|_h \leq C_{\text{div}} \nu^{-1} \|\pi_0(q_h)\|_{\Omega}. \quad (11)$$

Proof. Let $q_h \in Q_h$. From Proposition 1, $\exists \mathbf{v}_{\pi_0(q_h)} \in \mathbf{H}_0^1(\Omega) \mid \operatorname{div} \mathbf{v}_{\pi_0(q_h)} = \pi_0(q_h)$ and $\|\mathbf{v}_{\pi_0(q_h)}\|_{\mathbf{H}_0^1(\Omega)} \leq C_{\operatorname{div}} \|\pi_0(q_h)\|_{\Omega}$. Let $\mathbf{v}_{0,h} = -\nu^{-1} \Pi_h(\mathbf{v}_{\pi_0(q_h)})$. Integrating by part twice, we get: $\forall K \in \mathcal{T}_h$, $\int_K \operatorname{div} \mathbf{v}_{0,h} = -\nu^{-1} \int_K \operatorname{div} \mathbf{v}_{\pi_0(q_h)}$, hence (10). Using Proposition 3, we get (11). \square

We can also build an explicit right inverse of the discrete divergence operator from $Q_{1,h}$ to $\mathbf{X}_{T,h} = \{\mathbf{v}_h \in \mathbf{X}_{0,h} \mid \forall F \in \mathcal{F}_h^i, \mathbf{v}_h \cdot \mathbf{n}_F = 0\}$, if [20, Hyp. 4.1] holds:

Hypothesis 1. We suppose that \mathcal{T}_h is such that each element $K \in \mathcal{T}_h$ has at most $d - 1$ facets that lie on $\partial\Omega$.

Lemma 3 Assuming Hypothesis 1, $\forall q_{1,h} \in Q_{1,h}$, $\exists \mathbf{v}_{1,h} \in \mathbf{X}_{T,h}$ such that:

$$(\nabla q_{1,h}, \mathbf{v}_{1,h})_{\Omega} \gtrsim \nu^{-1} \sigma^{-2(d-2)} \|q_{1,h} - \pi_0(q_{1,h})\|_{\Omega}^2, \quad (12)$$

$$(\operatorname{div} \mathbf{v}_{1,h}, q_{0,h})_K = 0 \quad \forall q_{0,h} \in Q_{0,h}, \quad (13)$$

$$\|\mathbf{v}_{1,h}\|_h \lesssim \nu^{-1} \sigma^2 \|q_{1,h} - \pi_0(q_{1,h})\|_{\Omega}. \quad (14)$$

Proof. Let us consider $q_{1,h} \in Q_{1,h}$. For all $K \in \mathcal{T}_h$, we let $q_K := q_{1,h}|_K$ and $\hat{q}_K = q_K \circ T_K$. Since $q_{1,h} \in H^1(\Omega)$ we have (see e.g. [21, Prop. 2.2.10]): $\forall F \in \mathcal{F}_h^i \mid F = K \cap K'$, $\nabla q_K \times \mathbf{n}_F = \nabla q_{K'} \times \mathbf{n}_F$. We construct $\mathbf{v}_{1,h} \in \mathbf{X}_{T,h} \mid \forall F \in \mathcal{F}_h^i$, $\mathbf{v}_{1,h}(M_F) = \nu^{-1} h_F^2 \mathbf{n}_F \times (\nabla q_{1,h} \times \mathbf{n}_F)$, where h_F is the diameter of F . We have:

$$\forall K \in \mathcal{T}_h \quad (\nabla q_{1,h}, \mathbf{v}_{1,h})_K = \nu^{-1} (d+1)^{-1} |K| \sum_{F \in \mathcal{F}_K} h_F^2 |\nabla q_K \times \mathbf{n}_F|^2. \quad (15)$$

Let $\mathbf{n}_{\hat{F}_{F,K}}$ be the unit normal vector of the facet $\hat{F}_{F,K}$ outgoing from \hat{K} . We have: $\mathbf{n}_{F,K} = |K| (\mathbb{B}_K^{-1})^T |\hat{F}_{F,K}| |F|^{-1} \mathbf{n}_{\hat{F}_{F,K}}$.

