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# Stability of the $\mathbf{P}_{nc}^1 - (P^0 + P^1)$ element

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## 1 Stokes problem

A popular low-order method for solving the Stokes problem is the nonconforming Crouzeix-Raviart mixed finite element method [1, Example 4] (cf. [2, 3] for higher orders). The discrete velocity is piecewise affine, continuous in the barycentre of the interfaces between elements, but not globally continuous,  $\mathbf{P}_{nc}^1$ ; and the discrete pressure is piecewise constant,  $P^0$ . This method introduces some consistency error on the discrete velocity, which may lead to an unphysical solution when solving the Navier-Stokes equations, especially when the source field is a strong gradient [4, 5]. To recover consistency, the discrete pressure space can be generated by  $P^1$ -continuous plus  $P^0$  basis functions, cf. the TrioCFD code [6, 7]. This leads to the  $\mathbf{P}_{nc}^1 - (P^0 + P^1)$  mixed finite element method, which exhibits interesting numerical features. However, only an incomplete proof of the stability is available in [8, 9]. Our goal here is to gather evidence regarding the inf-sup condition. Let us consider Stokes problem set in an open, bounded, connected subset  $\Omega$  of  $\mathbb{R}^d$ ,  $d = 2, 3$ , with a Lipschitz boundary  $\partial\Omega$ :

$$\text{Find } (\mathbf{u}, p) \text{ such that } -\nu \Delta \mathbf{u} + \nabla p = \mathbf{f} \text{ and } \operatorname{div} \mathbf{u} = 0 \text{ in } \Omega. \quad (1)$$

with Dirichlet boundary conditions for the velocity  $\mathbf{u}$  and a normalisation condition for the pressure  $p$ :  $\mathbf{u} = 0$  on  $\partial\Omega$ ,  $\int_{\Omega} p = 0$ . Given  $D$  an open subset of  $\mathbb{R}^d$ , let

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us set  $\mathbf{L}^2(D) = (L^2(D))^d$ , and  $L^2_{zmv}(D) = \{q \in L^2(D) \mid \int_D q = 0\}$ . We denote by  $(\cdot, \cdot)_D$  the  $L^2(D)$  or  $\mathbf{L}^2(D)$  scalar product, and  $\|\cdot\|_D$  the associated norm. Let  $\mathbf{H}_0^1(D) = (H_0^1(D))^d$ , and  $\mathbf{H}^{-1}(D) = (H^{-1}(D))^d$  its dual space. Due Poincaré-Steklov inequality [10, Lemma 3.24], the semi-norm in  $H_0^1(\Omega)$  is equivalent to the natural norm, so that  $(v, w)_{H_0^1(\Omega)} = (\nabla v, \nabla w)_\Omega$  and  $\|v\|_{H_0^1(\Omega)} = \|\nabla v\|_\Omega$ . Let  $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$ . We denote by  $(v_i)_{i=1}^d$  the components of  $\mathbf{v}$ , and we set  $\underline{\nabla} \mathbf{v} = (\partial_j v_i)_{i,j=1}^d \in [L^2(\Omega)]^{d \times d}$ . Let us set  $\mathbf{V} = \{\mathbf{v} \in \mathbf{H}_0^1(\Omega) \mid \operatorname{div} \mathbf{v} = 0\}$  which is a closed subset of  $\mathbf{H}_0^1(\Omega)$ . We denote by  $\mathbf{V}^\perp$  the orthogonal of  $\mathbf{V}$  in  $\mathbf{H}_0^1(\Omega)$ .

**Proposition 1** *The operator  $\operatorname{div} : \mathbf{H}_0^1(\Omega) \rightarrow L^2(\Omega)$  is an isomorphism of  $\mathbf{V}^\perp$  onto  $L^2_{zmv}(\Omega)$ . We call  $C_{\operatorname{div}} > 1$  the constant such that:  $\forall p \in L^2_{zmv}(\Omega)$ ,  $\exists ! \mathbf{v}_p \in \mathbf{V}^\perp \mid \operatorname{div} \mathbf{v}_p = p$  and  $\|\mathbf{v}_p\|_{\mathbf{H}_0^1(\Omega)} \leq C_{\operatorname{div}} \|p\|_\Omega$  [11, Theorem 3.1 (e)].*

Let  $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$ . The variational formulation of Problem (1) reads:  
Find  $(\mathbf{u}, p) \in \mathbf{H}_0^1(\Omega) \times L^2_{zmv}(\Omega)$  such that  $\forall (\mathbf{v}, q) \in \mathbf{H}_0^1(\Omega) \times L^2_{zmv}(\Omega)$ :

$$v(\mathbf{u}, \mathbf{v})_{\mathbf{H}_0^1(\Omega)} - (p, \operatorname{div} \mathbf{v})_\Omega - (q, \operatorname{div} \mathbf{u})_\Omega = \langle \mathbf{f}, \mathbf{v} \rangle_{\mathbf{H}_0^1(\Omega)}. \quad (2)$$

One can prove that Problem (2) is well-posed with the T-coercivity theory [12, 13, 14], using Poincaré-Steklov inequality [10, Lemma 3.24] and Proposition 1.

## 2 Discretization

Consider  $(\mathcal{T}_h)_h$  a family of simplicial triangulations of  $\Omega$ . For a given  $\mathcal{T}_h := \{K\}$ , we call  $\mathcal{F}_h := \{F\}$  its set of  $(d-1)$ -simplices, called facets (faces if  $d=3$ , edges if  $d=2$ ), and  $\mathcal{V}_h := \{S\}$  its set of vertices. In the bounds, we will use  $\lesssim$  and  $\gtrsim$  for (hidden) constants independent of  $\mathcal{T}_h$ .

- For all  $K \in \mathcal{T}_h$ , we call  $h_K$  and  $\rho_K$  the diameters of  $K$  and its inscribed sphere respectively, and we let:  $\sigma_K = \frac{h_K}{\rho_K}$ . When the  $(\mathcal{T}_h)_h$  is a shape-regular triangulation family (see e.g. [10, def. 11.2]), there exists a constant  $\sigma > 1$ , called the shape regularity parameter, such that for all  $h$ , for all  $K \in \mathcal{T}_h$ ,  $\sigma_K \leq \sigma$ .
- Let  $\mathcal{F}_h = \mathcal{F}_h^i \cup \mathcal{F}_h^b$ , where  $\mathcal{F}_h^b := \{F \in \mathcal{F}_h \mid F \subset \partial\Omega\}$ ,  $\mathcal{F}_h^i = \mathcal{F}_h \setminus \mathcal{F}_h^b$ . For all  $K \in \mathcal{T}_h$ , we set  $\mathcal{F}_K^i = \{F \in \mathcal{F}_h^i \mid F \subset \partial K\}$ .

