# Optimal Stopping of Branching Diffusion Processes

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**Abstract**

This article explores an optimal stopping problem for branching diffusion processes. It consists in looking for optimal stopping lines, a type of stopping time that maintains the branching structure of the processes under analysis. By using a dynamic programming approach, we characterize the value function for a multiplicative cost that depends on the particle's label. We reduce the problem's dimensionality by setting a branching property and defining the problem in a finite-dimensional context. Within this framework, we focus on the value function, establishing polynomial growth and local Lipschitz properties, together with an innovative dynamic programming principle. This outcome leads to an analytical characterization with the help of a nonlinear elliptic PDE. We conclude by showing that the value function serves as the unique viscosity solution for this PDE, generalizing the comparison principle to this setting.

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# 1 Introduction

Since its introduction in the late sixties in [16, 14, 15, 28], the class of branching diffusion processes received a great deal of interest. This object is used to describe the evolution of a population where

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we are interested in a special feature, *e.g.*, the spatial motion, of identical particles that reproduce at random times.

These processes are well-suited in capturing a dual level of interaction. A macroscopic dynamics, marked by the branching aspect, is connected to a microscopic one, characterized by a stochastic differential equation. By establishing a link between macroscopic and microscopic facets, these dynamics prove applicable in a wide array of domains, from biology to finance. In the realm of biology, they prove invaluable for modeling phenomena such as parasite infection within cell populations (see, *e.g.*, [2, 20, 21]). Conversely, in the financial domain, these processes are used to characterize options related to cryptocurrencies (see, *e.g.*, [17]).

In the study of branching diffusion processes, a fundamental question emerges: at what juncture does it become optimal to halt such a process? This question delves into the determination of an opportune point in time to stop the evolution of a branching diffusion. This research line echoes the optimization of a given functional to trade-off between the diffusion and reproduction of these processes and a possible degradation of the reward. By investigating the optimal stopping time for branching diffusion processes, we aim to shed light on the decision-making process involved in terminating these dynamical systems, thereby enhancing our understanding of their behaviour and enabling more effective applications in various fields of study.

One possible approach to consider is looking at the entire branching diffusion process as a whole, as done in [25], and finding a universal stopping time that applies to all active branches simultaneously. This global stopping time serves as a comprehensive decision rule, enabling a synchronized halt to the progression of each branch in the system, regardless of their characteristics or temporal disparities.

Although the aforementioned approach has its appeal, it may not fully align with the intrinsic structure of such processes. Indeed, the fundamental nature of a branching process, even when studied as a collective entity, is fundamentally rooted in its ability to portray the trajectory and dynamics of a singular individual. Therefore, while a global perspective may offer valuable insights and provide a comprehensive overview of the system, it may inadvertently disregard the inherent individuality of the branches.

This dual mode between the individuality of the single component as opposed to the wholeness of the population is a key concept in cooperative game theory. For example, mean-field control literature (see, *e.g.*, [6, 7]) deals with the control of large-scale systems involving a multitude of interacting agents, assumed to be rational decision-makers who aim to optimize their objective functions. The goal is to find control strategies that maximize a specific objective at the population level, which aligns with the optimal behaviour of each agent, influenced by the collective behaviours of the entire population. An additional example illustrating the transformation of global behaviour into individual optimization can be observed in [11, 17, 24, 25]. These studies prove how control strategies are contingent upon the decisions made by each participant. Moreover, the concept of the branching property emerges as a means to reduce the complexity of the problem, consequently shifting the focus toward analyzing the dynamics of the individual agents.

To capture the decision-making process of individuals within a collective framework, we adopt the concept of stopping lines. This mathematical object, introduced in [8, 9], serves as the counterpart to stopping times in branching dynamics. Stopping lines are characterized by a subset of the process's genealogy, where no member can be traced back to another member, and we can see their use in applications such as [19].

Although stopping lines have been used in previous studies, the exploration of optimal stopping lines based on specific criteria remains, to the best of our knowledge, an open problem. This article aims to address this research gap by directing our attention to this exact issue.

An application of this optimal stopping problem is possible in the field of finance, specifically in the valuation of American options tied to cryptocurrencies (see, *e.g.*, [13]). This modeling is the one discussed in [17], in the case of super-replication. In the realm of biology, another pertinent utilization arises in the optimization of halting infections caused by parasites, a model explored by [2, 20, 21]. This application gains significance as the initial stage of a mutant invasion closely aligns with the characteristics of a branching process, as expounded in works like [4, 3, 1]. In this phase, critical stopping criteria become imperative, especially in the context of identifying the emergence and detection of cancer.

Within a branching diffusion process framework, we look for the characterization of the value function linked to an infinite horizon optimal stopping problem. Optimization is done over the set of stopping lines, where each branch becomes eligible for halting only if no preceding ancestor has been stopped before. We narrow our investigation to multiplicative rewards, similar to the approach taken in [11, 23]. Drawing inspiration from [10], we prove a fundamental branching property. This property provides conditional independence among the offspring branches subsequent to a given conditioning time. This allows working within a finite-dimensional setting, distinguishing it from the traditional approach that treats branching diffusion dynamics as measure-valued processes. This framework, additionally, yields polynomial growth and local Lipschitz properties for the value function.

We employ a dynamic programming approach to characterize the value function as a solution to a specific Partial Differential Equation (PDE). Establishing an original Dynamic Programming Principle (DPP), we extend the framework of the classical optimal stopping problem to our branching context. This outcome paves the way for an analytical characterization of the value function.

The corresponding PDE takes the form of an obstacle problem with a semilinear term, which involves a polynomial series associated with the branching mechanism and value functions related to offspring labels. Under the assumption that this series has an infinite radius of convergence, we show that the value function is a solution in the sense of viscosity to this PDE. It is worth noting that a global bound on the label for the test functions is needed within the viscosity properties. This condition serves to retrieve the martingale property for the compensated jump component of the branching diffusion dynamics.

To conclude the PDE characterization, we present a comparison theorem. The presence of the semilinear term, tied to the value functions associated with offspring labels, introduces a non-classical aspect to this PDE. We explore a multiplicative penalization, making the viscosity solutions go towards zero in the spatial variable as a result of the previously demonstrated polynomial growth. Then, using the assumption of vanishing rewards as the label goes to infinity, we establish the comparison principle for value functions related to sufficiently large starting labels. We finally extend this analysis to cover the remaining functions through a backward induction on the size of the label.

The remainder of the paper is structured as follows. Section 2 presents a detailed description of the model under examination, focusing on the characteristics of branching diffusion processes and stopping lines. Additionally, we discuss the continuity of these processes' trajectories and highlight a crucial branching property that will play a significant role in subsequent sections. In Section 3,

we introduce the optimal stopping problem and establish the regularity of the corresponding value function. Section 4 is dedicated to proving the dynamic programming principle, while Section 5 provides the characterization of the value function as the unique viscosity solution to an obstacle problem.

# 2 Branching diffusion processes formulation

**Label set** We start by introducing the Ulam-Harris notation. This is key in the description of the tree structure of the problem, identifying immediately the genealogy of a particle. For $n \geq 1$, we write $i = i_1 \dots i_n$ for the multi-integer $i = (i_1, \dots, i_n) \in \mathbb{N}^n$. For $n, m \geq 1$ and two multi-integers $i = i_1 \dots i_n \in \mathbb{N}^n$ and $j = j_1 \dots j_m \in \mathbb{N}^m$, we define their concatenation $ij \in \mathbb{N}^{n+m}$ as

$$ij \;\; := \;\; i_1 \dots i_n j_1 \dots j_m \; . \tag{2.1}$$

The evolution of the particle population can now be described with the help of the set of labels $\mathcal{I}$ defined as follows

$$\mathcal{I} \;\; := \;\; \{\varnothing\} \cup \bigcup_{n=1}^{+\infty} \mathbb{N}^n \; ,$$

where the label $\varnothing$ corresponds to the mother particle. We extend the concatenation (2.1) to the whole set $\mathcal{I}$ with $\varnothing i = i\varnothing = i$, for all $i \in \mathcal{I}$. For a particle $i \in \mathcal{I} \setminus \{\varnothing\}$, $i = i_1 \cdots i_n$ with $i_1, \dots, i_n \in \mathbb{N}$, we denote by $i-$ its parent defined by $i- = i_1 \cdots i_{n-1}$.