Consider $d = 2$. Changing the variable, assuming Hypothesis 1, we get:

$$\sum_{F \in \mathcal{F}_K} h_F^2 |\nabla q_K \times \mathbf{n}_F|^2 = \sum_{F \in \mathcal{F}_K} |\hat{F}_{F,K}|^2 |\nabla_{\hat{\mathbf{x}}} \hat{q}_K \times \mathbf{n}_{\hat{F}_{F,K}}|^2 \gtrsim |\nabla_{\hat{\mathbf{x}}} \hat{q}_K|^2. \quad (16)$$

Consider $d = 3$. Changing the variable, one now finds that $|\nabla q_K \times \mathbf{n}_F| = |\hat{F}_{F,K}| |F|^{-1} |\mathbb{B}_K \nabla_{\hat{\mathbf{x}}} \hat{q}_K \times \mathbf{n}_{\hat{F}_{F,K}}|$. Hence: $|\nabla q_K \times \mathbf{n}_F| \gtrsim |F|^{-1} \|\mathbb{B}_K^{-1}\|^{-1} |\nabla_{\hat{\mathbf{x}}} \hat{q}_K \times \mathbf{n}_{\hat{F}_{F,K}}|$. Notice that $h_F |F|^{-1} \|\mathbb{B}_K^{-1}\|^{-1} \gtrsim \sigma_K^{-1}$. Assuming Hypothesis 1, we then have:

$$\sum_{F \in \mathcal{F}_K} h_F^2 |\nabla q_K \times \mathbf{n}_F|^2 \gtrsim \sigma_K^{-2} |\nabla_{\hat{\mathbf{x}}} \hat{q}_K|^2. \quad (17)$$

Using Poincaré-Steklov inequality [10, Lemma 12.11] in \hat{K} , we have: $|K| |\nabla_{\hat{\mathbf{x}}} \hat{q}_K|^2 \gtrsim \|\hat{q}_K - \underline{q}_K\|_{\hat{K}}^2$ with $\underline{q}_K = \int_K q_K / |K|$. Using (16) and (17) in (15), it holds:

$$(\nabla q_{1,h}, \mathbf{v}_{1,h})_K \gtrsim \nu^{-1} \sigma_K^{-2(d-2)} |K| |\nabla_{\hat{\mathbf{x}}} \hat{q}_K|^2 \gtrsim \nu^{-1} \sigma_K^{-2(d-2)} \|q_K - \underline{q}_K\|_{\hat{K}}^2. \quad (18)$$

We obtain (12) by summation. Remark that $\nabla \psi_{F|K} = |K|^{-1} |F| \mathbf{n}_{F,K}$.

Let us prove (13). Recall the orthogonality $\nabla \psi_{F|K}(M_F) \perp \mathbf{v}_{1,h}(M_F)$, so that $(\operatorname{div} \mathbf{v}_{1,h}, q_{0,h})_K = |K| q_{0,h}(K) \sum_{F \in \mathcal{F}_K^i} \nabla \psi_{F|K}(M_F) \cdot \mathbf{v}_{1,h}(M_F) = 0$.

Let us prove (14). For $K \in \mathcal{T}_h$, $\underline{\nabla} \mathbf{v}_{1,h}|_K = \sum_{F \in \mathcal{F}_K^i} \mathbf{v}_{1,h}(M_F) \otimes \nabla \psi_{F|K}$.