For all  $F \in \mathcal{F}_h$ ,  $M_F$  denotes the barycentre of  $F$ , and  $\mathbf{n}_F$  its unit normal (outward oriented if  $F \subset \partial\Omega$ ). For all  $K \in \mathcal{T}_h$  and  $F \subset \partial K$ , we denote by  $\mathbf{n}_{F,K}$  the normal vector of  $F$  outgoing from  $K$ . Let  $\hat{K}$  be the reference simplex. For all  $K \in \mathcal{T}_h$ , we denote by  $T_K : \hat{K} \rightarrow K$  the geometric mapping such that  $\forall \hat{\mathbf{x}} \in \hat{K}$ ,  $\mathbf{x}_{|K} = T_K(\hat{\mathbf{x}}) = \mathbb{B}_K \hat{\mathbf{x}} + \mathbf{b}_K$ , where  $\mathbb{B}_K \in \mathbb{R}^{d \times d}$  and  $\mathbf{b}_K \in \mathbb{R}^d$ . Let us set  $J_K = \det(\mathbb{B}_K)$ . Recall that [10, Lemma 11.1]:  $|J_K| = |K|/|\hat{K}|$ ,  $\|\mathbb{B}_K\| \leq h_K/\rho_{\hat{K}}$ ,  $\|\mathbb{B}_K^{-1}\| \leq h_{\hat{K}}/\rho_K$ . For all  $v \in L^2(\Omega)$ , we set:  $v_K = v|_K$ ,  $\underline{v}_K = \int_K v / |K|$ ,  $\tilde{v}_K = v_K - \underline{v}_K$ ,  $\hat{v}_K = v \circ T_K$ . By changing the variable, we get:  $\|v\|_K^2 = |J_K| \|\hat{v}_K\|_{\hat{K}}^2$ . Let  $v \in H^1(K)$ . By changing the variable, we get:  $\nabla v|_K = (\mathbb{B}_K^{-1})^T \nabla_{\hat{\mathbf{x}}} \hat{v}$ , and  $\|\nabla v\|_K^2 \leq \|\mathbb{B}_K^{-1}\|^2 |J_K| \|\nabla_{\hat{\mathbf{x}}} \hat{v}\|_{\hat{K}}^2$ ,

$$\|\nabla_{\hat{\mathbf{x}}}\hat{v}\|_{\hat{K}}^2 \leq \|\mathbb{B}_K\|^2 |J_K|^{-1} \|\nabla v\|_K^2.$$

We set  $\mathcal{P}_h H^1 = \{v \in L^2(\Omega); \forall K \in \mathcal{T}_h, v|_K \in H^1(K)\}$ , endowed with:  $(v, w)_h := \sum_{K \in \mathcal{T}_h} (\nabla v, \nabla w)_K$ ,  $\|v\|_h^2 = \sum_{K \in \mathcal{T}_h} \|\nabla v\|_K^2$ . We define the operator  $\nabla_h$  such that:  $\forall q \in \mathcal{P}_h H^1$ ,  $\nabla_h q \in \mathbf{L}^2(\Omega)$  is such that  $\forall \mathbf{v} \in \mathbf{L}^2(\Omega)$ ,  $(\nabla_h q, \mathbf{v})_\Omega = \sum_{K \in \mathcal{T}_h} (\nabla q, \mathbf{v})_K$ . Let  $K \in \mathcal{T}_h$  and  $F \subset \partial K$ , we denote by  $T_{F,K}$  the geometric mapping such that  $\forall \mathbf{x} \in F$ ,  $T_{F,K}(T_K^{-1}(\mathbf{x})) = \mathbf{x}$ . We let  $\hat{F}_{F,K} = T_K^{-1}(F)$ . For all  $v \in \mathcal{P}_h H^1$ , for all  $K \in \mathcal{T}_h$  and for all  $F \subset \partial K$ , we set:  $v_{F,K} = (v|_K)|_F$ ,  $\underline{v}_{F,K} = \int_F v|_K / |F|$ ,  $\tilde{v}_{F,K} = v|_K - \underline{v}_{F,K}$ ,  $\hat{v}_{F,K} = v \circ T_{F,K}$ . By changing the variable, we get  $\|v_{F,K}\|_F^2 = (|F|/|\hat{F}_{F,K}|) \|\hat{v}_{F,K}\|_{\hat{F}_{F,K}}^2$ . According to Equation [1, (3.17)], it holds:

$$\forall K \in \mathcal{T}_h, \forall F \subset \partial K, \quad |F| |K|^{-1} \lesssim \rho_K^{-1}. \quad (3)$$

We set  $\mathcal{P}_h \mathbf{H}^1 = [\mathcal{P}_h H^1]^d$  and for all  $\mathbf{v} \in \mathcal{P}_h \mathbf{H}^1$ :  $\underline{\nabla}_h \mathbf{v} = ((\nabla_h v_i)^T)_{i=1}^d \in [L^2(\Omega)]^{d \times d}$ .

Let  $F \in \mathcal{F}_h^i$  such that  $F = \partial K_L \cap \partial K_R$  and  $\mathbf{n}_F = \mathbf{n}_{F,K_L}$ . The jump of  $v \in \mathcal{P}_h H^1$  across  $F$  is defined by:  $[v]_F := v|_{K_L} - v|_{K_R}$ . For  $F \in \mathcal{F}_h^b$ , we set:  $[v]_F := v|_F$ .

For a subset  $D$  of  $\mathbb{R}^d$ , for  $k \in \mathbb{N}$ , we call  $P^k(D)$  the set of degree  $k$  polynomials on  $D$ ,  $\mathbf{P}^k(D) = (P^k(D))^d$ , and  $\mathcal{P}_{disc}^k(\mathcal{T}_h) = \{q \in L^2(\Omega); \forall K \in \mathcal{T}_h, q|_K \in P^k(K)\}$ .

For  $k > 0$  we define  $\mathcal{P}^k(\mathcal{T}_h) = C^0(\overline{\Omega}) \cap \mathcal{P}_{disc}^k(\mathcal{T}_h)$ . We call  $\pi_0 : L^2(\Omega) \mapsto \mathcal{P}_{disc}^0(\mathcal{T}_h)$  the  $L^2$ -orthogonal projection onto  $\mathcal{P}_{disc}^0(\mathcal{T}_h)$ , such that  $\forall q \in L^2(\Omega)$ ,  $\pi_0(q) = \sum_{K \in \mathcal{T}_h} \pi_0(q)(K) \mathbb{1}_K$  where  $\forall K \in \mathcal{T}_h$ ,  $\pi_0(q)(K) = \int_K q / |K|$ .