When the particle $i = i_1 \dots i_n \in \mathbb{N}^n$ gives birth to $k$ particles, the off-springs are labelled $i0, \dots, i(k-1)$. By employing this method of generating the genealogy, we can establish a partial ordering $\preceq$ (resp. $\prec$) by

$$i \preceq j \;\; \Leftrightarrow \;\; \exists \ell \in \mathcal{I} \; : \; i = j\ell \qquad (\text{resp. } i \prec j \Leftrightarrow \exists \ell \in \mathcal{I} \setminus \{\varnothing\} \; : \; i = j\ell) \; ,$$

for all $i, j \in \mathcal{I}$. We say that $i \in \mathcal{I}$ is the parent of $j \in \mathcal{I}$ if $j = i\ell$ with $\ell \in \mathbb{N}$. Moreover, if $i = i_1 \dots i_n$, we say that $i$ belongs to the $n$-th generation of the population.

We endow $\mathcal{I}$ with the discrete topology, which is generated by the following distance $d_{\mathcal{I}}$

$$d^{\mathcal{I}}(i, j) := \sum_{\ell=p+1}^{n} (i_\ell + 1) + \sum_{\ell'=p+1}^{m} (j_{\ell'} + 1) \; ,$$

for $i = i_1 \cdots i_n \in \mathbb{N}^n, j = j_1 \cdots j_m \in \mathbb{N}^m$, where $p$ is the generation of the greatest common ancestor, *i.e.*, $p = \max\{\ell \geq 1 : i_\ell = j_\ell\}$. We next write $|i| := d^{\mathcal{I}}(i, \varnothing)$ for $i \in \mathcal{I}$. We define the following generation function $\mathfrak{g} : \mathcal{I} \to \mathbb{N}$ by $\mathfrak{g}(\varnothing) := 0$ and

$$\mathfrak{g}(i) \;\; := \;\; n \; , \tag{2.2}$$

for $i = i_1 \cdots i_n \in \mathcal{I}$ with $i_1, \dots, i_n \in \mathbb{N}$.

**Set of marked trees** In the following, we will denote as tree the family tree of the population. A tree $\omega^0$ is a subset of $\mathcal{I}$ that satisfies the following properties: $\varnothing \in \omega^0$,

$$ij \in \omega^0 \quad \Rightarrow \quad i \in \omega^0 \qquad \text{for } \; i, j \in \mathcal{I} \, ,$$

and, for any $i \in \omega^0$, there exists $\nu_i(\omega^0) \in \mathbb{N}$ such that

$$i\ell \in \omega^0 \quad \Rightarrow \quad 0 \leq \ell \leq \nu_i(\omega^0) - 1 \, , \tag{2.3}$$

with the convention that $\nu_i(\omega^0) = 0$, when $i\ell \notin \omega^0$ for all $\ell \in \mathbb{N}$.

We denote $\Omega^0$ the set of trees. We say that $i \in \mathcal{I}$ is a node of $\omega^0 \in \Omega^0$ if $i \in \omega^0$. For $i \in \mathcal{I}$, let $\Omega_i^0$ be the subset of trees having $i$ as a node, *i.e.*,

$$\Omega_i^0 \quad := \quad \left\{ \omega^0 \in \Omega^0 \; : \; i \in \omega^0 \right\} .$$

We notice that $\Omega_i^0$ is the domain of the map $\nu_i$ introduced in (2.3) for $i \in \mathcal{I}$.

For $m \in \mathbb{N}^*$, let $\Omega^1 := C^0(\mathbb{R}_+, \mathbb{R}^m) \times \mathbb{R}_+$ be the space of marked paths. We denote $B$ (resp. $\rho$) the projection map from $\Omega^1$ to its first (resp. second) component, that is

$$B(\omega^1, s) := \omega^{10}(s), \qquad \rho(\omega^1) := \omega^{11} \, ,$$

for $\omega^1 = (\omega^{10}, \omega^{11}) \in \Omega^1$ with $\omega^{10} \in \Omega^{10}$ and $\omega^{11} \in \Omega^{11}$.

The set of marked trees $\Omega$ is now defined as

$$\Omega \quad := \quad \left\{ \omega = \left( \omega^0, (\omega_i^1 \, , \; i \in \omega^0) \right) \, , \; \omega^0 \in \Omega^0 \, , \; \omega_i^1 \in \Omega^1 \right\} ,$$

and we denote $\pi^0$ the canonical projection from $\Omega$ to $\Omega^0$. For $i \in \mathcal{I}$, we set $\Omega_i = (\pi^0)^{-1}(\Omega_i^0)$ and still denote $\nu_i$ the map induced on $\Omega_i$ by $\pi^0$ and (2.3). For $i \in \mathcal{I}$, we define the canonical projection $\pi_i^1$ from $\Omega_i$ to $\Omega^1$ by

$$\pi_i^1(\omega) \quad := \quad \omega_i^1 \, , \qquad \text{for } \omega = \left( \omega^0, (\omega_j^1 \, , \; j \in \omega^0) \right) \in \Omega_i \, .$$

We can now extend $B$ and $\rho$ to $\Omega_i$, obtaining the map $B_i$ and $\rho_i$ as follows

$$B_i \quad := \quad B \circ \pi_i^1 \, , \qquad \rho_i \quad := \quad \rho \circ \pi_i^1 \, ,$$

for $i \in \mathcal{I}$.

One key property of trees that will be of se in the following is their self-similarity. This means that, when we zoom on a node and look at its offspring, we still have a tree, up to re-indexation of the labels. Therefore, we define the shift operator $T_{i,s}$, for $i \in \mathcal{I}$ and $s \in \mathbb{R}_+$, from $\Omega_i$ to $\Omega$. This operator is such that $T_{i,s}(\omega)$ is the subtree of $\omega$ starting from a particle $i$ alive at time $s$. More

precisely, we have

$$\begin{aligned}
\pi^0(T_{i,s}(\omega)) &:= \{j \in \mathcal{I} \,:\, ij \in \pi^0(\omega)\}\,, \\
\rho_\varnothing(T_{i,s}(\omega)) &:= \rho_i(\omega) - s \wedge \rho_i(\omega)\,, \\
B_\varnothing\big(T_{i,s}(\omega), t\big) &:= B_i\big(\omega, (s \wedge \rho_i(\omega)) + t\big) - B_i\big(\omega, s \wedge \rho_i(\omega)\big)\,, \text{ for } t \in [s \wedge \rho_i(\omega), \rho_i(\omega)]\,, \\
B_j(T_{i,s}(\omega)) &= B_{ij}(\omega)\,, \qquad \text{for } \omega \in \Omega_{ij}\,,\ j \neq \varnothing\,, \\
\rho_j(T_{i,s}(\omega)) &= \rho_{ij}(\omega)\,, \qquad \text{for } \omega \in \Omega_{ij}\,,\ j \neq \varnothing\,.
\end{aligned}$$

**Lifetime, birthtime and position.** When dealing with processes indexed on a tree, we have two notions of time to take into consideration. On one hand, the age of the particle, and consequently its time of death/reproduction. On the other hand, the calendar time expresses a notion of time for all the particles. Let $S_i$ be the birthtime of a particle $i \in \mathcal{I}$ such that $S_\varnothing = 0$ and, inductively on the generations,

$$S_i := S_{i-} + \rho_i\,.$$

With this notion, which encodes the calendar time for the population, we write $\mathcal{V}_t$ for the set of alive particles at time $t \in \mathbb{R}_+$, defined by

$$\mathcal{V}_t := \Big\{i \in \mathcal{I} \,:\, S_i \leq t < S_i + \rho_i\Big\}\,. \tag{2.4}$$

**$\sigma$-algebrae and filtrations.** As $\sigma$-algebrae describe the information we can access, we need to define filtration that will match the tree structure. First, on $\Omega^1$, let $\mathcal{H}^1 = (\mathcal{H}^1(t))_{t\in\mathbb{R}_+}$ be the right-continuous filtration generated by marginal projection $B$ and progressively enlarged by $\rho$

$$\mathcal{H}^1(t) := \bigcap_{\varepsilon>0} \sigma\big(B_s, \mathbb{1}_{\rho\leq s}\,,\ s \leq t+\varepsilon\big)\,, \quad t \geq 0\,.$$

Then, on $\Omega_i$, take $\mathcal{H}_i = (\mathcal{H}_i(t))_{t\in\mathbb{R}_+}$ to be the filtration associated with the evolution of the branch with label $i$, *i.e.*,

$$\mathcal{H}_i(t) := (\pi^1_i)^{-1}\big(\mathcal{H}^1(t)\big) \qquad \text{for } t \in \mathbb{R}_+\,.$$

As done for the birthtime, we consider the $\sigma$-algebrae associated with the ancestors of a particle $i$. Let $\mathcal{G}_\varnothing$ be the completed trivial $\sigma$-algebra on $\Omega$. For $i \in \mathcal{I}$, we consider the $\sigma$-algebra $\mathcal{G}_i$ on $\Omega_i$ defined inductively by

$$\mathcal{G}_i := \sigma(\mathcal{G}_{i-}, \mathcal{H}_{i-}(\rho_{i-})) \cap \Omega_i\,.$$

Finally, we introduce the filtration that stores all the information of the ancestors up to calendar time $t \in \mathbb{R}_+$. Let $\mathcal{A}_i = (\mathcal{A}_i(t))_{t\in\mathbb{R}_+}$ on $\Omega_i$ be

$$\mathcal{A}_i(t) := \sigma(\mathcal{G}_i, \mathcal{H}_i(t)), \qquad \text{for } t \in \mathbb{R}_+, i \in \mathcal{I}\,.$$

We observe that $B_i$ and $Z_i$ are $\mathcal{A}_i$-adapted for $i \in \mathcal{I}$. We endow $\Omega$ with the $\sigma$-algebra $\mathcal{F}$ generated

by $\mathcal{G}_i$ for $i \in \mathcal{I}$.