Hence: $\|\underline{\nabla} \mathbf{v}_{1,h}\|_K^2 \lesssim \nu^{-2} \sum_{F \in \mathcal{F}_K^i} |F|^2 |K|^{-1} h_F^4 |\nabla q_K \times \mathbf{n}_F|^2$. Using (3), and assuming Hypothesis 1, we get: $\|\underline{\nabla} \mathbf{v}_{1,h}\|_K^2 \lesssim \nu^{-2} \sigma_K^2 h_K^2 |K| \sum_{F \in \mathcal{F}_K^i} |\nabla q_K \times \mathbf{n}_F|^2 \lesssim \nu^{-2} \sigma_K^2 h_K^2 \|\nabla q_K\|_K^2$. From Eq. (6): $\|\underline{\nabla} \mathbf{v}_{1,h}\|_K \lesssim \nu^{-1} \sigma_K^2 \|q_K - \underline{q}_K\|_K$. \square

Let $b_h : \mathbf{X}_h \times Q_h \rightarrow \mathbb{R}$ be defined by $b_h(\mathbf{v}_h, q_h) = -(\operatorname{div}_h \mathbf{v}_h, q_{0,h})_\Omega + (\mathbf{v}_h, \nabla q_{1,h})_\Omega$ with $q_h = q_{0,h} + q_{1,h}$ such that $(q_{0,h}, q_{1,h}) \in Q_{0,h} \times Q_{1,h}$.

Theorem 1 (Stability) *The following continuity property holds:*

$$\forall (\mathbf{v}_h, q_h) \in \mathbf{X}_{0,h} \times Q_h, \quad |b_h(\mathbf{v}_h, q_h)| \lesssim \sigma \|\mathbf{v}_h\|_h \|q_h\|_\Omega. \quad (19)$$

Assuming Hypothesis 1, the following inf-sup condition holds:

$$\forall q_h \in Q_h, \exists \mathbf{v}_h \in \mathbf{X}_{0,h}, \quad b_h(\mathbf{v}_h, q_h) \gtrsim \frac{1}{\sqrt{2}} C_{\operatorname{div}}^{-2} \sigma^{-2d} \|\mathbf{v}_h\|_h \|q_h\|_\Omega. \quad (20)$$

Proof. Let us prove (19). Let $(\mathbf{v}_h, q_h) \in \mathbf{X}_{0,h} \times Q_h$. Integrating by parts, we have:

$$b_h(\mathbf{v}_h, q_h) = -(\operatorname{div}_h \mathbf{v}_h, \pi_0(q_h))_\Omega + \sum_{K \in \mathcal{T}_h} \sum_{F \subset \partial K} (\mathbf{v}_h \cdot \mathbf{n}_{F,K}, q_{1,h})_F. \quad (21)$$

Using Cauchy-Schwarz, we have: $|(\operatorname{div}_h \mathbf{v}_h, \pi_0(q_h))_\Omega| \leq \sqrt{d} \|\mathbf{v}_h\|_h \|\pi_0(q_h)\|_\Omega$. Using (9) in (21), we obtain (19) from Proposition 4. Let us prove (20), starting from the proof of [20, Lemma 4.2]. Let $q_h \in Q_h \setminus \{0\}$, where $q_h = q_{0,h} + q_{1,h}$ is such that $(q_{0,h}, q_{1,h}) \in Q_{0,h} \times Q_{1,h}$. Let $\mathbf{v}_h := \mathbf{v}_{0,h}$ be like in Lemma 2. Using (21) with $\mathbf{v}_h = \mathbf{v}_{0,h}$, (9) and (10), letting $\tilde{q}_{1,h} = q_h - \pi_0(q_h)$, we have:

$$b_h(\mathbf{v}_{0,h}, q_h) \gtrsim \nu^{-1} \|\pi_0(q_h)\|_\Omega^2 - \sigma \|\mathbf{v}_{0,h}\|_h \|\tilde{q}_{1,h}\|_\Omega. \quad (22)$$

Using (11) and Young inequality, we have for all $\varepsilon > 0$:

$$-\|\mathbf{v}_{0,h}\|_h \|\tilde{q}_{1,h}\|_\Omega \geq -\frac{1}{2} C_{\operatorname{div}} \nu^{-1} (\varepsilon \|\pi_0(q_h)\|_\Omega^2 + \varepsilon^{-1} \|\tilde{q}_{1,h}\|_\Omega^2). \quad (23)$$