For  $K \in \mathcal{T}_h$  and a vertex  $S$  of  $K$ ,  $\lambda_{S,K}$  denotes the barycentric coordinate related to  $S$ . We can endow  $\mathcal{P}^1(\mathcal{T}_h)$  with the basis  $(\phi_S)_{S \in \mathcal{V}_h}$  such that:  $\forall K \in \mathcal{T}_h$ ,  $\phi_S|_K = \lambda_{S,K}$  if  $S \in K$ , zero otherwise, so that  $\mathcal{P}^1(\mathcal{T}_h) = \text{vect}((\phi_S)_{S \in \mathcal{V}_h})$ .

Let  $k \in \mathbb{N}^*$ ,  $v \in H^1(K) \cap P^k(K)$ . Using the discrete trace inequality [15, Thm. 5] and Poincaré-Steklov inequality [10, Lemma 12.11], we have:

$$\|(\widehat{v}_{F,K})_{F,K}\|_{\hat{F}_{F,K}}^2 \lesssim \|(\widehat{v}_{F,K})_K\|_{\hat{K}}^2 \lesssim \|\nabla_{\hat{\mathbf{x}}}\hat{v}_K\|_{\hat{K}}^2. \quad (4)$$

**Proposition 2** For all  $K \in \mathcal{T}_h$ ,  $\forall v \in H^1(K) \cap P^1(K)$ ,  $\|\nabla v\|_K \lesssim (\rho_K)^{-1} \|\tilde{v}\|_K$ .

*Proof.* We follow the proof of [16, Lemma 2]: changing the variable, we have  $\|\nabla v\|_{L^2(K)} \lesssim \|\mathbb{B}_K^{-1}\| |K|^{1/2} \|\int_{\hat{K}} \nabla_{\hat{\mathbf{x}}}\hat{v}_K\|$ . Integrating by parts in the reference element, it holds:  $|\int_{\hat{K}} \nabla_{\hat{\mathbf{x}}}\hat{v}_K| = |\int_{\partial \hat{K}} \hat{v}_K \hat{\mathbf{n}}|_{\partial \hat{K}}$ . We get the result using Cauchy-Schwarz, [15, Thm. 5] and changing the variable again.  $\square$

We set  $\mathcal{P}_h \mathbf{H}(\text{div}) = \{\mathbf{v} \in \mathbf{L}^2(\Omega); \forall K \in \mathcal{T}_h, \mathbf{v}|_K \in \mathbf{H}(\text{div}; K)\}$ .

We define the operator  $\text{div}_h$  such that:  $\forall \mathbf{v} \in \mathcal{P}_h \mathbf{H}(\text{div})$ ,  $\text{div}_h \mathbf{v} \in L^2(\Omega)$  is such that  $\forall q \in L^2(\Omega)$ ,  $(\text{div}_h \mathbf{v}, q)_\Omega = \sum_{K \in \mathcal{T}_h} (\text{div} \mathbf{v}, q)_K$ .

Let  $\mathbf{X}_{0,h}$  be the space of the discrete velocity of the nonconforming Crouzeix-Raviart mixed finite element method, defined by:

$$\begin{aligned} \mathbf{X}_h &= (X_h)^d, \quad X_h = \{v_h \in \mathcal{P}_{disc}^1(\mathcal{T}_h); \forall F \in \mathcal{F}_h^i, \int_F [v_h] = 0\}; \\ \mathbf{X}_{0,h} &= (X_{0,h})^d, \quad X_{0,h} = \{v_h \in X_h; \forall F \in \mathcal{F}_h^b, \int_F v_h = 0\}. \end{aligned} \quad (5)$$

The condition on the jumps of  $v_h$  on the inner facets is often called the patch-test condition. Due to this condition, one can prove that [1, Lemma 2]:

**Proposition 3** *The broken norm  $v_h \rightarrow \|v_h\|_h$  is a norm over  $X_{0,h}$ .*

For all  $K \in \mathcal{T}_h$ , for all  $F \subset \partial K$ ,  $\lambda_{F,K}$  denotes the barycentric coordinate related to the vertex  $S_{F,K}$  opposite to facet  $F$  in  $K$ . We can endow  $X_h$  with the basis  $(\psi_F)_{F \in \mathcal{F}_h}$  such that for all  $K \in \mathcal{T}_h$ ,  $\psi_{F|K} = 1 - d\lambda_{F,K}$  if  $F \subset \partial K$  and zero otherwise. We then have  $\psi_{F|F} = 1$ , so that  $[\psi_F]_F = 0$  if  $F \in \mathcal{F}_h^i$ , and for all  $F' \neq F$ ,  $\int_{F'} \psi_F = 0$ . We have:  $X_h = \text{vect}((\psi_F)_{F \in \mathcal{F}_h})$  and  $X_{0,h} = \text{vect}((\psi_F)_{F \in \mathcal{F}_h^i})$ .

Let  $\Pi_h : \mathbf{H}^1(\Omega) \mapsto \mathbf{X}_h$  be the Crouzeix-Raviart interpolation operator, such that:  $\forall \mathbf{v} \in \mathbf{H}^1(\Omega)$ ,  $\Pi_h(\mathbf{v}) = \sum_{F \in \mathcal{F}_h} \Pi_F \mathbf{v} \psi_F$ ,  $\Pi_F \mathbf{v} = \underline{\mathbf{v}}_F = \int_F \mathbf{v} / |F|$ .

**Proposition 4** *The interpolation operator  $\Pi_h$  is such that [16, Lemma 2]:  $\forall \mathbf{v}_h \in \mathbf{H}_0^1(\Omega)$ ,  $\|\Pi_h(\mathbf{v}_h)\|_h \leq \|\mathbf{v}_h\|_{\mathbf{H}_0^1(\Omega)}$ .*

However, due to the consistency error on the discrete velocity, it is well-known [17, 18] that choosing the space of the discrete pressures equal to  $\mathcal{P}_{disc}^0(\mathcal{T}_h) \cap L_{zmv}^2(\Omega)$  yields inaccurate results when solving Navier-Stokes equations. Starting from [19], a new element is proposed in [8]. The space of the discrete pressures is then:  $Q_h = Q_{0,h} + Q_{1,h}$ , where  $Q_{0,h} := \mathcal{P}_{disc}^0(\mathcal{T}_h) \cap L_{zmv}^2$  and for  $k > 0$ ,  $Q_{k,h} := \mathcal{P}^k(\mathcal{T}_h) \cap L_{zmv}^2(\Omega)$ . One can build an  $L^2$ -orthogonal sum letting  $\tilde{Q}_{1,h} = \{q_h \in Q_h \mid \forall K \in \mathcal{T}_h, \int_K q_h = 0\}$ .

**Proposition 5** *We have:  $Q_h = Q_{0,h} \oplus \tilde{Q}_{1,h}$ . Let  $q_h \in Q_h$  and  $\tilde{q}_{1,h} := q_h - \pi_0(q_h)$ , it holds:  $\|q_h\|_Q^2 = \|\pi_0(q_h)\|_\Omega^2 + \|\tilde{q}_{1,h}\|_\Omega^2$ .*

To prove the discrete inf-sup condition of  $\mathbf{X}_{0,h} \times Q_h$ , we need some Lemmas.

