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# Quantum Toda lattice: a challenge for representation theory

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**Abstract.** Quantum Toda lattice may be solved by means of the Representation Theory of semisimple Lie groups, or alternatively by using the technique of the Quantum Inverse Scattering Method. A comparison of the two approaches sheds a new light on Representation Theory and leads to a number of challenging questions.

## 1. Introduction

Modern representation theory of Lie groups was started with the famous paper of Wigner on representations of the Poincaré group and developed by Bargmann and Gelfand and Naimark in the late 1940's and early 1950's. One of the fundamental problems of representation theory is the decomposition of natural representations of a Lie group, e.g. its regular representation, into irreducible components. For non-compact semisimple Lie groups this problem is equivalent to the study of common eigenfunctions of  $G$ -invariant differential operators (the Laplace operators) and has much in common with the quantum mechanical scattering theory. In a series of highly technical papers written in late 1950's Harish Chandra has almost solved the spectral problem associated with the Riemannian symmetric spaces, i.e. the spectral decomposition of  $L_2(G/K)$  [4]; he reduced it to the study of an explicit meromorphic function, *the Harish Chandra  $c$ -function*; in his initial paper Harish Chandra was unable, however, to evaluate this function or even to prove its key analytic properties. It came as a big surprise, a few years later, that the  $c$ -function may be computed explicitly; the famous Gindikin–Karpelevich formula [3] expresses it as a product of functions of one variable, namely, the classical Beta-functions. This fundamental discovery has led to an important paradigm: while the properties of the eigenfunctions of the Laplace operators are very complicated, their asymptotic behaviour appears to be much more simple and is described by factorized formulae reducing it to the rank one case and to meromorphic functions of one variable. This Harish Chandra philosophy culminated in his proof of the Plancherel theorem for non-compact semisimple Lie groups and admits various important generalizations.

Already in 1960's there was established a link between the representation theory of semisimple Lie groups and integrable systems; namely, the Hamiltonian of the so called Sutherland system (a special instance of the famous Calogero–Moser systems) coincides, up to a simple change of variables, with the radial part of the Laplace operator on  $G/K$  [7]. Hence the solution of the associated quantum model is reduced to Harish Chandra's theory of spherical functions. Another important example is the quantum Toda lattice; as noticed by Kostant in 1978 [5],



the Toda Hamiltonian may be identified with the radial part of the Laplace operator associated with the so called Whittaker vectors. In both cases, the eigenfunctions of the integrable models in question are appropriate matrix coefficients of irreducible representations of  $G$ . To prove the spectral theorem one has to study the asymptotic behaviour of these matrix coefficients which again is associated with factorized scattering [10].

In another development, which has started almost simultaneously, a new fundamental technique, the Quantum Inverse Scattering Method, has been developed. It provides a systematic way to construct quantum integrable models and to solve the associated spectral problems [1]. The application of QISM to the quantum Toda lattice was not immediate, but as we realise it now, it brings about a totally new picture of representation theory. The treatment of the Toda lattice based on QISM was started by the seminal papers of Pasquier and Gaudin [8] and Sklyanin [9]; it was continued by Kharchev, Lebedev and Gerasimov [6, 2]. The rather radical change of the old paradigm brought about by QISM consists in the following main points:

- (i) There exists a representation (in the sense of Dirac) in which the algebras of quantum integrals of motion for the Toda lattice, or of the Sutherland system are freely generated by first order difference operators.
- (ii) In this representation the common eigenvectors of these algebras are decomposable, i.e., they are products of functions of one variable.
- (iii) This realization of the algebras of quantum integrals of motion extends to a genuine representation of the full underlying Lie algebra, again acting by difference operators.
- (iv) The transformation operators which relate this representation to the standard coordinate representation may be constructed recursively and amount to a sequence of ordinary Fourier transforms.