We now insert the bound (23) in (22) to get:

$$b_h(\mathbf{v}_{0,h}, q_h) \gtrsim \nu^{-1} \left(\left(1 - \frac{\varepsilon}{2} C_{\operatorname{div}} \sigma\right) \|\pi_0(q_h)\|_\Omega^2 - \frac{C_{\operatorname{div}}}{2\varepsilon} \sigma \|\tilde{q}_{1,h}\|_\Omega^2 \right). \quad (24)$$

Let $\mathbf{v}_{1,h}$ be like in Lemma 3. Using (12) and (13), we get:

$$b_h(\mathbf{v}_{1,h}, q_h) = (\nabla q_{1,h}, \mathbf{v}_{1,h})_\Omega \gtrsim \nu^{-1} \sigma^{-2(d-2)} \|\tilde{q}_{1,h}\|_\Omega^2 \quad (25)$$

Last, we set $\mathbf{v}_h^* = \mu \mathbf{v}_{0,h} + \mathbf{v}_{1,h}$, where $\mu > 0$. Using (24) and (25), we have:

$$\begin{aligned} b_h(\mathbf{v}_h^*, q_h) &= \mu b_h(\mathbf{v}_{0,h}, q_h) + b_h(\mathbf{v}_{1,h}, q_h), \\ &\gtrsim \nu^{-1} \left(\mu \left(1 - \frac{\varepsilon}{2} C_{\operatorname{div}} \sigma\right) \|\pi_0(q_h)\|_\Omega^2 + \left(\sigma^{-2(d-2)} - \frac{\mu C_{\operatorname{div}}}{2\varepsilon} \sigma\right) \|\tilde{q}_{1,h}\|_\Omega^2 \right). \end{aligned}$$

Let us choose $\varepsilon = (C_{\operatorname{div}} \sigma)^{-1} < 1$ and $\mu = \sigma^{-2(d-2)} \varepsilon^2$. We obtain that:

$$b_h(\mathbf{v}_h^*, q_h) \gtrsim C_{\min} \nu^{-1} \|q_h\|_{\Omega}^2 \text{ with } C_{\min} = \varepsilon^2 \sigma^{-2(d-2)} = C_{\text{div}}^{-2} \sigma^{-2(d-1)}. \quad (26)$$

Let us bound $\|\mathbf{v}_h^*\|_h$ by $\|q_h\|_{\Omega}$. We have: $\|\mathbf{v}_h^*\|_h^2 \lesssim (\mu^2 \|\mathbf{v}_{0,h}\|_h^2 + \|\mathbf{v}_{1,h}\|_h^2)$. Using (11) and (14) $\|\mathbf{v}_h^*\|_h^2 \lesssim \nu^{-2} (\mu^2 C_{\text{div}}^2 \|\pi_0(q_h)\|_{\Omega}^2 + \sigma^4 \|\tilde{q}_{1,h}\|_{\Omega}^2)$, hence:

$$\|\mathbf{v}_h^*\|_h^2 \leq C_{\max}^2 \nu^{-2} \|q_h\|_{\Omega}^2 \text{ with } C_{\max} = \sqrt{2} \sigma^2. \quad (27)$$

Using (27) in (26), we get (20) since $C_{\min} C_{\max}^{-1} = \frac{1}{\sqrt{2}} C_{\text{div}}^{-2} \sigma^{-2d}$. \square

Let $\mathbf{V}_h = \{\mathbf{v}_h \in \mathbf{X}_{0,h} \mid \forall q_h \in Q_h, b_h(\mathbf{v}_h, q_h) = 0\}$. We know from [22, Lemma 3.1] that $\mathbf{V}_h \subset \{\mathbf{v}_h \in \mathbf{X}_{0,h} \mid \forall K \in \mathcal{T}_h, \text{div } \mathbf{v}_h|_K = 0\}$. Using $P^0 + P^1$ discrete pressures improves consistency. Indeed, integrating by parts, we have that for all $(\mathbf{v}_h, q_{1,h}) \in \mathbf{V}_h \times Q_{1,h}$, $0 = b_h(\mathbf{v}_h, q_{1,h}) = -(\text{div}_h \mathbf{v}_h, \pi_0(q_{1,h}))_{\Omega} + \sum_{F \in \mathcal{F}_h} \int_F q_{1,h} [\mathbf{v}_h] \cdot \mathbf{n}_F = 0 + \sum_{F \in \mathcal{F}_h} \int_F q_{1,h} [\mathbf{v}_h] \cdot \mathbf{n}_F$. Hence:

Lemma 4 For all $(\mathbf{v}_h, q_{1,h}) \in \mathbf{V}_h \times Q_{1,h}$, it holds: $\sum_{F \in \mathcal{F}_h} \int_F q_{1,h} [\mathbf{v}_h] \cdot \mathbf{n}_F = 0$.

Let $\ell_{\mathbf{f}} \in \mathcal{L}(\mathbf{X}_h, \mathbb{R})$ be such that $\forall \mathbf{v}_h \in \mathbf{X}_h$, $\ell_{\mathbf{f}}(\mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h)_{\Omega}$ if $\mathbf{f} \in \mathbf{L}^2(\Omega)$, $\ell_{\mathbf{f}}(\mathbf{v}_h) = \langle \mathbf{f}, \mathcal{I}_h(\mathbf{v}_h) \rangle_{\mathbf{H}_0^1(\Omega)}$ if $\mathbf{f} \notin \mathbf{L}^2(\Omega)$, where $\mathcal{I}_h : \mathbf{X}_{0,h} \rightarrow \mathbf{Y}_{0,h}$ is for instance an averaging operator [10, §22.4.1], with $\mathbf{Y}_{0,h} = \{\mathbf{v}_h \in \mathbf{H}_0^1(\Omega) \mid \forall K \in \mathcal{T}_h, \mathbf{v}_h|_K \in \mathbf{P}^k(K)\}$. The discretization of Problem (1) with the $\mathbf{P}_{nc}^1 - (P^0 + P^1)$ element reads: Find $(\mathbf{u}_h, p_h) \in \mathbf{X}_{0,h} \times Q_h$ such that $\forall (\mathbf{u}_h, p_h) \in \mathbf{X}_{0,h} \times Q_h$

$$v(\mathbf{u}_h, \mathbf{v}_h)_h + b_h(\mathbf{v}_h, p_h) + b_h(\mathbf{u}_h, q_h) = \ell_{\mathbf{f}}(\mathbf{v}_h). \quad (28)$$

Due to Proposition 2 and Theorem 1, Problem (28) is well posed.

3 Convergence Of The $\mathbf{P}_{nc}^1 - (P^0 + P^1)$ Element

Following the proofs of [1, Theorems 3, 4, 6], we can prove

Theorem 2 (Convergence) Suppose that Hypothesis 1 holds and that the solution of Problem (1) is such that $(\mathbf{u}, p) \in (\mathbf{V} \cap \mathbf{H}^2(\Omega)) \times (H^1(\Omega) \cap L_{zmv}^2(\Omega))$. It holds:

$$\|\mathbf{u} - \mathbf{u}_h\|_h \lesssim \sigma h \nu^{-1} \|\mathbf{f}\|_{\Omega}. \quad (29)$$

$$\text{If } \Omega \text{ is convex } \|\mathbf{u} - \mathbf{u}_h\|_{\Omega} \lesssim \sigma^2 h^2 \nu^{-1} \|\mathbf{f}\|_{\Omega}. \quad (30)$$

$$\|(p - p_h) - \pi_0(p - p_h)\|_{\Omega} \lesssim \nu \sigma^{(2d-1)} h \|\mathbf{f}\|_{\Omega}. \quad (31)$$

$$\|\pi_0(p - p_h)\|_{\Omega} \lesssim \nu C_{\text{div}} \sigma^{(2d+1)} h \|\mathbf{f}\|_{\Omega}. \quad (32)$$

In the case of inhomogeneous Dirichlet boundary conditions, one can also recover (29) and (30), following [1, §7], where a lifting of \mathbf{g} is used.

When $\Omega \subset \mathbb{R}^2$, we have moreover the two Lemmas below.