**Lemma 1** *Let  $k \in \mathbb{N}^*$  and  $(\mathbf{v}_h, q_{k,h}) \in \mathbf{X}_{0,h} \times Q_{k,h}$ . We then have:*

$$|\sum_{K \in \mathcal{T}_h} \sum_{F \subset \partial K} (\mathbf{v}_h \cdot \mathbf{n}_{F,K}, q_{k,h})_F| \lesssim \sigma \|\mathbf{v}_h\|_h \|\tilde{q}_{k,h}\|_\Omega. \quad (6)$$

*Proof.* We start with the proof of [19, Lemma 3.1]. Using the patch-test, one obtains:

$$\sum_{K \in \mathcal{T}_h} \sum_{F \subset \partial K} (\mathbf{v}_h \cdot \mathbf{n}_{F,K}, q_{k,h}|_K)_F = \sum_{K \in \mathcal{T}_h} \sum_{F \subset \partial K} (\tilde{\mathbf{v}}_{F,K} \cdot \mathbf{n}_{F,K}, \tilde{q}_{k,h}|_K)_F.$$

We can compute the following estimates, using (4) twice and (3):

$$|(\tilde{\mathbf{v}}_{F,K} \cdot \mathbf{n}_{F,K}, \tilde{q}_{k,h}|_K)_F| \lesssim |F| \|\widehat{\tilde{\mathbf{v}}}_{F,K}\|_{\hat{F}_{F,K}} \|\widehat{(\tilde{q}_{k,h})}_K\|_{\hat{F}_{F,K}} \lesssim \sigma_K \|\underline{\mathbf{v}}_h\|_K \|\tilde{q}_{k,h}\|_K.$$

Summing and using the discrete Cauchy-Schwarz inequality, we obtain (6).  $\square$

Let us set  $\mathbf{X}_{N,h} := \{\mathbf{v}_h \in \mathbf{X}_{0,h} \mid \forall F \in \mathcal{F}_h^i, \mathbf{v}_h \times \mathbf{n}_F = 0\}$ . We can build an explicit right inverse of  $\text{div}_h$  from  $Q_h$  to  $\mathbf{X}_{N,h}$ .

**Lemma 2** *For all  $q_h \in Q_h$ , there exists  $\mathbf{v}_{N,h} \in \mathbf{X}_{N,h}$  such that:*

$$-(\text{div}_h \mathbf{v}_{N,h}, \pi_0(q_h))_\Omega = v^{-1} \|\pi_0(q_h)\|_\Omega^2, \quad (7)$$

$$\|\mathbf{v}_{N,h}\|_h \leq C_{\text{div}} v^{-1} \|\pi_0(q_h)\|_\Omega. \quad (8)$$

*Proof.* Let  $q_h \in Q_h$ . From Proposition 1,  $\exists \mathbf{v}_N \in \mathbf{H}_0^1(\Omega) \mid \operatorname{div} \mathbf{v}_N = -\nu^{-1} \pi_0(q_h)$  and  $\|\mathbf{v}_N\|_{\mathbf{H}_0^1(\Omega)} \leq C_{\operatorname{div}} \nu^{-1} \|\pi_0(q_h)\|_{\Omega}$ . Let  $\mathbf{v}_{N,h} \in \mathbf{X}_{N,h} \mid \forall F \in \mathcal{F}_h^i, \mathbf{v}_{N,h}(M_F) = (\Pi_h(\mathbf{v}_N)(M_F) \cdot \mathbf{n}_F) \mathbf{n}_F$ . Integrating by part twice, we get:  $\forall K \in \mathcal{T}_h, \int_K \operatorname{div} \mathbf{v}_{N,h} = \int_K \operatorname{div} \mathbf{v}_N$ , hence (7). Using Proposition 4, we get (8).  $\square$

We can also build an explicit right inverse of the discrete divergence operator from  $Q_{1,h}$  to  $\mathbf{X}_{T,h} := \{\mathbf{v}_h \in \mathbf{X}_{0,h} \mid \forall F \in \mathcal{F}_h^i, \mathbf{v}_h \cdot \mathbf{n}_F = 0\}$ , if [19, Hyp. 4.1] holds:

**Hypothesis 1.** *We suppose that  $\mathcal{T}_h$  is such that  $\partial\Omega$  contains at most one edge for  $d = 2$  and at most two facets for  $d = 3$ , of the same element  $K \in \mathcal{T}_h$ .*

**Lemma 3** *Assuming Hypothesis 1,  $\forall q_{1,h} \in Q_{1,h}, \exists \mathbf{v}_{T,h} \in \mathbf{X}_{T,h}$  such that:*

$$(\nabla q_{1,h}, \mathbf{v}_{T,h})_{\Omega} \gtrsim \nu^{-1} \sigma^{-2(d-2)} \|\tilde{q}_{1,h}\|_{\Omega}^2, \quad (9)$$

$$(\operatorname{div} \mathbf{v}_{T,h}, q_{0,h})_K = 0 \quad \forall q_{0,h} \in Q_{0,h}, \quad (10)$$

$$\|\mathbf{v}_{T,h}\|_h \lesssim \nu^{-1} \sigma^2 \|\tilde{q}_{1,h}\|_{\Omega}. \quad (11)$$

*Proof.* Let us consider  $q_{1,h} \in Q_{1,h}$ . For all  $K \in \mathcal{T}_h$ , we let  $q_K := q_{1,h}|_K$  and  $\hat{q}_K = q_K \circ T_K$ . Since  $q_{1,h} \in H^1(\Omega)$  we have (see e.g. [20, Prop. 2.2.10]):  $\forall F \in \mathcal{F}_h^i \mid F = K \cap K', \nabla q_K \times \mathbf{n}_F = \nabla q_{K'} \times \mathbf{n}_F$ . We construct  $\mathbf{v}_{T,h} \in \mathbf{X}_{T,h} \mid \forall F \in \mathcal{F}_h^i, \mathbf{v}_{T,h}(M_F) = \nu^{-1} h_F^2 \mathbf{n}_F \times (\nabla q_{1,h} \times \mathbf{n}_F)$ , where  $h_F$  is the diameter of  $F$ . We have  $\forall K \in \mathcal{T}_h$ :

$$(\nabla q_{1,h}, \mathbf{v}_{T,h})_K = \nu^{-1} \frac{|K|}{d+1} \sum_{F \in \mathcal{F}_K^i} h_F^2 |\nabla q_K \times \mathbf{n}_F|^2. \quad (12)$$

Let  $\mathbf{n}_{\hat{F}_{F,K}}$  be the unit normal vector of the facet  $\hat{F}_{F,K}$  outgoing from  $\hat{K}$ . We have:  $\mathbf{n}_{F,K} = |K| (\mathbb{B}_K^{-1})^T |\hat{F}_{F,K}| |F|^{-1} \mathbf{n}_{\hat{F}_{F,K}}$ .