Finally, we consider the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$ generated by all the particles alive, with respect to the calendar time $t \in \mathbb{R}_+$

$$\mathcal{F}_t := \sigma\left(\mathcal{A}_i(t - S_i) : i \in \mathcal{V}_t\right) .$$

**Stopping lines** A stopping line is a collection of maps $(\tau_i,\ i \in \mathcal{I})$ such that

(i) $\tau_i :\ \Omega_i \to \mathbb{R}_+$ is a $\mathcal{A}_i$-stopping time for all $i \in \mathcal{I}$,

(ii) the random set $L_\tau$, defined by

$$L_\tau(\omega) \quad = \quad \Big\{ i \in \pi^0(\omega) \ :\ 0 \le \tau_i(\omega) < \rho_i(\omega) \Big\} , \quad \omega \in \Omega ,$$

satisfies the so-called *line property*, *i.e.*,

$$j \prec i \ \text{ and } \ i \in L_\tau \quad \Rightarrow \quad j \notin L_\tau \quad \text{ for } i, j \in \mathcal{I} .$$

This last property tells that the set $L_\tau$ cannot select two particles if one is the ancestor of the other.

We denote the set of stopping lines $\mathcal{SL}$. For $\tau \in \mathcal{SL}$, we define the set $D_\tau$ as

$$D_\tau \quad := \quad \{ i \in \mathcal{I} \ :\ \exists j \in \mathcal{I} \ ,\ j \prec i \ ,\ j \in L_\tau \} \ ,$$

which corresponds to the set of strict descendants of the line $L_\tau$. As for stopping times on the real line, the $\sigma$-algebra $\mathcal{F}_\tau$ related to a stopping line $\tau$ is defined as

$$\mathcal{F}_\tau \quad := \quad \sigma\Big( \{ i \notin D_\tau \} \cap \mathcal{A}_i(\tau_i) \ ,\ i \in \mathcal{I} \Big) .$$

With respect to the filtration $\mathbb{F}$, we see that $\mathcal{F}_t$ corresponds to the filtration generated by the stopping line $\tau^t$

$$\begin{aligned} \tau^t \quad &:= \quad t - S_i, \qquad \text{if } i \in \mathcal{V}_t, \\ \tau^t \quad &:= \quad \rho_i, \qquad\quad \text{else.} \end{aligned}$$

Moreover, we have $L_{\tau^t} = \mathcal{V}_t$.

**Probability law and branching property** We turn to the definition of the probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$. We follow the construction of [22] for Galton Watson Processes and extend to Brownian branching processes as done in [10]. Consider the auxiliary space

$$\Omega^* \quad := \quad \Big( C^0(\mathbb{R}_+, \mathbb{R}^m) \times \mathbb{R}_+ \times \mathbb{N} \Big)^{\mathcal{I}} ,$$

endowed with the $\sigma$-algebra

$$\mathcal{F}^* \quad = \quad \Big( \mathcal{B}(C^0(\mathbb{R}_+, \mathbb{R}^m)) \otimes \mathcal{B}(\mathbb{R}_+))^{\otimes\mathbb{N}} \Big)^{\otimes\mathcal{I}} .$$

Then, define on $(\Omega^*, \mathcal{F}^*)$ the probability measure $\mathbb{P}^*$ by

$$\mathbb{P}^* \;=\; \Big(\mathbb{P}^0 \otimes \mathcal{E}(\alpha) \otimes \sum_{n\in\mathbb{N}} p_n \delta_{\{n\}}\Big)^{\otimes \mathcal{I}} ,$$

where $\mathbb{P}^0$ stands for the Wiener measure on $C^0(\mathbb{R}_+, \mathbb{R}^m)$, $\mathcal{E}(\alpha)$ is the exponential law with parameter $\alpha$, and $p_n$, with $n \in \mathbb{N}$, is such that $p_n \in [0,1]$, for $n \in \mathbb{N}$, and $\sum_{n\in\mathbb{N}} p_n = 1$. For $i \in \mathcal{I}$, define on $\Omega^*$ the projections $\nu_i^*$, $B_i^*$ and $\rho_i^*$ as follows

$$\begin{aligned} B_i^*(\omega^*) &:= \omega_i^{*,1} \in C^0(\mathbb{R}_+, \mathbb{R}^m) , \\ \rho_i^*(\omega^*) &:= \omega_i^{*,2} \in \mathbb{R}_+ , \\ \nu_i^*(\omega^*) &:= \omega_i^{*,3} \in \mathbb{N} , \end{aligned}$$

for $\omega^* = \big(\omega_i^{*,1}, \omega_i^{*,2}, \omega_i^{*,3}\big)_{i\in\mathcal{I}} \in \Omega^*$.

Let $\Phi : \Omega^* \to \Omega$ be the map such that $\Phi(\omega^*)$ is the tree of $\Omega$ starting from $B_\varnothing(\omega^*)$, for $\omega^* \in \Omega^*$, and each node $i \in \mathcal{I}$ has $\nu_i^*(\omega^*)$ offspring with $B_{i0}^*, \dots, B_{i(\nu_i^*(\omega^*)-1)}^*$ trajectories and $\rho_{i0}^*, \dots, \rho_{i(\nu_i^*(\omega^*)-1)}^*$ respective extinction times. We endow $(\Omega, \mathcal{F})$ with the probability measure $\mathbb{P}$, which is defined as the image measure of $\mathbb{P}^*$ by $\Phi$.

We have the following result on the laws of $B_i$ and $\rho_i$ for $i \in \mathcal{I}$.

**Proposition 2.1.** *Given $\Omega_i$, $\nu_i$, $B_j$ and $\rho_j$, $j \preceq i$ are independent and follow respectively the laws $\sum_n p_n \delta_{\{n\}}$, $\mathbb{P}^0$ and $\mathcal{E}(\alpha)$ for any $i \in \mathcal{I}$.*

*Proof.* Fix $i = i_1 \dots i_n \in \mathcal{I}$ with $i_1, \dots, i_n \in \mathbb{N}$, $A_j \times B_j \in \mathcal{B}(C^0(\mathbb{R}_+, \mathbb{R}^m)) \times \mathcal{B}(\mathbb{R}_+)$ for $j \preceq i$ and $k \in \mathbb{N}$. We then have

$$\begin{aligned} \mathbb{P}\big((B_j, \rho_j) \in A_j \times B_j,\ j \preceq i\ , \nu_i = k \mid \Omega_i\big) &= \\ \frac{\mathbb{P}\big((B_j, \rho_j) \in A_j \times B_j\ , j \preceq i,\ \nu_i = k,\ \ \nu_\varnothing \geq i_1 + 1, \dots,\ \nu_{i_1 \dots i_{n-1}} \geq i_n + 1\big)}{\mathbb{P}\big(\nu_\varnothing \geq i_1 + 1, \dots,\ \nu_{i_1 \dots i_{n-1}} \geq i_n + 1\big)} &= \\ \frac{\mathbb{P}^*\big((B_j^*, \rho_j^*) \in A_j \times B_j\ , j \preceq i,\ \nu_i^* = k,\ \nu_\varnothing^* \geq i_1 + 1, \dots,\ \nu_{i_1 \dots i_{n-1}}^* \geq i_n + 1\big)}{\mathbb{P}\big(\nu_\varnothing^* \geq i_1 + 1, \dots,\ \nu_{i_1 \dots i_{n-1}}^* \geq i_n + 1\big)} &= \\ p_k \prod_{j\preceq i} \mathbb{P}^0(A_j) \int_{B_j} \alpha e^{-\alpha u} du\ , \end{aligned}$$

where the last equality comes from the definition of $\mathbb{P}^*$. □

**Branching diffusion processes** We can now define the object of diffusion processes on trees. Let $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ be measurable functions and $x \in \mathbb{R}^d$ a starting point. Let $X_\varnothing$ be defined on $\Omega$ by

$$\begin{aligned} X_\varnothing^x(0) &= x && (2.5) \\ dX_\varnothing^x(s) &= b(X_\varnothing^x(s))ds + \sigma(X_\varnothing^x(s))dB_\varnothing(s)\ , \quad s \geq 0\ , && (2.6) \end{aligned}$$

and $X_i^x$, $i \in \mathcal{I} \setminus \{\varnothing\}$, defined on $\Omega_i$ by

$$\begin{aligned} X_i^x(0) &= X_{i-}^x(\rho_{i-}) && (2.7)\\ dX_i^x(s) &= b(X_i^x(s))ds + \sigma(X_i^x(s))dB_i(s)\ , \quad s \geq 0\ . && (2.8)\end{aligned}$$