Lemma 5 For all $(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_{2,h}$, $(\nabla q_h, \mathbf{v}_h)_{\Omega} = 0$.

Proof. Let $(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_{2,h}$. Integrating by parts and using Lemma 4, we get that $\forall q_{1,h} \in Q_{1,h}$: $(\nabla q_h, \mathbf{v}_h)_\Omega = -\sum_{F \in \mathcal{F}_h} \int_F q_h [\mathbf{v}_h] \cdot \mathbf{n}_F = -\sum_{F \in \mathcal{F}_h} \int_F (q_h - q_{1,h}) [\mathbf{v}_h] \cdot \mathbf{n}_F = \sum_{F \in \mathcal{F}_h} \int_F (q_h - q_{1,h}) [\mathbf{v}_h - \mathbf{v}_h(M_F)] \cdot \mathbf{n}_F$. Hence, we obtain: $(\nabla q_h, \mathbf{v}_h)_\Omega = -\sum_{K \in \mathcal{T}_h} \sum_{F \subset \partial K} \int_F (q_h - q_{1,h}) (\mathbf{v}_h - \mathbf{v}_h(M_F)) \cdot \mathbf{n}_{F,K}$. Choose $q_{1,h}$ such that $\forall S \in \mathcal{V}_h$, $q_{1,h}(S) = q_{2,h}(S)$. Then, $\forall F \in \mathcal{F}_h$, the degree 3 polynomial $(q_h - q_{1,h})(\mathbf{v}_h - \mathbf{v}_h(M_F))$ vanishes at the quadrature points of Simpson's rule. Since it is exact for degree 3 polynomials, $\forall F \in \mathcal{F}_h$, $\int_F (q_h - q_{1,h})(\mathbf{v}_h - \mathbf{v}_h(M_F)) \cdot \mathbf{n}_{F,K} = 0$ and $(\nabla q_h, \mathbf{v}_h)_\Omega = 0$. \square

Lemma 6 Let $\phi \in H^3(\Omega) \cap L^2_{zmv}(\Omega)$ such that $|\phi|_{H^3(\Omega)} \neq 0$.

Then, $\forall \mathbf{v}_h \in \mathbf{V}_h$, $(\nabla \phi, \mathbf{v}_h)_\Omega \lesssim \sigma h^2 |\phi|_{H^3(\Omega)} \|\mathbf{v}_h\|_\Omega$.

Proof. Let $\mathbf{v}_h \in \mathbf{V}_h$. Using Lemma 5 and Cauchy-Schwarz, it holds: $|(\nabla \phi, \mathbf{v}_h)_\Omega| = |(\nabla(\phi - q_h), \mathbf{v}_h)_\Omega|$ for all $q_h \in Q_{2,h}$. We then use [10, Lemma 11.9] to conclude. \square

Theorem 3 (Pressure Robustness) Suppose that Ω is convex. Let $p \in H^3(\Omega) \cap L^2_{zmv}(\Omega)$ such that $|p|_{H^3(\Omega)} \neq 0$. Consider Problem (1) with $\mathbf{f} = \nabla p$. Then (\mathbf{u}_h, p_h) , the solution of Problem (28) is such that (where $\delta p = p - p_h$):

$$\nu \|\mathbf{u}_h\|_h \lesssim \sigma^2 h^3 |p|_{H^3(\Omega)}, \quad \nu \|\mathbf{u}_h\|_\Omega \lesssim \sigma^3 h^4 |p|_{H^3(\Omega)}. \quad (33)$$