Consider  $d = 2$ . Changing the variable, assuming Hypothesis 1, we get:

$$\sum_{F \in \mathcal{F}_K^i} h_F^2 |\nabla q_K \times \mathbf{n}_F|^2 = \sum_{F \in \mathcal{F}_K^i} |\hat{F}_{F,K}|^2 |\nabla_{\hat{\mathbf{x}}} \hat{q}_K \times \mathbf{n}_{\hat{F}_{F,K}}|^2 \gtrsim |\nabla_{\hat{\mathbf{x}}} \hat{q}_K|^2. \quad (13)$$

Consider  $d = 3$ . It holds that  $|\nabla q_K \times \mathbf{n}_F| = |\hat{F}_{F,K}| |F|^{-1} \|\mathbb{B}_K \nabla_{\hat{\mathbf{x}}} \hat{q}_K \times \mathbf{n}_{\hat{F}_{F,K}}\|$  changing the variable. Hence:  $|\nabla q_K \times \mathbf{n}_F| \gtrsim |F|^{-1} \|\mathbb{B}_K^{-1}\|^{-1} |\nabla_{\hat{\mathbf{x}}} \hat{q}_K \times \mathbf{n}_{\hat{F}_{F,K}}|$ . Notice that  $h_F |F|^{-1} \|\mathbb{B}_K^{-1}\|^{-1} \gtrsim \sigma_K^{-1}$ . Assuming Hypothesis 1, we then have:

$$\sum_{F \in \mathcal{F}_K^i} h_F^2 |\nabla q_K \times \mathbf{n}_F|^2 \gtrsim \sigma_K^{-2} |\nabla_{\hat{\mathbf{x}}} \hat{q}_K|^2. \quad (14)$$

Using Poincaré-Steklov inequality [10, Lemma 12.11] in  $\hat{K}$ , we have:  $|\nabla_{\hat{\mathbf{x}}} \hat{q}_K|^2 \gtrsim \|\hat{q}_K - \underline{\hat{q}}_K\|_{\hat{K}}^2$ . Using (13) and (14) in (12), it holds:

$$(\nabla q_{1,h}, \mathbf{v}_{T,h})_K \gtrsim \nu^{-1} \sigma_K^{-2(d-2)} |K| |\nabla_{\hat{\mathbf{x}}} \hat{q}_K|^2 \gtrsim \nu^{-1} \sigma_K^{-2(d-2)} \|\tilde{q}_K\|_{\hat{K}}^2. \quad (15)$$

We obtain (9) by summation. Remark that  $\nabla \psi_{F|K} = |K|^{-1} |F| \mathbf{n}_{F,K}$ . Let us prove (10). Since  $\nabla \psi_{F|K} \perp \mathbf{v}_{T,h}(M_F)$ ,  $(\operatorname{div} \mathbf{v}_{T,h}, q_{0,h})_K = |K| q_{0,h}(K) \sum_{F \in \mathcal{F}_K^i} \nabla \psi_{F|K} \cdot \mathbf{v}_{T,h}(M_F) = 0$ . Let us prove (11). For all  $K \in \mathcal{T}_h$ ,  $\underline{\mathbf{v}}_{T,h|K} = \sum_{F \in \mathcal{F}_K^i} \mathbf{v}_{T,h}(M_F) \otimes$

$\nabla\psi_{F|K}$ . Hence:  $\|\underline{\nabla}\mathbf{v}_{T,h}\|_K^2 \lesssim \nu^{-2} \sum_{F \in \mathcal{F}_K^i} |F|^2 |K|^{-1} h_F^4 |\nabla q_K \times \mathbf{n}_F|^2$ . Using (3), and assuming Hypothesis 1, we get:  $\|\underline{\nabla}\mathbf{v}_{T,h}\|_K^2 \lesssim \nu^{-2} \sigma_K^2 h_K^2 |K| \sum_{F \in \mathcal{F}_K^i} |\nabla q_K \times \mathbf{n}_F|^2 \lesssim \nu^{-2} \sigma_K^2 h_K^2 \|\nabla q_K\|_K^2$ . From Proposition 2:  $\|\underline{\nabla}\mathbf{v}_{T,h}\|_K \lesssim \nu^{-1} \sigma_K^2 \|\tilde{q}_K\|_K$ .  $\square$

Let  $b_h : \mathbf{X}_h \times \mathcal{Q}_h \mapsto \mathbb{R} \mid \forall (\mathbf{v}_h, q_h) \in \mathbf{X}_h \times \mathcal{Q}_h, b_h(\mathbf{v}_h, q_h) = -(\operatorname{div}_h \mathbf{v}_h, q_{0,h})_\Omega + (\mathbf{v}_h, \nabla q_{1,h})_\Omega$  with  $q_h = q_{0,h} + q_{1,h}$  such that  $(q_{0,h}, q_{1,h}) \in \mathcal{Q}_{0,h} \times \mathcal{Q}_{1,h}$ .

**Theorem 1** *The following continuity property holds:*

$$\forall (\mathbf{v}_h, q_h) \in \mathbf{X}_{0,h} \times \mathcal{Q}_h, \quad |b_h(\mathbf{v}_h, q_h)| \lesssim \sigma \|\mathbf{v}_h\|_h \|q_h\|_\Omega. \quad (16)$$

Assuming Hypothesis 1, the following inf-sup condition holds:

$$\forall q_h \in \mathcal{Q}_h, \exists \mathbf{v}_h \in \mathbf{X}_{0,h}, \quad b_h(\mathbf{v}_h, q_h) \gtrsim \frac{1}{\sqrt{2}} C_{\operatorname{div}}^{-2} \sigma^{-2d} \|\mathbf{v}_h\|_h \|q_h\|_\Omega. \quad (17)$$

*Proof.* Let us prove (16). Let  $(\mathbf{v}_h, q_h) \in \mathbf{X}_{0,h} \times \mathcal{Q}_h$ . Integrating by parts, we have:

$$b_h(\mathbf{v}_h, q_h) = -(\operatorname{div}_h \mathbf{v}_h, \pi_0(q_h))_\Omega + \sum_{K \in \mathcal{T}_h} \sum_{F \subset \partial K} (\mathbf{v}_h \cdot \mathbf{n}_{F,K}, q_{1,h})_F. \quad (18)$$

