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Elementary students demonstrate proportional and protoquantitative reasoning in models for sharing costs fairly

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Mathematical modelling is a process in which students investigate authentic problems and everyday situations using mathematics. In these non-routine tasks, students bring their multiple mathematical knowledge bases and cultural funds of knowledge into their solution strategies. During a modelling task called “Abuelo’s Birthday”, 297 elementary students in grades 3-5 decided how to split the cost of a gift “fairly” and justified their work using early notions of proportional reasoning. We aligned their strategies with previous work on student understanding of proportional reasoning and found that most students consistently attended to fittingness and covariation principles. We argue that these young students were successful with a complex modelling task due to the way the realistic context connected to students’ lived experiences and funds of knowledge.

Keywords: Elementary school education, mathematical modelling, ratio & proportional reasoning.

Conceptual framework

Mathematical modelling is internationally regarded as a beneficial mathematical practice for students across all grade levels (Schukajlow et al., 2018; Sriraman & English, 2010; Verschaffel & De Corte, 1997). At the elementary level, open-ended mathematics tasks that align with the modelling process can connect to children’s mathematical thinking in a variety of content areas, such as ratio and proportional reasoning. Moreover, the real-world contexts of modelling tasks can connect to disciplines and situations outside of the mathematics classroom. In the following sections, we explore how the mathematical modelling process, students’ funds of knowledge, and proportional reasoning strategies intersect in one rich task.

Mathematical modelling

Mathematical modelling is defined in multiple ways in research literature and education standards. Modelling is a process for connecting the real world to the world of mathematics (Blum & Borromeo Ferri, 2009), or “a process in which students consider and make sense of an everyday situation that will be analyzed using mathematics for the purpose of understanding, explaining, or predicting something” (Anhalt et al., 2018, p. 202). During the modelling process, authentic situations are made sense of, simplified, modelled, analyzed, interpreted, and generalized. Tasks that encourage the modelling process are open-ended tasks that foreground diverse solution strategies and connections to multiple mathematical knowledge bases (Turner et al., 2012). In this way, mathematical modelling benefits students in multiple ways, including fostering creativity, problem solving, sense-making, and communication (e.g., Chamberlin et al., 2022). At the elementary level, research has shown that mathematical modelling is accessible to young children and to students from a diverse range of

mathematical and cultural backgrounds (e.g., English & Watters, 2005; Sawatzki et al., 2019; Verschaffel & De Corte, 1997).

Proportional and protoquantitative reasoning

Research literature documents elementary and middle grades students' struggles with ratio and proportional reasoning problems, illustrating how students sometimes rely on additive rather than multiplicative reasoning (e.g., Lo & Watanabe, 1997; Steinhorsdottir & Sriraman, 2009; Van Dooren et al., 2010). Proportional reasoning is typically introduced in middle (6-8) grades, yet students have limited exposure to this notion of multiplicative reasoning prior to this (e.g., Sawatzki et al., 2019). Thus, it is crucial to introduce proportional reasoning in upper elementary (grades 3-6), particularly through real-world contexts (Askew, 2018; Sawatzki et al., 2019).

Researchers have explored how children as young as first grade (about age 6) can understand key aspects of proportional reasoning, such as the Fittingness Principle and Covariation (Askew, 2018; Resnick & Singer, 1993). Fittingness refers to the notion that two (or more) things go together because their sizes or amounts are appropriate for one other. When one quantity exists in an ordered series, it will covary, directly or inversely, with the other ordered quantity. This early reasoning is sometimes called *protoquantitative* because it describes qualities or sets of objects, rather than precisely determining equivalent ratios with numeric values (Resnick & Singer, 1993).