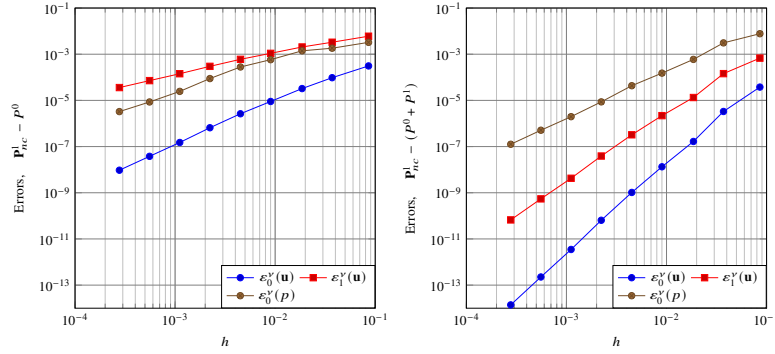
$$\|\delta p - \pi_0(\delta p)\|_\Omega \lesssim \sigma^4 h^2 |p|_{H^2(\Omega)}, \quad \|\pi_0(\delta p)\|_\Omega \lesssim C_{\text{div}} \sigma^6 h^2 |p|_{H^2(\Omega)}. \quad (34)$$

Proof. Setting $\mathbf{v}_h = \mathbf{u}_h$ as test-function in Problem (28) and using Lemma 6, it holds: $\nu \|\mathbf{u}_h\|_h^2 = (\nabla \phi, \mathbf{u}_h)_\Omega \lesssim \sigma h^2 |\phi|_{H^3(\Omega)} \|\mathbf{u}_h\|_\Omega$. From [1, Theorem 4], we have: $\|\mathbf{u}_h\|_\Omega \lesssim \sigma h \|\mathbf{u}_h\|_h$. We deduce that $\nu \|\mathbf{u}_h\|_h \lesssim \sigma^2 h^3 |\phi|_{H^3(\Omega)}$. Using this estimate, we get that $\nu \|\mathbf{u}_h\|_\Omega \lesssim \sigma^3 h^4 |\phi|_{H^3(\Omega)}$. The proof of (34) is similar to that of Eqs (31)-(32), using (33) and noticing that $\forall K \in \mathcal{T}_h$, $\|\nabla(\phi - \pi_1 \phi)\|_K \lesssim \sigma_K h_K |\phi|_{H^2(K)}$. \square

4 Numerical Results

To get $p_h \in L^2_{zmv}(\Omega)$, we eliminate a degree of freedom in $\mathcal{P}^0_{disc}(\mathcal{T}_h)$ and in $\mathcal{P}^1(\mathcal{T}_h)$ and post-process by subtracting the mean-value. Let us set: $\|(\mathbf{u}, p)\|_\nu^2 := \|\nabla \mathbf{u}\|_\Omega^2 + \nu^{-2} \|p\|_\Omega^2$. Let $\delta_h \mathbf{u} = \Pi_h \mathbf{u} - \mathbf{u}_h$, $\delta p = p - p_h$ and π^1_{disc} be the L^2 -orthogonal projection onto $\mathcal{P}^1_{disc}(\mathcal{T}_h)$. We compute the discrete errors values: $\varepsilon_0^\nu(\mathbf{u}_h) := \|\delta_h \mathbf{u}\|_\Omega / \|(\mathbf{u}, p)\|_\nu$, $\varepsilon_1^\nu(\mathbf{u}_h) := \|\delta_h \mathbf{u}\|_h / \|(\mathbf{u}, p)\|_\nu$ and $\varepsilon_0^\nu(p_h) := \nu^{-1} (\|\pi_0(\delta p)\|_\Omega^2 + \|\pi^1_{disc}(\delta p) - \pi_0(\delta p)\|_\Omega^2)^{\frac{1}{2}} / \|(\mathbf{u}, p)\|_\nu$. Let $\tau_{0,\mathbf{u}}$, $\tau_{1,\mathbf{u}}$ and τ_p be the averaged convergence rates of $\varepsilon_0^\nu(\mathbf{u})$, $\varepsilon_1^\nu(\mathbf{u})$ and $\varepsilon_0^\nu(p)$.

We first consider a 2D case illustrating Thm. 3, with the prescribed solution $\mathbf{u} = 0$, $\mathbf{f} = \nabla \psi$ with $\psi(x, y) = \exp(-10(1 - x + 2y))$. Fig. 1 shows $\varepsilon_0^\nu(\mathbf{u}_h)$, $\varepsilon_1^\nu(\mathbf{u}_h)$ and $\varepsilon_0^\nu(p_h)$ against h , for $\nu = 10^{-2}$, with P^0 (left) or $P^0 + P^1$ (right) discrete pressures, comparing the $\mathbf{P}^1_{nc} - P^0$ [1, Example 4] and $\mathbf{P}^1_{nc} - (P^0 + P^1)$ schemes. Tab. 1 reports observed convergence rates, which are better for the $\mathbf{P}^1_{nc} - (P^0 + P^1)$ scheme.