While these results were obtained already about 20 years ago, they still remain almost unknown in the representation theory community and their implications for the representation theory of semisimple Lie groups should be a matter of much future work. The purpose of this review talk is to draw attention to this highly promising area.

## 2. A brief reminder

Let us start with a brief reminder of the basic facts on representation theory of semisimple Lie groups. Let  $G$  be a semisimple (or reductive) Lie group,  $\mathfrak{g}$  its Lie algebra,  $\mathcal{Z}_G \subset U(\mathfrak{g})$  the center of its universal enveloping algebra,  $K \subset G$  its maximal compact subgroup.  $\mathcal{Z}_G$  naturally acts in  $L_2(G)$  and commutes with left and right translations; they are usually called generalized Laplace operators. The spectral analysis of Laplace operators in  $L_2(G)$  is tantamount to the decomposition of  $L_2(G)$  into irreducible components.

In general, generalized Laplace operators are not elliptic. The standard way to circumvent this difficulty is to consider  $K$ -finite eigenfunctions, i.e. to combine the spectral decomposition of  $\mathcal{Z}_G$  with the decomposition into  $K$ -irreducible components, in other words, to consider generalized spherical functions. The structure of the joint spectrum of  $\mathcal{Z}$  in  $L_2(G)$  is rather complicated. The discrete spectrum of  $\mathcal{Z}$  is non-empty if and only if  $G$  contains a compact Cartan subgroup. Besides, there are several branches of continuous spectrum; each branch is associated with an equivalence class of cuspidal parabolic subgroups  $P = MAN \subset G$ . The simplest case, which yields the main branch of the continuous spectrum, is associated with minimal parabolics; in that case  $M \subset K$  and the dimension of the continuous spectrum is maximal (it is equal to the split rank of  $G$ ). In this case we may take the trivial representations of both  $M$  and  $K$ ; this gives the class 1 principal series representations of  $G$  and the corresponding spherical functions,

$$\phi_\lambda(x) = \int_K e^{(i\lambda + \rho, H(xk))} dk.$$

Another important matrix coefficient of the principal series representation is the Whittaker function. Let  $B = MAN$  be the minimal parabolic,  $V \subset G$  the opposite unipotent subgroup,  $\psi$  a principal character of  $V$ . Let  $\mathcal{H}$  be a representation space of  $G$ ,  $T$  the corresponding representation. A vector  $w = w_\psi \in \mathcal{H}$  is called a Whittaker vector if

$$T(v)w = \psi(v) \cdot w \quad \text{for all } v \in V.$$

The Whittaker function is the matrix coefficient

$$W_\lambda(g) = \langle T_\lambda(g)w_\psi, \mathbf{1} \rangle_{\mathfrak{h}_\lambda}. \quad (1)$$

Some care is needed, since the Whittaker vector does not lie in  $L_2(V)$ ; one can show that for principal characters  $\psi$  the Whittaker function admits an analytic continuation and is in fact an entire function of  $\lambda$ .

The Whittaker function  $W_\lambda$  satisfies functional equation

$$W_\lambda(kgv) = \psi(v)W_\lambda(g) \quad \text{for all } k \in K, v \in V; \quad (2)$$

The standard integral representation for  $W_\lambda$  is quite similar to (2),

$$W_\lambda(g) = \int_V e^{\langle i\lambda + \rho, H(vg) \rangle} \psi(v) dv \quad (3)$$

Spherical functions satisfy a simple functional equations which bring into play the fundamental symmetry with respect to the Weyl group,

$$\phi_\lambda(x) = \phi_{s\lambda}(x); \quad (4)$$

one can show that after a proper normalization the Whittaker functions satisfy the same equation. Both spherical and Whittaker functions are completely determined by their restrictions to  $A = \exp \mathfrak{a}$ . Radial parts of generalized Laplace operators induced by this restriction give rise to two famous quantum integrable systems. Specifically, for  $G = GL(N)$  let us put  $H_S = J^{-1}\Delta_A J$ , where we use the parametrization  $a = e^x, x \in \mathfrak{a}$ , for the split Cartan subgroup and put

$$J = \prod_{k < j} \sinh^{1/2}(x_k - x_j)$$

Then

$$H_S = -\Delta + \sum_{k < j} \frac{1}{\sinh^2(x_k - x_j)}. \quad (5)$$

This is the so called *Sutherland Hamiltonian* describing a system of  $N$  particles on the line. Thus up to a simple factor the eigenfunctions of  $H_S$  coincide with *spherical functions* on  $G$ .