In other words, young children can reason protoquantitatively using fittingness and covariation schemas. Early studies on proportional reasoning showed that protoquantitative multiplication does not appear in children's strategies until around age 10 (Resnick & Singer, 1993). Thus, elementary teachers and curricula should consider different approaches to help children develop flexible command of multiplicative structures earlier, such as including non-routine tasks (i.e., not missing-value problems) (Van Dooren et al., 2010) or discussing multiple solutions to story problems (Resnick & Singer, 1993). For instance, in a problem about sharing fairly, students may consider nonmathematical aspects of the situation to make sense of the quantities and their relations.

To this end, we pose the following two research questions: (1) How do elementary students demonstrate informal and intuitive notions of proportional reasoning through a modelling task? And (2) How do elementary students use proportional reasoning strategies to justify their notion of fairness in the context of solving a modelling task?

Methods

Setting and participants

The research presented here is part of a multi-institutional project "Mathematical Modeling with Cultural and Community Contexts" in two diverse regions of the U.S. We worked with teachers and students at elementary schools that served multi-racial, multilingual, multicultural, and working-class communities. This project was motivated by prior research claiming that mathematics interest and learning improves when students can draw on their knowledge and experiences and connect classroom mathematics to their everyday lives (e.g., Turner et al., 2012). We drew on elementary students' cultural funds of knowledge to create modelling tasks that fostered their mathematical thinking and their community-based ways of learning (e.g., Civil, 2002).

Teachers participated in a week-long summer workshop and ongoing monthly professional development sessions over two one-year cycles. During workshops, teachers engaged in solving and discussing modelling tasks; the Abuelo's Birthday task was one task out of several that teachers were introduced to and encouraged to use in their classrooms. In this task, students must create a fair plan for four grandchildren to share the costs of a birthday gift for their grandfather. The children are of different ages and have different weekly earnings, which challenges students to formulate models that differ from dividing the total cost into four equal parts (see Figure 1).

Abuelo's Birthday Task (adapted from Aguirre & Zavala, 2013)

It is Señor Aguirre's 70th birthday. Four of his grandchildren want to buy him a photo printer so he can print photos of his family members. They found the printer on sale for \$120 (USD).

Alex, a 9th grader, earns \$15-\$20 each week from babysitting.

Sam, a 6th grader, earns \$10 each week taking care of his neighbor's pets.

Elena, a 4th grader, earns \$5 each week doing jobs for an aunt.

Jaden, a 1st grader, has no weekly job but has saved \$8 in her piggy bank.

One of the grandchildren says they should split the cost of the printer between them and each pay the same amount. Another grandchild says that is not fair and they should each pay different amounts.

What do you think? What is fair in this situation?

Figure 1. The task prompt for the Abuelo's Birthday Task

The Abuelo's Birthday task is a complex task for elementary students because it relies on proportional reasoning, which they likely have not been introduced to in formal classroom settings. In Lamon's (1993) semantic classification of proportional reasoning problems, this task is most similar to the "associated sets" and "well-chunked measure" (here, income or "earnings" is a well-known descriptor of the rate of money per time) types of problems. However, because the task involves the abstract quantity of income, it can be more challenging for students to represent the relationship with higher-level proportional strategies (Lamon, 1993).

The Abuelo's Birthday task is not a traditional missing value task, where students need to set up and solve a single proportion; rather, it involves multiple relationships between quantities that covary. This structure requires students to create composite units, or "units of units" (Lamon, 1993), and decide how to iterate them for an unspecified amount of time. The openness of this task adds to its complexity, as students must make assumptions and decisions about how much each child should contribute and how long they will earn money toward the gift. The flexibility of strategies and solutions encourages students to engage in the modelling process, constantly making sense of, formulating, analyzing, and interpreting their model for "fairness".