The branching diffusion process starting from $x$ is a map $X^x :\ \Omega \to \Omega$ such that

$$\pi^0\big(X^x(\omega)\big) \;=\; \pi^0(\omega) \;=\; \omega^0$$

and

$$\begin{aligned} \rho_i \circ X^x(\omega) &= \rho_i(\omega)\ ,\\ B_i \circ X^x(\omega) &= X_i^x(\omega)\ ,\end{aligned}$$

for $i \in \mathcal{I}$ and $\omega \in \Omega_i$.

We make the following assumptions on the coefficients $b$ and $\sigma$ and on the law $p = (p_k)_{k\geq 0}$.

**Assumption A1.** *(i) The functions $b$ and $\sigma$ are Lipschitz continuous, i.e., there exists a constant $L > 0$ such that*

$$|b(x) - b(x')| + |\sigma(x) - \sigma(x')| \;\leq\; L|x - x'|, \qquad (2.9)$$

*for all $x, x' \in \mathbb{R}^d$.*

*(ii) The coefficients $(p_k)_{k\geq 0}$ satisfy*

$$M \;:=\; \sum_{k\geq 0} k p_k \;<\; +\infty\ .$$

We recall that the generation function $\mathfrak{g}$ is given by (2.2).

**Proposition 2.2.** *Suppose that Assumption A1 holds.*

*(i) There exists a unique process $(X_i^x)_{i\in\mathcal{I}}$ solution to* (2.5)-(2.6)-(2.7)-(2.8).

*(ii) For $p \geq 1$, there exists two constants $\alpha_p > 0$ and $C_p > 0$ such that*

$$\mathbb{E}\left[\sup_{s\in[0,\rho_i]} |X_i^x(s)|^{2p}\middle|\Omega_i\right] \;\leq\; \left(\sum_{k=1}^{\mathfrak{g}(i)+1} (C_p)^k\right)\big(1 + |x|^{2p}\big)\ , \qquad (2.10)$$

$$\mathbb{E}\left[\sup_{s\in[0,\rho_i]} |X_i^x(s) - X_i^{x'}(s)|^{2p}\middle|\Omega_i\right] \;\leq\; (C_p)^{\mathfrak{g}(i)+1}|x - x'|^{2p}\ , \qquad (2.11)$$

*for $x, x' \in \mathbb{R}^d$, $i \in \mathcal{I}$, and $\alpha \geq \alpha_p$.*

*Proof. (i)* Since $B_i$ follows $\mathbb{P}^0$ given $\Omega_i$, Assumption A1(i) gives the existence and uniqueness of a process $X_i$ defined on $\Omega_i$ satisfying (2.7)-(2.8) for all $i \in \mathcal{I}$ (see, *e.g.*, [18, Theorem 2.5.7]). This ensures the good definition of the map $X^x$.

(ii) We prove (2.10)-(2.11) by induction on the generation. Fix $x, x' \in \mathbb{R}^d$. From [18, Corollary 10, Section 5, Chapter 2] and [18, Theorem 9, Section 5, Chapter 2], we have that there exists a

constant $\bar{C}_p > 0$, which depends only on $q$ and $L$ from the growth condition consequence of (2.9), such that

$$\begin{aligned}\mathbb{E}\left[\sup_{s\in[0,t]}\left|X^x_\varnothing(s)\right|^{2p}\right] &\leq \bar{C}_p t^{p-1}e^{\bar{C}_p t}\left(1+|x|^{2p}\right),\\ \mathbb{E}\left[\sup_{s\in[0,t]}\left|X^x_\varnothing(s)-X^{x'}_\varnothing(s)\right|^{2p}\right] &\leq \bar{C}_p e^{\bar{C}_p t}\left(\left|x-x'\right|^{2p}\right),\end{aligned}$$

for $t \geq 0$. Since $B_\varnothing$ and $\rho_\varnothing$ are independent and $\rho_\varnothing$ is distributed as an exponential random variable with parameter $\alpha$, we have that

$$\begin{aligned}\mathbb{E}\left[\sup_{s\in[0,\rho_\varnothing]}|X^x_\varnothing(s)|^{2p}\right] &\leq \bar{C}_p\int_0^\infty t^{p-1}e^{\bar{C}_p t}\alpha e^{-\alpha t}dt\left(1+|x|^{2p}\right),\\ \mathbb{E}\left[\sup_{s\in[0,\rho_\varnothing]}|X^x_\varnothing(s)-X^{x'}_\varnothing(s)|^{2p}\right] &\leq \bar{C}_p\int_0^\infty e^{\bar{C}_p t}\alpha e^{-\alpha t}dt\left|x-x'\right|^{2p}.\end{aligned}$$

Therefore, define $\alpha_p := \bar{C}_p + \delta$ for $\delta > 0$, from comparing the previous expression with the gamma distribution with parameters $p$ and $\delta$, we get (2.10)-(2.11), with $C_p := \frac{\alpha\bar{C}_p}{(\alpha-\bar{C}_p)^q}\Gamma(q)$ for $\alpha > \alpha_p$. Therefore, the property holds for the label $\varnothing$.

Suppose that (2.10)-(2.11) hold for all labels up to generation $n-1$ with $n \geq 1$. Fix $i \in \mathcal{I}$ with $\mathfrak{g}(i) = n$. We notice that, given $\Omega_i$, the process $\left(X^x_i(s)\right)_{s\geq 0}$ (resp. $\left(X^{x'}_i(s)\right)_{s\geq 0}$) is a diffusion process starting from $X^x_{i-}(\rho_{i-})$ (resp. $X^x_{i-}(\rho_{i-})$) and driven by $B_i$. From Proposition 2.1 $X^{x'}_{i-}(\rho_{i-})$ (resp. $X^{x'}_{i-}(\rho_{i-})$), $B_i$ and $\rho_i$ are independent. We can therefore apply the arguments used for the label $\varnothing$ and get

$$\begin{aligned}\mathbb{E}\Big[\sup_{s\in[0,\rho_i]}\left|X^x_i(s)\right|^{2p}\Big|\Omega_i\Big] &\leq C_p\left(1+\mathbb{E}\Big[\left|X^x_{i-}(\rho_{i-})\right|^{2p}\Big|\Omega_i\Big]\right),\\ \mathbb{E}\left[\sup_{s\in[0,\rho_i]}|X^x_i(s)-X^{x'}_i(s)|^{2p}\middle|\Omega_i\right] &\leq C_p\mathbb{E}\left[|X^x_{i-}(\rho_{i-})-X^{x'}_{i-}(\rho_{i-})|^{2p}\middle|\Omega_i\right].\end{aligned}$$

Using the identity $\Omega_i = \Omega_{i-}\cap\{\nu_{i-} \geq i_n\}$, together with Proposition 2.1, we obtain

$$\begin{aligned}\mathbb{E}\Big[\left|X^x_{i-}(\rho_{i-})\right|^{2p}\Big|\Omega_i\Big] &= \mathbb{E}\Big[\left|X^x_{i-}(\rho_{i-})\right|^{2p}\Big|\Omega_{i-}\Big],\\ \mathbb{E}\left[|X^x_{i-}(\rho_{i-})-X^{x'}_{i-}(\rho_{i-})|^{2p}\middle|\Omega_i\right] &= \mathbb{E}\left[|X^x_{i-}(\rho_{i-})-X^{x'}_{i-}(\rho_{i-})|^{2p}\middle|\Omega_{i-}\right].\end{aligned}$$

Finally, combining this with the induction assumption, we entail the result.