Using Cauchy-Schwarz, we have:  $|(\operatorname{div}_h \mathbf{v}_h, \pi_0(q_h))_\Omega| \leq \sqrt{d} \|\mathbf{v}_h\|_h \|\pi_0(q_h)\|_\Omega$ . Using (6) in (18), we obtain (16) from Proposition 5. Let us prove (17), starting from the proof of [19, Lemma 4.2]. Let  $q_h \in \mathcal{Q}_h \setminus \{0\}$ , where  $q_h = q_{0,h} + q_{1,h}$  is such that  $(q_{0,h}, q_{1,h}) \in \mathcal{Q}_{0,h} \times \mathcal{Q}_{1,h}$ . Let  $\mathbf{v}_{N,h} := \mathbf{v}_{N,h}(q_h)$  be like in Lemma 2. Using (18) with  $\mathbf{v}_h = \mathbf{v}_{N,h}$ , (6) and (7), we have:

$$b_h(\mathbf{v}_{N,h}, q_h) \gtrsim \nu^{-1} \|\pi_0(q_h)\|_\Omega^2 - \sigma \|\mathbf{v}_{N,h}\|_h \|\tilde{q}_{1,h}\|_\Omega. \quad (19)$$

Using (8) and Young inequality, we have for all  $\varepsilon > 0$ :

$$-\|\mathbf{v}_{N,h}\|_h \|\tilde{q}_{1,h}\|_\Omega \geq -\frac{1}{2} C_{\operatorname{div}} \nu^{-1} (\varepsilon \|\pi_0(q_h)\|_\Omega^2 + \varepsilon^{-1} \|\tilde{q}_{1,h}\|_\Omega^2). \quad (20)$$

We now insert the bound (20) in (19) to get:

$$b_h(\mathbf{v}_{N,h}, q_h) \gtrsim \nu^{-1} \left( \left(1 - \frac{\varepsilon}{2} C_{\operatorname{div}} \sigma\right) \|\pi_0(q_h)\|_\Omega^2 - \frac{C_{\operatorname{div}}}{2\varepsilon} \sigma \|\tilde{q}_{1,h}\|_\Omega^2 \right). \quad (21)$$

Let  $\mathbf{v}_{T,h} := \mathbf{v}_{T,h}(q_{1,h})$  be like in Lemma 3. Using (9) and (10), we get:

$$b_h(\mathbf{v}_{T,h}, q_{1,h}) = (\nabla q_{1,h}, \mathbf{v}_{T,h})_\Omega \gtrsim \nu^{-1} \sigma^{-2(d-2)} \|\tilde{q}_{1,h}\|_\Omega^2 \quad (22)$$

Last, we consider  $\mathbf{v}_h^* = \mu \mathbf{v}_{N,h} + \mathbf{v}_{T,h}$ , where  $\mu > 0$ . Using (21) and (22), we have:

$$\begin{aligned} b_h(\mathbf{v}_h^*, q_h) &= \mu b_h(\mathbf{v}_{N,h}, q_h) + b_h(\mathbf{v}_{T,h}, q_h), \\ &\gtrsim \nu^{-1} \left( \mu \left(1 - \frac{\varepsilon}{2} C_{\operatorname{div}} \sigma\right) \|\pi_0(q_h)\|_\Omega^2 + \left(\sigma^{-2(d-2)} - \frac{\mu C_{\operatorname{div}}}{2\varepsilon} \sigma\right) \|\tilde{q}_{1,h}\|_\Omega^2 \right). \end{aligned}$$

Let us choose  $\varepsilon = (C_{\operatorname{div}} \sigma)^{-1} < 1$  and  $\mu = \sigma^{-2(d-2)} \varepsilon^2$ . We obtain that:

$$b_h(\mathbf{v}_h^*, q_h) \gtrsim C_{\min} \nu^{-1} \|q_h\|_{\Omega}^2 \text{ with } C_{\min} = \varepsilon^2 \sigma^{-2(d-2)} = C_{\text{div}}^{-2} \sigma^{-2(d-1)}. \quad (23)$$

We will now bound  $\|\mathbf{v}_h^*\|_h$  by  $\|q_h\|_{\Omega}$ . We have:  $\|\mathbf{v}_h^*\|_h^2 \leq 2(\mu^2 \|\mathbf{v}_{N,h}\|_h^2 + \|\mathbf{v}_{T,h}\|_h^2)$ . Using (8) and (11)  $\|\mathbf{v}_h^*\|_h^2 \leq 2\nu^{-2}(\mu^2 C_{\text{div}}^2 \|\pi_0(q_h)\|_{\Omega}^2 + \sigma^4 \|\tilde{q}_{1,h}\|_{\Omega}^2)$ , hence:

$$\|\mathbf{v}_h^*\|_h^2 \leq C_{\max}^2 \nu^{-2} \|q_h\|_{\Omega}^2 \text{ with } C_{\max} = \sqrt{2} \sigma^2. \quad (24)$$

Using (24) in (23), we get (17) with  $C_{\min} C_{\max}^{-1} = \frac{1}{\sqrt{2}} C_{\text{div}}^{-2} \sigma^{-2d}$ .  $\square$

Let  $\mathbf{V}_h := \{\mathbf{v}_h \in \mathbf{X}_{0,h} \mid \forall q_h \in Q_h, b_h(\mathbf{v}_h, q_h) = 0\}$ . Let  $(\mathbf{v}_h, q_{1,h}) \in \mathbf{V}_h \times Q_{1,h}$ . Integrating by parts, we have  $0 = b_h(\mathbf{v}_h, q_{1,h}) = -(\text{div}_h \mathbf{v}, \pi_0(q_{1,h}))_{\Omega} + \sum_{F \in \mathcal{F}_h} \int_F q_{1,h}[\mathbf{v}_h] \cdot \mathbf{n}_F = 0 + \sum_{F \in \mathcal{F}_h} \int_F q_{1,h}[\mathbf{v}_h] \cdot \mathbf{n}_F$ . Hence:

**Lemma 4** For all  $(\mathbf{v}_h, q_{1,h}) \in \mathbf{V}_h \times Q_{1,h}$ , it holds:  $\sum_{F \in \mathcal{F}_h} \int_F q_{1,h}[\mathbf{v}_h] \cdot \mathbf{n}_F = 0$ .