In a similar way the radial part of the Laplace–Beltrami operator associated with Whittaker functions yields the Toda lattice Hamiltonian, as noticed by Kazhdan and Kostant in 1978.

$$H_T = -\Delta + V(x), \quad V(x) = \sum_n e^{x_n - x_{n+1}}. \quad (6)$$

The configuration space of the Toda lattice is identified with the Cartan subalgebra of  $gl(N)$ ; linear functions  $\alpha_n(x) = x_n - x_{n+1}$  which appear in the potential are its simple roots. In the periodic case one more root  $\alpha_0(x) = x_N - x_1$  is added; this brings into play the affine root system  $\tilde{A}_{N-1}$ .

While the integral representations for the spherical and Whittaker functions do not admit any simplification, their asymptotic behaviour for large values of the argument is much more simple. In particular, we have

$$e^{\rho(x)}\phi_\lambda(e^x) \sim \sum_{w \in W} c(w\lambda)e^{iw\lambda(x)}, \quad x \in \mathfrak{a}, \tag{7}$$

where the sum is taken over the Weyl group and  $c(\lambda)$  is the Harish-Chandra  $c$ -function defined by the integral

$$c(\lambda) = \int_V e^{\langle i\lambda + \rho, H(v) \rangle} dv, \tag{8}$$

where again  $V = \bar{N}$  is the opposite nilpotent subgroup and  $H(v)$  is the split Cartan component in the Iwasawa decomposition  $v = ke^{H(v)}n$ . This integral is explicitly given by the Gindikin–Karpelevich formula,

$$c(\lambda) = \prod_{\alpha \in \Delta_+} c_\alpha(\lambda), \quad c_\alpha(\lambda) = \frac{\Gamma(i\lambda_\alpha)\Gamma(1/2)}{\Gamma(i\lambda_\alpha + 1/2)}, \quad \lambda_\alpha = \frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle}, \quad i = \sqrt{-1}. \tag{9}$$

The asymptotics (7) holds for large  $x \in \mathfrak{a}_+$  lying inside the positive Weyl chamber; a similar asymptotics holds for the Whittaker functions when  $x$  lies in the Weyl chamber where  $x_n - x_{n+1} \rightarrow -\infty$  for all  $n$ . As already noticed, this simple asymptotic behaviour is the basis of the old paradigm in Representation theory referred to above.

### 3. Introduction to QISM

Let  $q_m, p_m, m = 1, \dots, n$ , be canonical position and momentum operators acting in a Hilbert space  $\mathfrak{H}$  with commutation relations

$$[p_j, q_k] = -i\delta_{jk}, \quad i = \sqrt{-1}.$$

We introduce a family of  $2 \times 2$ -matrices with operator entries depending on a complex parameter  $u$ ,

$$L_m(u) = \begin{pmatrix} u - p_m & -e^{q_m} \\ e^{-q_m} & 0 \end{pmatrix}. \tag{10}$$

The space  $\mathfrak{H}$  is commonly referred to as the *quantum space* and the space  $\mathbb{C}^2$  as the *auxiliary space*. The success of the QISM machinery relies on a clever way to encode the commutation relations in the quantum space by an appropriate linear transformation in the auxiliary space. Let  $R(u)$  be a linear operator acting in  $\mathbb{C}^2 \otimes \mathbb{C}^2$ ,

$$R(u) = uI - iP, \tag{11}$$

where  $I$  is the identity operator and  $P \in \text{End}(\mathbb{C}^2 \otimes \mathbb{C}^2)$  is the permutation operator,  $P(v \otimes w) = w \otimes v$ .