□

We shall assume in the sequel that $\alpha > \alpha_4$. We now study the law of the shifted diffusion trees. The following result provides a conditional independence property also called branching property.

**Theorem 2.1** (Branching property)**.** *For a stopping line $\tau = (\tau_i)_{i\in\mathcal{I}}$, given $\mathcal{F}_\tau$, the shifted diffusion trees $T_{i,\tau_i}\circ X^x$, $i \in L_\tau$, are independent and follow the law of a diffusion tree starting from $X^x_i(\tau_i)$.*

*Moreover, we have*

$$\mathbb{E}\left[\prod_{i\in L_\tau} f_i(T_{i,\tau_i}\circ X^x)\middle|\mathcal{F}_\tau\right] \;=\; \prod_{i\in L_\tau}\mathbb{E}\left[f_i(X^{x_i})\right]\Big|_{x_i=X_i^x(\tau_i)}, \tag{2.12}$$

*for any family $(f_i)_{i\in\mathcal{I}}$ of non-negative $\mathcal{F}$-measurable functions.*

*Proof.* Let $(X_i^{*x})_{i\in\mathcal{I}}$ be the process such that $X_\varnothing^{*x}$ is defined on $\Omega^*$ by

$$\begin{aligned} X_\varnothing^{*x}(0) &= x\,, &(2.13)\\ dX_i^{*x}(s) &= b(X_i^{*x}(s))ds+\sigma(X_i^{*x}(s))dB_i^*(s)\,,\quad s\geq 0\,, &(2.14)\end{aligned}$$

and $X_i^{*x}$, $i\in\mathcal{I}\setminus\{\varnothing\}$, is defined on $\Omega^*$ by

$$\begin{aligned} X_i^{*x}(0) &= X_{i-}^{*x}(\rho_{i-}^*)\,, &(2.15)\\ dX_i^{*x}(s) &= b(X_i^{*x}(s))ds+\sigma(X_i^{*x}(s))dB_i^*(s)\,,\quad s\geq 0\,. &(2.16)\end{aligned}$$

We have $X^x\big(\Phi(\omega^*)\big)=\Phi\big(X_i^{*x}(\omega^*)\big)$ for all $\omega^*\in\Omega^*$ as $\mathbb{P}=\mathbb{P}^*\circ\Phi^{-1}$. Applying [27, Theorem 10.4], $X_i^x$ can be written as

$$X_i^x(s) \;=\; \Psi(X_{i-}^x(\rho_{i-}),B_i)(s)\,,\quad s\geq 0\,,$$

for some progressive function $\Psi:\ \mathbb{R}^d\times C^0(\mathbb{R}_+,\mathbb{R}^m)\to C^0(\mathbb{R}_+,\mathbb{R}^d)$. Still using [27, Theorem 10.4], we get that $\Psi(X_{i-}^{*x}(\rho_{i-}^*),B_i^*)$ is also solution to (2.15)-(2.16). We therefore get by an induction that $X^{*i}=\Psi(X_{i-}^{*x}(\rho_{i-}^*),B_i^*)$.
For $i\in\mathcal{I}$ and $s\in\mathbb{R}_+$, we define the translation map $T_{i,s}^*:\ \Omega^*\cap\{\rho_i^*\geq s\}\to\Omega^*$ by

$$\begin{aligned} B_\varnothing^*\big(T_{i,s}^*(\omega^*),t\big) &= B_i^*\big(\omega^*,t+s\big)-B_i^*\big(\omega^*,s\big)\,, &(2.17)\\ B_j^*\big(T_{i,s}^*(\omega^*),t\big) &= B_{ij}^*\big(\omega^*,t\big)\,,\ j\neq\varnothing\,, &(2.18)\\ \rho_\varnothing^*\big(T_{i,s}^*(\omega^*)\big) &= \rho_i^*(\omega^*)-s\,, &(2.19)\\ \rho_j^*\big(T_{i,s}^*(\omega^*)\big) &= \rho_{ij}^*(\omega^*)\,,\ j\neq\varnothing\,, &(2.20)\\ \nu_j^*\big(T_{i,s}^*(\omega^*)\big) &= \nu_{ij}^*\big(\omega^*\big)\,, &(2.21)\end{aligned}$$

for $j\in\mathcal{I}$ and $t\in\mathbb{R}_+$. Then, the operators $T_{i,s}$ and $T_{i,s}^*$ are related by

$$T_{i,s}\circ\Phi \;=\; \Phi\circ T_{i,s}^*\quad\text{on}\quad\Phi^{-1}\big(\Omega_i\cap\{s\leq\rho_i\}\big)\,, \tag{2.22}$$

for $i\in\mathcal{I}$ and $s\in\mathbb{R}_+$.

Fix now a stopping line $\tau=(\tau_i)_{i\in\mathcal{I}}$ and define the map $\tau^*=(\tau_i^*)_{i\in\mathcal{I}}$ by

$$\tau_i^* \;=\; \tau_i\circ\Phi\ \text{ on }\ \Phi^{-1}(\Omega_i)\,.$$

Also take the random set $L_{\tau^*}^*$ as follows

$$\begin{aligned} L_{\tau^*}^*(\omega^*) &:= \Big\{i\in\pi^0(\Phi(\omega^*))\ :\ 0\leq\tau_i^*(\omega^*)<\rho_i^*(\omega^*)\Big\}\\ &= L_\tau(\Phi(\omega^*))\,,\end{aligned}$$

for $\omega^* \in \Omega^*$. In view of (2.17) to (2.21) , the maps $T^*_{i,\tau^*} \circ X^{*x}$, $i \in L^*_{\tau^*}$ are mutually independent given $\mathcal{F}^*_{\tau^*} = \Phi^{-1}(\mathcal{F}_\tau)$ and the law of $T^*_{i,\tau^*} \circ X^{*x}$ given $\mathcal{F}^*_{\tau^*}$ is given by

$$\mathcal{L}(T^*_{i,\tau^*} \circ X^{*x} | \mathcal{F}^*_{\tau^*}) \;=\; \mathcal{L}(X^{*x_i})\big|_{x_i = X^{*x}_{\tau^*_i}} .$$

Since $L^*_{\tau^*} \in \mathcal{F}^*_{\tau^*}$, we have

$$\mathbb{E}^* \left[ \prod_{i \in B} f_i(T^*_{i,\tau^*} \circ X^{*x}) \Big| \mathcal{F}^*_{\tau^*} \right] \;=\; \prod_{i \in B} \mathbb{E}^* \left[ f_i(X^{*x_i}_i) \right] \Big|_{x_i = X^{*x}_i(\tau^*_i)}$$

on $\{B \subset L^*_{\tau^*}\}$ for any finite subset $B$ of $\mathcal{I}$. Using (2.22), we get

$$\mathbb{E} \left[ \prod_{i \in B} f_i(T_{i,\tau} \circ X^x) \Big| \mathcal{F}_\tau \right] \;=\; \prod_{i \in B} \mathbb{E} \left[ f_i(X^{x_i}_i) \right] \Big|_{x_i = X^x_i(\tau_i)}$$

on $\{B \subset L_\tau\}$ for any finite subset $B$ of $\mathcal{I}$. Since $L_\tau$ is $\mathcal{F}_\tau$-measurable, we get the result. □

# 3 The optimal stopping problem

Prior to delving into the optimal stopping problem, we consider a modified version of the preceding scenario. Specifically, akin to the standard context, we incorporate an actualization component. This discount is applied to each branch, aligning with the type of cost function observed in [11] within the context of the optimal control setting. This augmentation involves the introduction of an extra temporal dimension into the previously introduced framework.

Let us now extend the dimension of the problem to $\mathbb{R}^{d+1}$ instead of $\mathbb{R}^d$. We consider elements of this space to be denoted as $\tilde{x} = \binom{x}{y}$, with $x \in \mathbb{R}^d$ and $y \in \mathbb{R}$. We define now each particle $\tilde{X}^{\tilde{x}}_i = \binom{X^x_i}{Y^y_i}$ to satisfy (2.5)-(2.8), with respect to $\left(\tilde{b}, \tilde{\sigma}\right)$ defined as follows

$$\tilde{b}(\tilde{x}) = \binom{b(x)}{1}, \quad \tilde{\sigma}(\tilde{x}) = \binom{\sigma(x)}{0} ,$$

with $b$ and $\sigma$ satisfying (2.9). It is clear that under these assumptions, we have that $Y^y_i(s) = S_i + s$. Fix now the functions $g_i : \mathbb{R}^d \to \mathbb{R}_+$ for $i \in \mathcal{I}$ satisfying the following assumption.