Let  $\ell_{\mathbf{f}} \in \mathcal{L}(\mathbf{X}_h, \mathbb{R})$  be such that  $\forall \mathbf{v}_h \in \mathbf{X}_h$ ,  $\ell_{\mathbf{f}}(\mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h)_{\Omega}$  if  $\mathbf{f} \in \mathbf{L}^2(\Omega)$ ,  $\ell_{\mathbf{f}}(\mathbf{v}_h) = \langle \mathbf{f}, \mathcal{I}_h(\mathbf{v}_h) \rangle_{\mathbf{H}_0^1(\Omega)}$  if  $\mathbf{f} \notin \mathbf{L}^2(\Omega)$ , where  $\mathcal{I}_h : \mathbf{X}_{0,h} \rightarrow \mathbf{Y}_{0,h}$  is for instance an averaging operator [10, §22.4.1], with  $\mathbf{Y}_{0,h} = \{\mathbf{v}_h \in \mathbf{H}_0^1(\Omega) \mid \forall K \in \mathcal{T}_h, \mathbf{v}_h|_K \in \mathbf{P}^k(K)\}$ . The discretization of Problem (1) with the  $\mathbf{P}_{nc}^1 - (P^0 + P^1)$  element reads: Find  $(\mathbf{u}_h, p_h) \in \mathbf{X}_{0,h} \times Q_h$  such that  $\forall (\mathbf{u}_h, p_h) \in \mathbf{X}_{0,h} \times Q_h$

$$\nu(\mathbf{u}_h, \mathbf{v}_h)_h + b_h(\mathbf{v}_h, p_h) + b_h(\mathbf{u}_h, q_h) = \ell_{\mathbf{f}}(\mathbf{v}_h). \quad (25)$$

Due to Proposition 3 and Theorem 1, Problem (25) is well posed. We can derive error estimates similar to those given by [1, Theorems 3, 4, 6].

### 3 Convergence estimates

#### 3.1 Optimal estimates in 2D

When  $\Omega \subset \mathbb{R}^2$ , we have the two Lemmas below.

**Lemma 5** For all  $(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_{2,h}$ ,  $(\nabla q_h, \mathbf{v}_h)_{\Omega} = 0$ .

*Proof.* Let  $(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_{2,h}$ . Integrating by parts and using Lemma 4, we get that  $\forall q_{1,h} \in Q_{1,h}$ :  $(\nabla q_h, \mathbf{v}_h)_{\Omega} = -\sum_{F \in \mathcal{F}_h} \int_F q_h[\mathbf{v}_h] \cdot \mathbf{n}_F = -\sum_{F \in \mathcal{F}_h} \int_F (q_h - q_{1,h})[\mathbf{v}_h] \cdot \mathbf{n}_F = \sum_{F \in \mathcal{F}_h} \int_F (q_h - q_{1,h})[\mathbf{v}_h - \mathbf{v}_h(M_F)] \cdot \mathbf{n}_F$ . Hence, we obtain:  $(\nabla q_h, \mathbf{v}_h)_{\Omega} = -\sum_{K \in \mathcal{T}_h} \sum_{F \subset \partial K} \int_F (q_h - q_{1,h})(\mathbf{v}_h - \mathbf{v}_h(M_F)) \cdot \mathbf{n}_{F,K}$ . Choose  $q_{1,h}$  such that  $\forall S \in \mathcal{V}_h$ ,  $q_{1,h}(S) = q_{2,h}(S)$ . Then,  $\forall F \in \mathcal{F}_h$ , the degree 3 polynomial  $(q_h - q_{1,h})(\mathbf{v}_h - \mathbf{v}_h(M_F))$  vanishes at the quadrature points of Simpson's rule. Since it is exact for degree 3 polynomials,  $\forall F \in \mathcal{F}_h$ ,  $\int_F (q_h - q_{1,h})(\mathbf{v}_h - \mathbf{v}_h(M_F)) \cdot \mathbf{n}_{F,K} = 0$  and  $(\nabla q_h, \mathbf{v}_h)_{\Omega} = 0$ .  $\square$

**Lemma 6** Let  $\phi \in H^3(\Omega) \cap L_{zmv}^2(\Omega)$  such that  $|\phi|_{H^3(\Omega)} \neq 0$ .

Then,  $\forall \mathbf{v}_h \in \mathbf{V}_h$ ,  $(\nabla \phi, \mathbf{v}_h)_{\Omega} \lesssim \sigma h^2 |\phi|_{H^3(\Omega)} \|\mathbf{v}_h\|_{\Omega}$ .



*Proof.* Let  $\mathbf{v}_h \in \mathbf{V}_h$ . Using Lemma 5 and Cauchy-Schwarz, it holds:  $|(\nabla\phi, \mathbf{v}_h)_\Omega| = |(\nabla(\phi - q_h), \mathbf{v}_h)_\Omega|$  for all  $q_h \in \mathcal{Q}_{2,h}$ . We then use Bramble-Hilbert Lemma [10, Lemma 11.9] to conclude.  $\square$

**Theorem 2** *Suppose that  $\Omega$  is convex. Let  $\phi \in H^3(\Omega) \cap L^2_{zmv}(\Omega)$  such that  $|\phi|_{H^3(\Omega)} \neq 0$ . Let  $(\mathbf{u}_h, p_h)$  be the solution of Problem (25) with  $\mathbf{f} = \nabla\phi$ . The solution of Problem (1) is then  $(\mathbf{u}, p) = (0, \phi)$ . It holds:*

$$\nu \|\mathbf{u}_h\|_h \lesssim \sigma^2 h^3 |\phi|_{H^3(\Omega)}; \quad \nu \|\mathbf{u}_h\|_\Omega \lesssim \sigma^3 h^4 |\phi|_{H^3(\Omega)}.$$

*Proof.* Setting  $\mathbf{v}_h = \mathbf{u}_h$  as test-function in Problem (25) and using Lemma 6, it holds:  $\nu \|\mathbf{u}_h\|_h^2 = (\nabla\phi, \mathbf{u}_h)_\Omega \lesssim \sigma h^2 |\phi|_{H^3(\Omega)} \|\mathbf{u}_h\|_\Omega$ . From [1, Theorem 4], have:  $\|\mathbf{u}_h\|_\Omega \lesssim \sigma h \|\mathbf{u}_h\|_h$ . We deduce that  $\nu \|\mathbf{u}_h\|_h \lesssim \sigma^2 h^3 |\phi|_{H^3(\Omega)}$ . Using this estimate, we get that  $\nu \|\mathbf{u}_h\|_\Omega \lesssim \sigma^3 h^4 |\phi|_{H^3(\Omega)}$ .  $\square$

### 3.2 Extension to 3D

When  $\Omega \subset \mathbb{R}^3$ , let  $\mathcal{E}_h$  be the set of edges of  $\mathcal{T}_h$ . Let  $E := S_i S_j \in \mathcal{E}_h$ . We set  $\phi_E := 4\lambda_i \lambda_j$  where  $\lambda_i$  (resp.  $\lambda_j$ ) is the barycentric coordinate related to vertex  $S_i$  (resp.  $S_j$ ). Let  $\tilde{P}^2(\mathcal{T}_h) = \text{vect}((\tilde{\phi}_E)_{E \in \mathcal{E}_h})$ , where for every edge  $E$ ,  $\tilde{\phi}_E$  is the canonical extension of  $\phi_E$  to all simplices intersecting with  $\text{int}(E)$ . In order to obtain optimal results in 3D similar to those obtained in 2D, the discrete pressure space can be  $\mathcal{Q}_h = \mathcal{Q}_{0,h} + \mathcal{Q}_{1,h} + \tilde{\mathcal{Q}}_{2,h}$ , where  $\tilde{\mathcal{Q}}_{2,h}$  is a subspace of  $\tilde{P}^2(\mathcal{T}_h) \cap L^2_{zmv}(\Omega)$  [9]. One can then prove Lemma 5 in 3D using a 7-point quadrature rule on a triangle, exact for polynomials of degree 3. The quadrature points are the barycentre of the triangle, its vertices and the midpoints of its edges. Lemma 6 follows. Finally, in [9, Theorem 4.4.6, page 92], it is claimed that one obtains a convergence result identical to the one of Theorem 2, now in 3D.