We have:

(i) 
$$[p_m, e^{\pm q_k}] = \mp i e^{\pm q_k} \delta_{mk}, \quad i = \sqrt{-1}. \tag{12}$$

(ii) Put  $L_m^1(u) = L_m(u) \otimes I, L_m^2(u) = I \otimes L_m(u)$ ; we regard  $L_m^1(u), L_m^2(u)$  as operators in  $\text{End}(\mathbb{C}^2 \otimes \mathbb{C}^2) \otimes \text{End}(\mathfrak{H})$ . The commutation relations (12) are equivalent to the operator identities

$$L_m^2(v)L_m^1(u) = R(u - v)L_m^1(u)L_m^2(v)R(u - v)^{-1}, \quad L_m^1(u)L_k^2(v) = L_k^2(v)L_m^1(u) \text{ for } m \neq k.$$

(iii) Let  $T_N(u) = L_N(u)L_{N-1}(u) \dots L_1(u)$  be the monodromy matrix associated with ‘local’ Lax matrices  $L_m(u)$ ; we put as usual  $T^1(u) = T(u) \otimes I$ ,  $T^2(u) = I \otimes T(u)$ . The monodromy matrices satisfy

$$T_N^2(v)T_N^1(u) = R(u-v)T_N^1(u)T_N^2(v)R(u-v)^{-1}. \tag{13}$$

(iv) We write

$$T_N(u) = \begin{pmatrix} A_N(u) & B_N(u) \\ C_N(u) & D_N(u) \end{pmatrix}, \quad t(u) = \text{tr} T_N(u) = A_N(u) + D_N(u). \tag{14}$$

Commutation relations (13) imply that

$$[t(u), t(v)] = 0. \tag{15}$$

Clearly,  $t(u)$  is a polynomial in  $u$ ,

$$t(u) = u^N + t_1u^{N-1} + t_2u^{N-2} + \dots,$$

where in particular  $-t_1 = P_N = p_1 + \dots + p_N$  is the full momentum and

$$t_2 = \frac{1}{2}P_N^2 - \frac{1}{2}(p_1^2 + \dots + p_N^2) - e^{x_2-x_1} - \dots - e^{x_N-x_{N-1}} - e^{x_1-x_N}. \tag{16}$$

is the Hamiltonian of the *periodic Toda lattice* of length  $N$ .

For applications to *open Toda lattice* (and hence eventually to Representation theory) we need other coefficients of the monodromy matrix. Commutation relations (13) imply that

$$\begin{aligned} [A_N(u), A_N(v)] &= [B_N(u), B_N(v)] = [C_N(u), C_N(v)] = [D_N(u), D_N(v)] = 0, \\ (u-v+i)A_N(v)C_N(u) &= (u-v)C_N(u)A_N(v) + iA_N(u)C_N(v). \end{aligned} \tag{17}$$

We can write

$$A_N(u) = u^N + \sum_{m=1}^N X_m u^{N-m}, \quad D_N(u) = \sum_{m=2}^N Y_m u^{N-m}, \tag{18}$$

where  $X_m, Y_m$  are two sets of mutually commuting operator coefficients. The coefficients  $X_1 = -P$  and  $X_2 = \frac{1}{2}P^2 - H$  give the total momentum and the Hamiltonian for the open Toda lattice; the coefficients  $Y_m$  add up to  $X_m$  to yield the quantum integrals of motion for the periodic Toda.