**Assumption A2.** *(i) The functions $g_i$, for $i \in \mathcal{I}$, are non-negative and vanish uniformly in $x$ as $i$ goes to $\infty$, i.e.,*

$$\lim_{|i| \to +\infty} \sup_{x \in \mathbb{R}^d} g_i(x) \;=\; 0 . \tag{3.23}$$

*(ii) The functions $g_i$, for $i \in \mathcal{I}$, are Lipschitz continuous uniformly in $i \in \mathcal{I}$, i.e., there exists a constant $L > 0$ such that*

$$|g_i(x) - g_i(x')| \;\leq\; L|x - x'| , \tag{3.24}$$

*for all $i \in \mathcal{I}$ and $x, x' \in \mathbb{R}^d$.*

**Remark 3.1.** *The initial assumption captures a degradation in reward as we extend farther from the parent particle, considering both the generation and the number of offspring. This setup enables the formulation of a system of differential equations indexed on a tree, similar to the approach in [17]. This kind of assumption can be also seen in biological applications like [20, 21], where reproduction is associated with the transmission of parasites from mother to daughter cells. Consider a binary division scenario, i.e., $p_0, p_2 \in (0,1)$, and $p_0 + p_2 = 1$. The parasites' partitioning kernel at division can be modeled in two way: either by a discontinuation of the starting condition* (2.7) *by a factor $\theta$ taking values in $[0,1]$, as in [20, 21], or by considering a discontinuity of the reward function from mother to daughter.*

*In particular, set this discount from parent to child to be deterministic and, for example, equal to $\frac{1}{2}$. If $\bar{g} \in C^0(\mathbb{R}^d; \mathbb{R}_+)$ represents the infectiousness rate for the cell labeled $\varnothing$, we define $g_i$, for $i \in \mathcal{I}$, as*

$$g_i(x) = \frac{1}{2} g_{i-}(x) = \left(\frac{1}{2}\right)^{\mathfrak{g}(i)} \bar{g}(x) \, .$$

*This different point of view could correspond to a deterministic parasites' partitioning kernel at division.*

Fix now a constant $\gamma > 0$. For the label $\varnothing$, we define the reward function $J_\varnothing$ by

$$J_\varnothing(x,\tau) \;:=\; \mathbb{E}\left[\prod_{j\in L_\tau} e^{-\gamma Y_j^0(\tau_j)} g_j\big(X_j^x(\tau_j)\big)\right] \;=\; \mathbb{E}\left[\prod_{j\in L_\tau} e^{-\gamma(S_j+\tau_j)} g_j\big(X_j^x(\tau_j)\big)\right] \, ,$$

for $x \in \mathbb{R}^d$ and $\tau \in \mathcal{SL}$. Consequently, using the symmetry highlighted in Theorem 2.1, we define the reward function, starting at $i \in \mathcal{I}$, as follows

$$J_i(x,\tau) \;:=\; \mathbb{E}\left[\prod_{j\in L_\tau} e^{-\gamma Y_j^0(\tau_j)} g_{ij}\big(X_j^x(\tau_j)\big)\right] \;=\; \mathbb{E}\left[\prod_{j\in L_\tau} e^{-\gamma(S_j+\tau_j)} g_{ij}\big(X_j^x(\tau_j)\big)\right] \, , \quad (3.25)$$

for $i \in \mathcal{I}$, $x \in \mathbb{R}^d$, and $\tau \in \mathcal{SL}$. Let $v_i : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ be the following value function

$$v_i(x) \;=\; \sup_{\tau \in \mathcal{SL}} J_i(x,\tau) \, , \quad (3.26)$$

for $i \in \mathcal{I}$ and $x \in \mathbb{R}^d$. Our goal is to provide an analytic characterization of the family of functions $(v_i)_{i\in\mathcal{I}}$. We first state the basic properties of this family.

**Proposition 3.3.** *Suppose that Assumptions A1 and A2 hold. There exist $\underline{\gamma} > 0$ such that for any $\gamma \geq \underline{\gamma}$, we have the following.*
(i) *The functions $v_i$ are well defined, nonnegative and there exists $p \geq 1$ and a constant $C > 0$ such that*

$$0 \;\leq\; v_i(x) \;\leq\; C\big(1 + |x|^p\big), \quad \text{for } i \in \mathcal{I},\ x \in \mathbb{R}^d \, . \quad (3.27)$$

(ii) *There exists a constant* $\bar{L} > 0$ *such that*

$$|v_i(x) - v_i(x')| \;\leq\; \bar{L}|x - x'|\left(1 + |x|^p + |x'|^p\right), \quad \textit{for } i \in \mathcal{I},\ x, x' \in \mathbb{R}^d. \,. \tag{3.28}$$

(iii) *The functions* $v_i$ *vanish uniformly in* $x$ *as* $i$ *goes to* $\infty$*, i.e.,*

$$\lim_{|i|\to+\infty} \sup_{x\in\mathbb{R}^d} v_i(x) \;=\; 0. \tag{3.29}$$

*Proof.* We suppose $i = \varnothing$. The general case for $i \in \mathcal{I}$ is proven by renaming the functions $g_j$ with $g_{ij}$ for any $j \in \mathcal{I}$.

*(i)* Fix $x \in \mathbb{R}^d$ and a stopping line $\tau$. Let $N$ be an integer such that

$$\sup_{x\in\mathbb{R}^d} g_i(x) \;\leq\; \frac{1}{2}\,, \tag{3.30}$$

for $|i| \geq N$. Then, we have

$$\mathbb{E}\left[\prod_{i\in L_\tau} e^{-\gamma(S_i+\tau_i)} g_i\big(X_i^x(\tau_i)\big)\right] \;\leq\; \mathbb{E}\left[\prod_{i\in L_\tau, |i|\leq N} e^{-\gamma(S_i+\tau_i)} g_i\big(X_i^x(\tau_i)\big)\right].$$

We notice that

$$\#\{i \in L_\tau, |i| \leq N\} \;\leq\; \#\{i \in \mathcal{I}, |i| \leq N\} := \bar{N} < \infty\,, \tag{3.31}$$

where # stands for the cardinality of the set. Therefore, using the inequality

$$\prod_{i=1}^p a_i \;\leq\; \sum_{i=1}^p a_i^p\,, \qquad \text{for } p \in \mathbb{N}\setminus\{0\}\,,\ a_1, \ldots, a_p \in \mathbb{R}_+\,, \tag{3.32}$$

we get

$$\mathbb{E}\left[\prod_{i\in L_\tau} e^{-\gamma(S_i+\tau_i)} g_i\big(X_i^x(\tau_i)\big)\right] \;\leq\; \sum_{i\in\mathcal{I}, |i|\leq N} \mathbb{E}\left[\big|g_i\big(X_i^x(\tau_i)\big)\big|^{\bar{N}} \big|\Omega_i\right] \mathbb{P}(\Omega_i)\,.$$

Using Assumption A2 (ii), we get a constant $C$ such that

$$\mathbb{E}\left[\prod_{i\in L_\tau} g_i\big(X_i^x(\tau_i)\big)\right] \;\leq\; C \sum_{i\in\mathcal{I}, |i|\leq N} \Big(1 + \mathbb{E}\left[\big|X_i^x(\tau_i)\big|^{\bar{N}} \big|\Omega_i\right]\Big)\mathbb{P}(\Omega_i)\,.$$

From (2.10), there exists a constant $C'$ such that

$$\mathbb{E}\left[\prod_{i\in L_\tau} e^{-\gamma(S_i+\tau_i)} g_i\big(X_i^x(\tau_i)\big)\right] \;\leq\; C'\Big(1 + |x|^{\bar{N}}\Big)\,,$$

for all $x \in \mathbb{R}^d$. Therefore, $J_\varnothing(x, \tau)$ is well defined for any $x \in \mathbb{R}^d$ and $\tau \in \mathcal{SL}$ and we get (3.27) with $p = \bar{N}$.