Data collection and analysis

For this paper, we focus on the dataset of classroom artifacts from the Abuelo's Birthday task, which consists of individual and group posters from student work during class. We collected artifacts from 297 students in grades 3-5 (ages 8-11) across 14 classrooms in two diverse regions of the U.S. We analyzed 119 student work samples, often in the form of group posters, and excluded instances where teachers altered the original task for their classrooms or the student work was illegible. We drew on Hatch's (2002) notion of artifact analysis to engage in multiple rounds of coding student work for

their mathematical strategies and other salient features of their solutions, such as explanations of “fairness” or connections to funds of knowledge. We initially used Lamon’s (1993) framework for proportional strategies as a guide, but after iterative coding and discussions, we adapted our categories to more accurately describe the strategies we observed (see Table 1).

Code	Description
All Equal	The three older kids or all four kids contribute the same amount, with no connection to earnings; this can include Jaden or not.
All Equal with Adjustments	Initially used strategy “All equal” where everyone paid the same amount (either all \$30 with Jaden, or older three all pay \$40). Then, they realized that this will not work and made some adjustments to the existing strategy; they do not just start over with a new strategy (e.g., they see that Jaden cannot pay her \$30 and have Alex “cover” it).
“Roughly” Proportional	Must abide by protoquantitative principles, where children’s contributions are in the order of their earnings. May or may not include the number of weeks it takes to earn the money; includes solutions that start out using some proportional thinking, but final solution is not proportional due to major adjustments.
Proportional with Adjustments	Initially used a Proportional strategy, then adjusted plan (in a minor way) to make numbers friendly or sum to \$120 (e.g., all 3 older kids contribute for 3 weeks, and then Elena doesn’t contribute for week 4). Or, used a Proportional strategy but then re-distributed the excess earnings back to the grandchildren disproportionately.
Proportional	The three older kids’ contributions must be proportional to their earnings. Can use addition or multiplication and any representation; this can include Jaden or not. Solution should not include errors in the setup or selection of amounts.

Table 1: Codebook for classifying students’ proportional strategies

Findings

We found that 90 of the 119 student work samples (76%) showed evidence of protoquantitative reasoning, attending to both the Covariation of Units and the Fittingness Principle (Resnick & Singer, 1993). As described above, the Abuelo’s Birthday task is a complex proportional reasoning task; thus, it is significant that the majority of elementary students in our study were successful with proportional strategies. Table 2 presents the distribution of strategies across grades 3-5. It is noteworthy that students in all grade levels used proportional strategies.

In the following subsections, we describe select student work samples by math strategy. We focus on the three strategies that demonstrated proportional (or protoquantitative) reasoning: Proportional, Proportional with Adjustments, and Roughly Proportional. These samples were chosen as representatives of the variation within the math strategies, as well as representatives of the ways students drew on their funds of knowledge and mathematized fairness. An important assumption of this work is that students are always drawing on their experiences as they learn, but in some cases, such as these posters of student work, we have explicit evidence.

	GRADE 3	GRADE 4	GRADE 5	TOTAL
PROPORTIONAL	10	15	12	37
PROPORTIONAL W/ ADJUSTMENTS	9	15	1	25
ROUGHLY PROPORTIONAL	18	10	0	28
ALL EQUAL W/ ADJUSTMENTS	0	2	2	4
ALL EQUAL	1	10	1	12
OTHER	4	7	2	13
TOTAL	42	59	18	119

Table 2: Frequency of math strategies across all student work artifacts

Proportional

In Proportional strategies, the amount that the three older grandchildren contribute must be proportional to their weekly earnings. One key variation within this strategy was how students decided to include Jaden or not in their model. In an example from a group of third graders (see the left image in Figure 2), Jaden's money was not included in the plan ("Jaden will not pay."), and so the older three children contributed proportional to their earnings over four weeks to earn exactly \$120. This poster clearly demonstrated a two-step process in making sense of the units involved, with Alex's, Sam's, and Elena's weekly earnings in the vertical bar diagrams and the subtotals for each week shown across the rows of the calendar. They create a unit of units, or composite unit, to iterate over the four weeks. This is a sophisticated proportional approach to solving this problem because it involves multiple uses of rates to reach the total.