## 4 Numerical results for the stationary Navier-Stokes model

Let us consider stationary Navier-Stokes equations set in  $\Omega = (0, 1)^d$ :

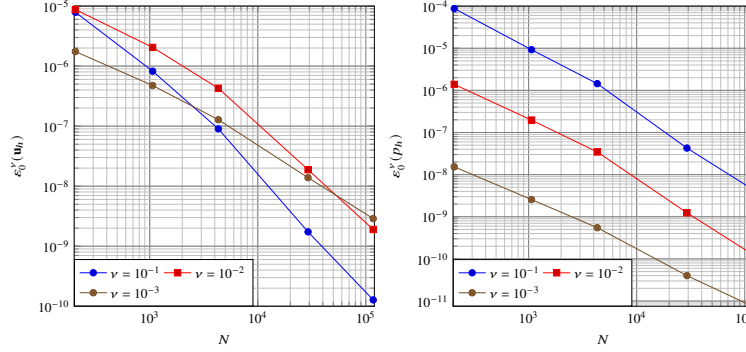
$$\text{Find } (\mathbf{u}, p) \text{ such that } -\nu \Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mathbf{f} \text{ and } \text{div } \mathbf{u} = 0 \text{ in } \Omega, \quad (26)$$

with Dirichlet boundary conditions on the velocity. Computations are done with the TrioCFD code [6, 7] with the  $\mathbf{P}_{nc}^1 - (P^0 + P^1)$  finite element. The convection term is discretized using the MUSCL scheme [6]. Let us set:  $\|(\mathbf{u}, p)\|_\nu^2 := \|\nabla \mathbf{u}\|_\Omega^2 + \nu^{-2} \|p\|_\Omega^2$ . The discrete errors values are  $\varepsilon_0^\nu(\mathbf{u}_h) := \|\mathbf{u} - \mathbf{u}_h\|_\Omega / \|(\mathbf{u}, p)\|_\nu$  for the velocity and  $\varepsilon_0^\nu(p_h) := \nu^{-1} \|p - p_h\|_\Omega / \|(\mathbf{u}, p)\|_\nu$  for the pressure.

We first consider the 2D case and the prescribed solution  $\mathbf{u} = (y, -x)^T$ ,  $p = (x^2 + y^2)/2 - 1/3$  so that  $\mathbf{f} = 0$ . Figure 1 shows  $\varepsilon_0^\nu(\mathbf{u}_h)$  (left) and  $\varepsilon_0^\nu(p_h)$  (right) against

the number of degrees of freedom  $N$ , for  $\nu = 10^{-1}$ ,  $\nu = 10^{-2}$  and  $\nu = 10^{-3}$ . Table

**Fig. 1** 2D case. Plots of  $\varepsilon_0^\nu(\mathbf{u}_h)$  and  $\varepsilon_0^\nu(p_h)$  for  $\nu = 10^{-1}$ ,  $\nu = 10^{-2}$  or  $\nu = 10^{-3}$ .



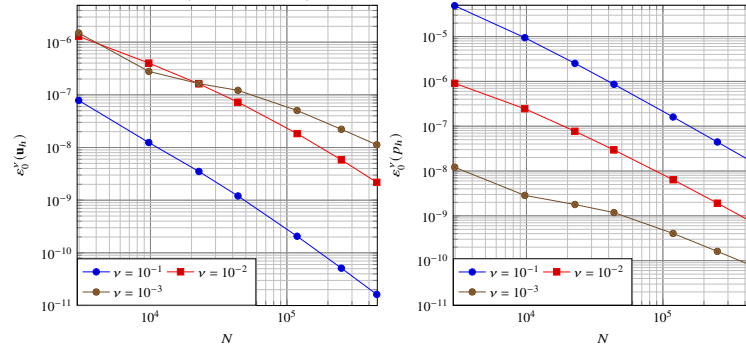
1 shows the convergence rates for the velocity  $\tau_{\mathbf{u}}$  (left) and for the pressure  $\tau_p$  (right), where  $h \approx N^{-1/2}$ . We then consider the 3D case and the prescribed solution

**Table 1** 2D case, convergence rates.

$\nu$	$10^{-1}$	$10^{-2}$	$10^{-3}$	$\nu$	$10^{-1}$	$10^{-2}$	$10^{-3}$
$\tau_{\mathbf{u}}$	3.9	3.3	2.3	$\tau_p$	3.4	3.4	2.5

$\mathbf{u} = (y - z, z - x, x - y)^T$ ,  $p = \frac{1}{2}(x^2 + y^2 + z^2) - xy - xz - yz - 1/4$  so that  $\mathbf{f} = 0$ . Figure 2 shows  $\varepsilon_0^\nu(\mathbf{u}_h)$  (left) and  $\varepsilon_0^\nu(p_h)$  (right) against the number of degrees of freedom  $N$ , for  $\nu = 10^{-1}$ ,  $\nu = 10^{-2}$  and  $\nu = 10^{-3}$ . Table 2 shows the convergence

**Fig. 2** 3D case. Plots of  $\varepsilon_0^\nu(\mathbf{u}_h)$  and  $\varepsilon_0^\nu(p_h)$  for  $\nu = 10^{-1}$ ,  $\nu = 10^{-2}$  or  $\nu = 10^{-3}$ .



rates for the velocity  $\tau_{\mathbf{u}}$  (left) and for the pressure  $\tau_p$  (right), where  $h \approx N^{-1/3}$ . More

**Table 2** 3D case, convergence rates.

$\nu$	$10^{-1}$	$10^{-2}$	$10^{-3}$	$\nu$	$10^{-1}$	$10^{-2}$	$10^{-3}$
$\tau_u$	5.6	4.7	3.3	$\tau_p$	5.2	4.8	3.7

numerical results are available in [8, 9, 6, 21]. The  $\mathbf{P}_{nc}^1 - (P^0 + P^1)$  scheme has good approximation properties and can therefore be used to solve the Navier-Stokes equations in the context of industrial simulation. We plan to study discrete pressure enrichment to higher order nonconforming Crouzeix-Raviart mixed FEM [2, 3].

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