*(ii)* Fix $x, x' \in \mathbb{R}^d$ and a stopping line $\tau$. We have

$$\left|J_\varnothing(x,\tau) - J_\varnothing(x',\tau)\right| \;\leq\; \mathbb{E}\left[\left|\prod_{i\in L_\tau} e^{-\gamma(S_i+\tau_i)} g_i\left(X_i^x(\tau_i)\right) - \prod_{i\in L_\tau} e^{-\gamma(S_i+\tau_i)} g_i\left(X_i^{x'}(\tau_i)\right)\right|\right] .$$

Consider $|\tau| = \sup\{|i| \;:\; i \in L_\tau\}$. From Assumption A2 (i), we have

$$J_\varnothing(x,\tau) \;=\; J_\varnothing(x',\tau) \;=\; 0 \quad \text{on} \quad \{|\tau| = +\infty\} .$$

We now work on $\{|\tau| < +\infty\}$. Using the inequality

$$\begin{aligned}\left|\prod_{i=1}^p a_i - \prod_{i=1}^p b_i\right| \;&\leq\; \sum_{i=1}^p |a_i - b_i| \prod_{j=1}^{i-1} a_j \prod_{j=i+1}^p b_j \\ &\leq\; \sum_{i=1}^p |a_i - b_i| \prod_{\substack{j=1\\ j\neq i}}^p a_j \vee b_j ,\end{aligned}$$

for $p \in \mathbb{N}\setminus\{0\}$ and $a_1, b_1, \ldots, a_p, b_p \in \mathbb{R}_+$, we have that

$$\begin{aligned}&\left|\prod_{i\in L_\tau} e^{-\gamma(S_i+\tau_i)} g_i\left(X_i^x(\tau_i)\right) - \prod_{i\in L_\tau} e^{-\gamma(S_i+\tau_i)} g_i\big(X_i^{x'}(\tau_i)\big)\right| \\ \leq\; &\sum_{i\in L_\tau}\left[\prod_{\substack{j\in L_\tau,\\ j\neq i, |j|\leq N}} e^{-\gamma(S_j+\tau_j)}\left(g_i(X_j^x(\tau_j)) \vee g_i(X_j^{x'}(\tau_j))\right)\right] e^{-\gamma(S_i+\tau_i)}\left|g_i\big(X_i^x(\tau_i)\big) - g_i(X_i^{x'}(\tau_i))\right| ,\end{aligned}$$

with $N$ as in (3.30).

Taking expectation in the previous equation and applying Cauchy-Schwarz inequality, we get

$$\begin{aligned}\left|J_\varnothing(x,\tau) - J_\varnothing(x',\tau)\right| \;\leq\; \sum_{i\in\mathcal{I}} \mathbb{E}&\left[\prod_{\substack{j\in L_\tau,\\ j\neq i, |j|\leq N}} e^{-2\gamma(S_j+\tau_j)}\left(\left|g_i\big(X_j^x(\tau_j)\big)\right|^2 \vee \left|g_i\big(X_j^{x'}(\tau_j)\big)\right|^2\right)\right]^{1/2} \\ &\mathbb{E}\left[e^{-2\gamma(S_i+\tau_i)}\left|g_i\left(X_i^x(\tau_i)\right) - g_i\big(X_i^{x'}(\tau_i)\big)\right|^2 \mathbb{1}_{i\in L_\tau}\right]^{1/2} .\end{aligned}$$

Let $C$ denote a positive constant which may change from line to line in the next computations.

Using (3.32) we have

$$\mathbb{E}\left[\prod_{\substack{j\in L_\tau,\\ j\neq i,|j|\leq N}} e^{-2\gamma(S_j+\tau_j)}\left(\left|g_i\big(X_j^x(\tau_j)\big)\right|^2 \vee \left|g_i\big(X_j^{x'}(\tau_j)\big)\right|^2\right)\right] \;\leq$$

$$\mathbb{E}\left[\sum_{\substack{j\in L_\tau,\\ j\neq i,|j|\leq N}} e^{-2\gamma(S_j+\tau_j)\bar{N}}\left(\left|g_i\big(X_j^x(\tau_j)\big)\right|^{2\bar{N}} + \left|g_i\big(X_j^{x'}(\tau_j)\big)\right|^{2\bar{N}}\right)\right] \;\leq\; C(1+|x|^{2\bar{N}}+|x'|^{2\bar{N}})\,,$$

where the last inequality comes from Assumption A2 (ii) and (2.10). We therefore get

$$\left|J_\varnothing(x,\tau)-J_\varnothing(x',\tau)\right| \;\leq\; C\left(1+|x|^{\bar{N}}+|x'|^{\bar{N}}\right)\sum_{i\in\mathcal{I}}\mathbb{E}\left[e^{-2\gamma(S_i+\tau_i)}\left|g_i\big(X_i^x(\tau_i)\big)-g_i\big(X_i^{x'}(\tau_i)\big)\right|^2 \mathbb{1}_{i\in L_\tau}\right]^{1/2}\,.$$

Therefore, we obtain (3.28) if we prove that there exists $\underline{\gamma}$ such

$$\mathbb{E}\left[\sum_{i\in L_\tau} e^{-2\gamma(S_i+\tau_i)}\left|g_i\big(X_i^x(\tau_i)\big)-g_i\big(X_i^{x'}(\tau_i)\big)\right|^2\right] \;\leq\; C|x-x'|^2\,. \tag{3.33}$$

for any $\gamma\geq\underline{\gamma}$ and $\tau\in\mathcal{SL}$.

Using (3.24), the l.h.s. in (3.33) can be written as

$$\begin{aligned}
&\mathbb{E}\left[\sum_{i\in L_\tau} e^{-2\gamma(S_i+\tau_i)}\left|g_i\big(X_i^x(\tau_i)\big)-g_i\big(X_i^{x'}(\tau_i)\big)\right|^2\right]\\
\leq\;& C\sum_{i\in\mathcal{I}}\mathbb{E}\left[e^{-2\gamma(S_i+\tau_i)}\left|X_i^x(\tau_i)-X_i^{x'}(\tau_i)\right|^2\mathbb{1}_{i\in L_\tau}\right]\\
\leq\;& C\sum_{i\in\mathcal{I}}\mathbb{E}\left[e^{-4\gamma S_i}\middle|\Omega_i\right]^{1/2}\mathbb{E}\left[\left|X_i^x(\tau_i)-X_i^{x'}(\tau_i)\right|^4\middle|\Omega_i\right]^{1/2}\mathbb{P}\left(\Omega_i\right)\,,
\end{aligned}$$

where, in the last inequality, we applied again Cauchy-Schwarz inequality and used $\{i\in L_\tau\}\subseteq\Omega_i$. For $i=i_1\cdots i_n$, we have that

$$\mathbb{P}(\Omega_i)=\mathbb{P}\left(\nu_\varnothing\geq i_1,\ldots,\nu_{i_1\cdots i_{n-1}}\geq i_n\right)=\left(\sum_{k\geq i_1}p_k\right)\cdots\left(\sum_{k\geq i_n}p_k\right)=\bar{p}_{i_1}\cdots\bar{p}_{i_n}\,,$$

with $\bar{p}_\ell:=\sum_{k\geq\ell}p_k$ for $\ell\geq 0$. Moreover, since $S_i$ is the sum of $n$ independent exponential random variables with parameter $\alpha$, $S_i$ is gamma-distributed with shape $n$ and scale $\alpha$. This means that its exponential moment is given by

$$\mathbb{E}\left[e^{-4\gamma S_i}\middle|\Omega_i\right] \;=\; \frac{\alpha^n}{(\alpha+4\gamma)^n}\,.$$

Combining these two results with (2.11) for $p = 4$, we get for a constant $C > 0$

$$\begin{aligned}
&\mathbb{E}\left[\sum_{i\in L_\tau} e^{-2\gamma(S_i+\tau_i)}\left|g_i\big(X_i^x(\tau_i)\big) - g_i\big(X_i^{x'}(\tau_i)\big)\right|^2\right] \\
\leq\ & C\sum_{n\geq 0}\frac{\alpha^{n/2}}{(\alpha+4\gamma)^{n/2}}\left(\sum_{i_1\geq 0}\bar{p}_{i_1}\right)\cdots\left(\sum_{i_n\geq 0}\bar{p}_{i_n}\right)C^{(n+1)/2}|x-x'|^2 \\
\leq\ & C\left(\sum_{n\geq 0}\frac{\alpha^{n/2}M^nC^{n/2}}{(\alpha+4\gamma)^{n/2}}\right)|x-x'|^2\ .
\end{aligned}$$

Therefore, for $\underline{\gamma} = \frac{\alpha|M^2C_4-1|}{4} + \delta$ for $\delta > 0$, we get (3.33) and, a fortiori, (3.28).

*(iii)* The condition (3.29) is a direct consequence of Assumption (A2).

□

We shall assume in the sequel that $\gamma > \underline{\gamma}$ and we denote $\beta_s^i = e^{-\gamma(S_i+s)}$ for $s \geq 0$ and $i \in \mathcal{I}$.

# 4 Dynamic programming principle

In this section, we aim at showing the dynamic programming principle for the previously introduced optimization problem. To attain this objective, we will leverage the established regularity of the value function as proved in Proposition 3.3 and approximate the value functions using $\varepsilon$-optimal stopping lines. This technique is widely employed in the stochastic control literature (see, *e.g.*, [11, 5]) when minimal regularity conditions of the value function are known. It serves as an alternative approach, circumventing the need for measurable selection results, which can often be intricate and complex.