Fig. 1 2D case. Plots of $\varepsilon_0^v(\mathbf{u}_h)$, $\varepsilon_1^v(\mathbf{u}_h)$ and $\varepsilon_0^v(p_h)$ for $\nu = 10^{-2}$.**Table 1** 2D case, averaged convergence rates on the last five meshes.

	P^0	$P^0 + P^1$		P^0	$P^0 + P^1$		P^0	$P^0 + P^1$		
$\tau_{0,\mathbf{u}}$	2.02	4.03		$\tau_{1,\mathbf{u}}$	1.01	3.07		τ_p	1.60	2.10

Let us consider stationary Navier-Stokes equations set in $\Omega = (0, 1)^3$: Find (\mathbf{u}, p) such that $-\nu \Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mathbf{f}$ and $\text{div } \mathbf{u} = 0$ in Ω , with inhomogeneous Dirichlet boundary conditions. Computations are done with the TrioCFD code [6, 7] with the $\mathbf{P}_{nc}^1 - (P^0 + P^1)$ scheme. The convection term is discretized using the MUSCL scheme [6]. We study the prescribed solution $\mathbf{u} = (y - z, z - x, x - y)^T$, $p = \frac{1}{2}(x^2 + y^2 + z^2) - xy - xz - yz - 1/4$ so that $\mathbf{f} = 0$. Fig. 2 shows $\varepsilon_0^v(\mathbf{u}_h)$ (left) and $\varepsilon_0^v(p_h)$ (right) against h , for $\nu = 10^{-1}$; 10^{-2} ; 10^{-3} .

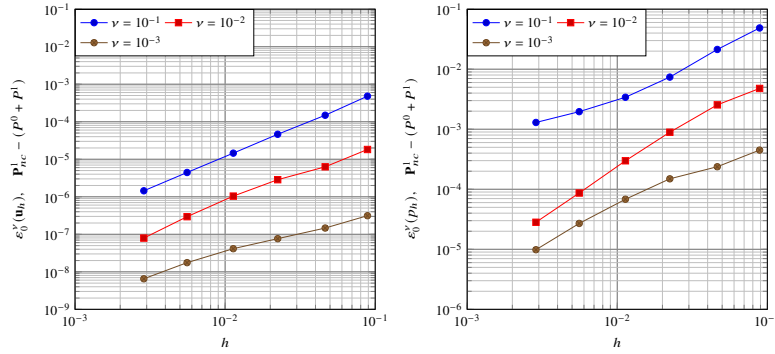
Fig. 2 3D case. Plots of $\varepsilon_0^v(\mathbf{u}_h)$ and $\varepsilon_0^v(p_h)$ for $\nu = 10^{-1}$, $\nu = 10^{-2}$ or $\nu = 10^{-3}$.

Table 2 reports the convergence rates $\tau_{0,\mathbf{u}}$ (left), and τ_p (right). More numerical results are available in [8, 9, 6, 23]. The $\mathbf{P}_{nc}^1 - (P^0 + P^1)$ scheme shows good approximation properties and should therefore be suitable for solving the Navier-Stokes

Table 2 3D case, averaged convergence rates on the last three meshes.

ν	10^{-1}	10^{-2}	10^{-3}	ν	10^{-1}	10^{-2}	10^{-3}
$\tau_{0,\mathbf{u}}$	1.68	1.88	1.41	τ_p	0.70	1.72	1.35

equations in the context of industrial simulation. More generally, it seems interesting to design and study discrete pressure enrichment for higher order nonconforming Crouzeix-Raviart mixed FEM [2, 3].

Acknowledgements We would like to thank P.-E. Angeli for his contribution to §4.

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