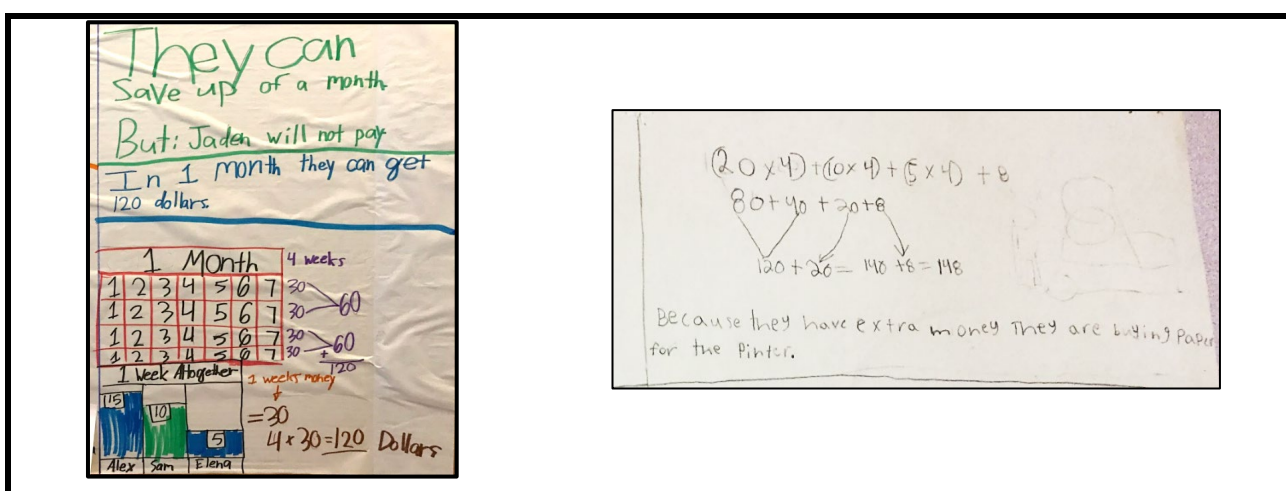


Figure 2. Two proportional strategies to share the cost of the gift fairly.

In a zoomed-in view of another poster (see the right image in Figure 2), grade 3 students represented their model with an equation; the parentheses emphasized their understanding that each term represented the contributions of a different grandchild. Elsewhere on their poster, they stated their assumptions: "I assumed that taxes are included. They have been saving for 4 weeks." This group chose to include Jaden's savings, and their equation demonstrated understanding that the \$8 represents a constant amount, not a rate of earnings per week. The total earnings for four weeks summed to \$148, exceeding the required \$120, so this group decided that the grandchildren "are buying paper for the printer". Their realistic considerations connect to their funds of knowledge, which led to incorporating additional elements into the task (i.e., tax and paper for the printer). In this example, the elements in addition to the task prompt helped students make sense of the remainder and interpret their solution in the context of the gift-buying situation.

Proportional with adjustments

In the Proportional with Adjustments category, students started with a proportional approach and then adjusted in a minor way, often referring to the realistic elements of the task context. One common strategy in this category involved decreasing the amount that Elena (the second youngest child) contributed so that Jaden could be included. One group of grade 4 students clearly wrote, "Fair: Little

kids pay less. Big kids pay more,” which emphasizes their understanding of fittingness. In their model, Alex and Sam contributed earnings for four weeks (\$60 and \$40, respectively), but Elena only contributed for three weeks (\$15) to allow Jaden to contribute \$5 of her savings. This group made the assumption that it is important for all grandchildren to contribute, even if this makes the plan no longer proportional.