Moreover,

$$A_N(u) = (u - p_N)A_{N-1}(u) + e^{-x_N}C_{N-1}(u), \quad C_N(u) = e^{-q_N}A_{N-1}(u). \tag{19}$$

Note that  $A_{N-1}(u)$ , which is the generating function for the Hamiltonians of the open Toda lattice of length  $N - 1$ , commutes with  $q_N, p_N$ . Clearly,  $C_N(u)$  is a polynomial in  $u$  and hence it may be written in factorized form,

$$C_N(u) = e^{-q_N} \prod_{m=1}^{N-1} (u - \hat{\lambda}_m), \tag{20}$$

where  $\hat{\lambda}_m$ ,  $m = 1, \dots, N - 1$ , are mutually commuting self-adjoint operators (“the operator-valued roots of  $C_N(u)$ ”).

We are now able to introduce our key definition:

- Operators  $\hat{\lambda}_m$  are called quantum separated variables
- Spectral representation for these operators is called separation representation.

In order to justify this definition we need to introduce some operator calculus. We define operators  $\Lambda_j^\pm, j = 1, \dots, N - 1$  by setting

$$\Lambda_j^- = A_N(u)|_{u=\hat{\lambda}_j} = \hat{\lambda}_j^N + \sum_{m=1}^{N-2} X_m \hat{\lambda}_j^{N-2-m}, \quad \Lambda_j^+ = D_N(u)|_{u=\hat{\lambda}_j} = \sum_{m=2}^N Y_m \hat{\lambda}_j^{N-m}, \quad (21)$$

where  $X_m, Y_m$  are defined in (18). Note that  $X_m, Y_m$  do not necessarily commute with  $\hat{\lambda}_j$ ; the definition (21) fixes the ‘left ordering’ of the coefficients.

The coefficients of the monodromy matrix may be restored from  $\hat{\lambda}_j, \Lambda_j^\pm$  by means of the Lagrange interpolation formulae,

$$A_N(u) = \left( u - P + \hat{\lambda}_1 + \dots + \hat{\lambda}_{N-1} \right) \prod_{j=1}^{N-1} (u - \hat{\lambda}_j) + \sum_{j=1}^{N-1} \left( \prod_{\substack{k=1 \\ k \neq j}}^{N-1} \frac{u - \hat{\lambda}_k}{\hat{\lambda}_j - \hat{\lambda}_k} \Lambda_j^- \right), \quad (22)$$

$$D_N(u) = \sum_{j=1}^{N-1} \left( \prod_{\substack{k=1 \\ k \neq j}}^{N-1} \frac{u - \hat{\lambda}_k}{\hat{\lambda}_j - \hat{\lambda}_k} \Lambda_j^+ \right).$$

Basic commutation relations (13), (17) imply that

$$\left[ \hat{\lambda}_j, \hat{\lambda}_k \right] = \left[ \Lambda_j^\pm, \Lambda_k^\pm \right] = 0, \quad \Lambda_j^\pm \hat{\lambda}_k = \left( \hat{\lambda}_k \pm i \delta_{jk} \right) \Lambda_j^\pm, \quad i = \sqrt{-1}. \quad (23)$$

Thus in the spectral representation for  $\hat{\lambda}_k$  the operators  $\Lambda_j^\pm$  act as translation operators affecting only one variable. Explicitly, this spectral representation is given by

$$\begin{aligned} \hat{\lambda}_j f(p, \lambda_1, \lambda_2, \dots, \lambda_{N-1}) &= \lambda_j \cdot f(k, \lambda_1, \lambda_2, \dots, \lambda_{N-1}), \\ \Lambda_j^\pm f(p, \lambda_1, \lambda_2, \dots, \lambda_{N-1}) &= i^{\pm N} f(k, \lambda_1, \lambda_2, \dots, \lambda_j \pm i, \dots, \lambda_{N-1}); \end{aligned} \quad (24)$$

where we have added one more variable  $p$ , the eigenvalue of the total momentum  $P$ . The phase factor  $i^{\pm N}$  is introduced for convenience. Together with the conjugate variable  $x_N$  the momentum  $P$  completes the set of independent variables to yield a representation of the rank  $N$  Heisenberg algebra.

The action of  $P, x_N$  in the spectral representation is given by

$$\begin{aligned} P f(p, \lambda_1, \lambda_2, \dots, \lambda_{N-1}) &= p f(p, \lambda_1, \lambda_2, \dots, \lambda_{N-1}), \\ e^{-x_N} f(p, \lambda_1, \lambda_2, \dots, \lambda_{N-1}) &= f(p - i, \lambda_1, \lambda_2, \dots, \lambda_{N-1}). \end{aligned} \quad (25)$$

*Remark.* The roots of the polynomial equation (20) are defined only up to a permutation; it is thus natural to assume that functions  $f$  in the spectral representation (24) are symmetric with respect to the variables  $\lambda_j$ . With this symmetry condition imposed the Toda wave functions will automatically satisfy the simple functional equation (4).



### 3.1. Toda wave functions in separation representation.

In the separation representation the simultaneous diagonalization of the quantum integrals of motion for the open Toda lattice is reduced to a system of difference equations. We can parameterize the common eigenfunctions  $\psi_\alpha$ ,  $\alpha = (\alpha_1, \dots, \alpha_N)$  so that

$$A_N(u)\psi_\alpha = \prod_{j=1}^N (u - \alpha_j) \cdot \psi_\alpha. \quad (26)$$

The commutative pencil  $A(u)$  is completely determined by its values  $A(\lambda_j) = \Lambda_j^-$ ,  $j = 1, \dots, N-1$  together with the value of the total momentum. Since in the separation representation the total momentum acts as multiplication operator, the common eigenfunctions of  $A(u)$  have the form

$$\psi_\alpha(P, \lambda_1, \dots, \lambda_{N-1}) = \delta(P - \alpha_1) \phi_\alpha(\lambda_1, \dots, \lambda_{N-1}), \quad (27)$$

where  $\phi_\alpha$  satisfies the system of difference equations

$$\Lambda_j^- \phi_\alpha(\lambda_1, \dots, \lambda_{N-1}) = \prod_{k=1}^N (\lambda_j - \alpha_k) \phi_\alpha(\lambda_1, \dots, \lambda_{N-1}), \quad j = 1, \dots, N-1. \quad (28)$$

The explicit solution of this system is given by

$$\phi_\alpha(\lambda_1, \dots, \lambda_{N-1}) = \prod_{j=1}^{N-1} \prod_{k=1}^N \Gamma\left(\frac{\lambda_j - \alpha_k}{i}\right). \quad (29)$$

In order to complete the solution of the open Toda lattice we must describe the transformation operator which relates the separation representation to the standard coordinate representation (in which the Toda Hamiltonian is a 2nd order differential operator). Its description is already implicit in formula (19) which states that the diagonalization of the operators  $C_N(u)$  is equivalent to the solution of the rank  $N-1$  open Toda lattice (up the introduction of an extra coordinate variable). Speaking accurately, the wave functions which correspond to the operators with continuous spectrum (of course, such wave functions do not lie in the Hilbert space) should be interpreted as distribution kernels of transformation operators which rely different sets of quantum observables. Following Dirac, we shall label different “coordinate systems” in the space of quantum states (alias, quantum representations) by indicating the complete sets of observables which are diagonal, i.e., act by multiplication. The relevant representations are

- Coordinate representation in which the diagonal operators are the standard coordinates.
- Separation representation which diagonalizes operator roots  $\hat{\lambda} = (\hat{\lambda}_1, \dots, \hat{\lambda}_{N-1})$  of  $C_N(u)$ .
- Spectral representation operator which diagonalizes roots of  $A_N(u)$ .