Contrastingly, another group of grade 4 students had Elena pay disproportionately more than the other kids. This group decided that they needed to reach a total of \$133 to purchase balloons, wrapping paper, and a birthday card, as well as cover sales tax. Although their initial strategy was proportional with the three older children contributing over four weeks (\$60, \$40, and \$20, respectively), they decided to have Elena work for an extra week. They reasoned that Elena’s additional \$5 with Jaden’s saved \$8 gave them enough money to cover the additional elements that they brought to this task, likely connecting to their experiences with buying a gift for a party.

Roughly proportional

The Roughly Proportional strategy aligned with protoquantitative reasoning (Resnick & Singer, 1993), and in this way, was stronger than some of Lamon’s (1993) pre-proportional strategies. Student groups who exhibited this strategy clearly expressed their notion of fairness to accompany their mathematical work. One group of grade 4 students wrote, “They pay half their payment until they pay \$120. *It is not fair* if they split it up [equally] because Jaden only has \$8.” In this model, the children contributed (roughly) half of their weekly earnings, likely because they want to save some of their earnings to buy other things. This assumption is not explicitly stated but is a possible implicit idea from student experiences with family savings. The students acknowledged that their model requires more weeks to earn the money since they save some of their earnings each week.

Another group of grade 4 students wrote, “*The most fair plan* is they pay different amounts of money. I think this because Alex has more money than all the siblings and the other siblings will take longer [to earn the money] than Alex.” In their solution, the children’s contributions follow the same order of their earnings, but Alex pays disproportionately more than the others. The students justified their decision: “I think this because Alex is the oldest and every week he gets \$20.” They reasoned that it is fair for the oldest sibling to contribute most and work for more time than the other children. Rather than focusing on the lack of proportionality, we noted the attention to the Fittingness Principle and the students’ notion that the oldest child takes on more responsibility, likely influenced by family experiences with siblings and cousins.

Discussion and conclusion

We reiterate our finding that 90 of the 119 student work samples (76%) showed evidence of protoquantitative reasoning, attending to both the Fittingness Principle and the Covariation of Units (Resnick & Singer, 1993). This demonstrates that many young children enter the mathematics classroom with an understanding of quantity and covariation, which lays the foundation for multiplicative reasoning and proportional strategies. The task structure of listing grandchildren of different ages may have strengthened the notion of covariation, since the order of their ages aligned with the order of their earnings. Due to the open-ended nature of the Abuelo’s Birthday task, students were able to draw on multiple strategies, representations, and prior mathematics concepts while

attending to a realistic situation. This highlights the ability of tasks that are aligned with the modelling process to assess and revisit previous concepts but also introduce and build intuitive understanding of new math concepts. Our task, rooted in the authentic situation of sharing costs fairly, elicited informal notions of proportional reasoning that students had not experienced in formal classroom instruction (see Sawatzki et al., 2019). This is consistent with prior research claiming that mathematics problems involving realistic contexts help students develop deeper and stronger mathematical understandings (Lo & Watanabe, 1997; Verschaffel & De Corte, 1997).

Students clearly communicated their model and justifications in the form of assumptions or sense-making about remainders (see Verschaffel, & De Corte, 1997). In our analysis of different strategies, we highlighted the strengths of students' work, rather than concluding that students were not demonstrating (formal) proportional reasoning. While students' mathematical solutions were justified by multiplicative and proportional strategies, their rationales were not exclusively mathematics-based, suggesting they were drawing on other mathematical knowledge bases (Turner et al., 2012). Most of the student artifacts that exhibited evidence of students' funds of knowledge also included Jaden in their plans. We claim that the non-routine structure of the Abuelo's Birthday task (i.e., including Jaden as a non-proportional component) added complexity to the task in a way that students had to draw on their experiences to make sense of. Our study provides evidence that young children across diverse cultural and linguistic backgrounds are capable of sophisticated mathematical reasoning while solving complex modelling tasks. We recommend further development of non-routine tasks that elicit notions of fittingness and covariation in elementary settings, as well as tasks within existing proportional reasoning lessons that draw authentically on students' prior knowledge and everyday lives.

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