In Dirac notation the wave function which diagonalizes the open Toda Hamiltonians may be written simply as  $\psi_\alpha(x_1, \dots, x_N) = \langle x | \alpha \rangle$ . In a similar way, the notation for  $\Psi_{\lambda, \lambda_N}(x_1, \dots, x_N)$  may be condensed to

$$\Psi_{\lambda, \lambda_N}(x_1, \dots, x_N) = \langle x | \lambda \rangle$$

with  $\lambda = (\lambda_1, \dots, \lambda_N)$ . Finally, the wave function (27) in separation representation is condensed to

$$\psi_\alpha(P, \lambda_1, \dots, \lambda_{N-1}) = \langle \lambda, P | \alpha \rangle.$$

This interpretation immediately yields the following integral formula for the wave function of the open Toda lattice in the coordinate representation:

$$\psi_\alpha(x_1, \dots, x_N) = \int e^{(iP-i\sum_{m=1}^{N-1}\lambda_m)x_N} \prod_{j=1}^{n-1} \prod_{k=1}^N \Gamma\left(\frac{\lambda_j - \alpha_k}{i}\right) \cdot \psi_\lambda(x_1, \dots, x_{N-1}) \mu(\lambda) d\lambda dP. \quad (30)$$

The measure density  $\mu(\lambda)$  is determined by the symmetry conditions for the monodromy,  $T_N^*(u) = T_N(\bar{u})$  which leads to a system of finite difference equations for  $\mu$  [9]; its explicit solution is given by

$$\mu(\lambda_1, \dots, \lambda_{N-1}) = \prod_{\substack{k=1 \\ j < k}}^{N-1} \frac{1}{\left| \Gamma\left(\frac{\lambda_j - \lambda_k}{i}\right) \right|^2}. \quad (31)$$

Formula (31) is equivalent to the Gindikin-Karpelevich formula, although its derivation is totally different. From (30) we immediately get a simple recursion which ends up with an explicit formula for  $\psi_\alpha(x_1, \dots, x_N)$  in terms of multiple Fourier transforms of an aggregate composed of Gamma functions. This is the famous Mellin–Barnes type formula first obtained by Kharchev and Lebedev [6]. In order to write down this formula explicitly let us rearrange the integration variables into a triangular array  $\lambda_n = (\lambda_{n1}, \lambda_{n2}, \dots, \lambda_{nn})$ ,  $n = 1, \dots, N$ ; the upper row  $(\lambda_{N1}, \lambda_{N2}, \dots, \lambda_{NN})$  is identified with our spectral variables  $\alpha = (\alpha_1, \dots, \alpha_N)$ . With this notation we get

$$\psi_{\lambda_N}(x_1, \dots, x_N) = \int \prod_{n=1}^{N-1} \frac{\prod_{k=1}^n \prod_{m=1}^{n+1} \Gamma\left(\frac{\lambda_{nk} - \lambda_{n+1,m}}{i}\right)}{\prod_{\substack{s \neq p, \\ s, p \leq n}} \Gamma\left(\frac{\lambda_{ns} - \lambda_{n,p}}{i}\right)} \exp\left\{i \sum_{n=1}^N \sum_{k=1}^{n-1} (\lambda_{nk} - \lambda_{n-1,k}) x_n\right\} \prod_{\substack{n=1 \\ j \leq n}}^{N-1} d\lambda_{nj}. \quad (32)$$

Formula (32) represents the main result of the QISM technique as applied to the open Toda lattice. Extending the fundamental result (29), Gerasimov, Kharchev and Lebedev have shown that the entire Lie algebra  $\mathfrak{g}$  admits a realization by first order difference operators (acting in an appropriate space of meromorphic functions of several variables) In this realization both spherical and Whittaker vectors are factorizable, hence the spherical functions also admit a Mellin–Barnes type representation similar to (32). Explicit description of these results may be found in [6], [11]. A generalization of these results to other irreducible representations of  $\mathfrak{g}$  induced from cuspidal parabolics and to semisimple Lie groups over various local and global fields remains unknown and represents a major challenge for the representation theory.

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