**Theorem 4.2.** *Under Assumptions A1, A2, we have*

$$v_i(x) \ = \ \sup_{\tau\in\mathcal{SL}}\mathbb{E}\left[\prod_{j\in L_\theta\setminus D_\tau}\Big(\beta^j_{\theta_j}v_{ij}\left(X_j^x(\theta_j)\right)\Big)^{\mathbb{1}_{\{\theta_j\leq\tau_j\}}}\prod_{j\in L_\tau\setminus D_\theta}\Big(\beta^j_{\tau_j}g_{ij}\left(X_j^x(\tau_j)\right)\Big)^{\mathbb{1}_{\{\tau_j<\theta_j\}}}\right] \quad (4.34)$$

*for any $x \in \mathbb{R}^d$ and $\theta \in \mathcal{SL}$.*

*Proof.* Without loss of generality, we can suppose $i = \varnothing$. Fix a stopping line $\theta$ and denote $\bar{v}(x)$ on the r.h.s. of (4.34). We first show that $v_\varnothing(x) \leq \bar{v}(x)$. Fix $\tau \in \mathcal{SL}$. The idea to follow is to divide the set $L_\tau$ between the particles that have already been stopped when looking at $L_\theta$ and the ones that have not yet been stopped. It is clear that

$$L_\tau = (L_\tau \setminus (D_\theta \cup L_\theta)) \cup (L_\tau \cap (L_\theta \cup D_\theta))\ .$$

Separating the stopping line $\tau$ between the branches that are stopped before and after $\theta$, we get

$$\mathbb{E}\left[\prod_{i\in L_\tau}\beta^i_{\tau_i}g_i\left(X_i^x(\tau_i)\right)\right] \ = \ \mathbb{E}\left[\prod_{i\in L_\tau\setminus(D_\theta\cup L_\theta)}\beta^i_{\tau_i}g_i\left(X_i^x(\tau_i)\right)\prod_{i\in L_\tau\cap(L_\theta\cup D_\theta)}\beta^i_{\tau_i}g_i\left(X_i^x(\tau_i)\right)\right] \quad (4.35)$$

Taking the conditional expectation given $\mathcal{F}_\theta$, we get

$$\mathbb{E}\left[\prod_{i\in L_\tau}\beta^i_{\tau_i}g_i\left(X^x_i(\tau_i)\right)\right] \;=\; \mathbb{E}\left[\prod_{i\in L_\tau\setminus(D_\theta\cup L_\theta)}\beta^i_{\tau_i}g_i\left(X^x_i(\tau_i)\right)\mathbb{E}\left[\prod_{i\in L_\tau\cap(L_\theta\cup D_\theta)}\beta^i_{\tau_i}g_i\left(X^x_i(\tau_i)\right)\Big|\mathcal{F}_\theta\right]\right] .$$

We have by definitions that $\tau_i<\theta_i$ for $i\in L_\tau\setminus(D_\theta\cup L_\theta)$. Therefore, we get

$$\mathbb{E}\left[\prod_{i\in L_\tau}\beta^i_{\tau_i}g_i\left(X^x_i(\tau_i)\right)\right] \;=\; \mathbb{E}\left[\prod_{i\in L_\tau\setminus(D_\theta\cup L_\theta)}\left(\beta^i_{\tau_i}g_i\left(X^x_i(\tau_i)\right)\right)^{\mathbb{1}_{\{\theta_i>\tau_i\}}}\mathbb{E}\left[\prod_{i\in L_\tau\cap(L_\theta\cup D_\theta)}\beta^i_{\tau_i}g_i\left(X^x_i(\tau_i)\right)\Big|\mathcal{F}_\theta\right]\right] .$$

We split the product on $L_\tau\cap L_\theta$ as follows

$$\prod_{i\in L_\tau\cap L_\theta}\beta^i_{\tau_i}g_i\left(X^x_i(\tau_i)\right) \;=\; \prod_{i\in L_\tau\cap L_\theta}\left(\beta^i_{\tau_i}g_i\left(X^x_i(\tau_i)\right)\right)^{\mathbb{1}_{\{\theta_i>\tau_i\}}}\prod_{i\in L_\tau\cap L_\theta}\left(\beta^i_{\tau_i}g_i\left(X^x_i(\tau_i)\right)\right)^{\mathbb{1}_{\{\theta_i\le\tau_i\}}} .$$

This gives

$$\mathbb{E}\left[\prod_{i\in L_\tau}\beta^i_{\tau_i}g_i\left(X^x_i(\tau_i)\right)\right] \;=$$

$$\mathbb{E}\left[\prod_{i\in L_\tau\setminus D_\theta}\left(\beta^i_{\tau_i}g_i\left(X^x_i(\tau_i)\right)\right)^{\mathbb{1}_{\{\theta_i>\tau_i\}}}\mathbb{E}\left[\prod_{i\in L_\tau\cap L_\theta}\left(\beta^i_{\tau_i}g_i\left(X^x_i(\tau_i)\right)\right)^{\mathbb{1}_{\{\theta_i\le\tau_i\}}}\prod_{i\in L_\tau\cap D_\theta}\beta^i_{\tau_i}g_i\left(X^x_i(\tau_i)\right)\Big|\mathcal{F}_\theta\right]\right] .$$

Notice that

$$L_\tau\cap D_\theta \;=\; \bigcup_{i\in L_\theta\setminus D_\tau}\{j\in L_\tau : i\prec j\} .$$

Combining this with Theorem 2.1, we get that (4.35) can be rewritten as

$$\mathbb{E}\left[\prod_{i\in L_\tau}\beta^i_{\tau_i}g_i\big(X^x_i(\tau_i)\big)\right] \;=\; \mathbb{E}\left[\prod_{i\in L_\tau\setminus D_\theta}\left(\beta^i_{\tau_i}g_i\left(X^x_i(\tau_i)\right)\right)^{\mathbb{1}_{\{\tau_i<\theta_i\}}}\prod_{i\in L_\theta\setminus D_\tau}\left(\beta^i_{\theta_i}J_i\left(X^x_i(\theta_i),\tau^{i,\theta_i}\right)\right)^{\mathbb{1}_{\{\theta_i\le\tau_i\}}}\right] ,$$

with $\tau^{i,s}$ the stopping line defined on $\Omega\cap\{\tau_i\ge s\}$ as

$$\tau^{i,s}_\varnothing \;:=\; (\tau_i-s)\mathbb{1}_{s\le\tau_i<\rho_i}+(\rho_i-s)\mathbb{1}_{\tau_i=\rho_i}$$

and $\tau^{i,s}_j:=\tau_{ij}$, for $j\in\mathcal{I}\setminus\{\varnothing\}$. From the definition of the value function $v$, we get

$$\mathbb{E}\left[\prod_{i\in L_\tau}\beta^i_{\tau_i}g_i\big(X^x_i(\tau_i)\big)\right] \;\le\; \mathbb{E}\left[\prod_{i\in L_\tau\setminus D_\theta}\left(\beta^i_{\tau_i}g_i\left(X^x_i(\tau_i)\right)\right)^{\mathbb{1}_{\{\tau_i<\theta_i\}}}\prod_{i\in L_\theta\setminus D_\tau}\left(\beta^i_{\theta_i}v_i\left(X^x_i(\theta_i)\right)\right)^{\mathbb{1}_{\{\theta_i\le\tau_i\}}}\right] ,$$

and

$$v_\varnothing(x) \;\le\; \bar v(x) .$$

We now turn to the reverse inequality. Fix an open ball $B(x,r)$ for $r>0$ . Define the stopping line $\theta^r$ by

$$\begin{aligned}\theta^r_\varnothing &= \inf\left\{s\geq 0 : X^x_\varnothing(s)\notin B(x,r)\right\}\wedge\theta_\varnothing\wedge\rho_\varnothing,\\ \theta^r_i &= \begin{cases}\inf\left\{s\geq 0 : X^x_i(s)\notin B(x,r)\right\}\wedge\theta_i\wedge\rho_i, & \text{if } \theta^r_j=\rho_j \text{ for any } j\prec i,\\ \theta^r_i=\rho_i, & \text{else.}\end{cases}\end{aligned}$$

With this stopping line, consider the following function

$$\bar v_r(x) := \sup_{\tau\in\mathcal{SL}}\mathbb{E}\left[\prod_{i\in L_{\theta^r}\setminus D_\tau}\left(\beta^i_{\theta^r_i}v_i\left(X^x_i(\theta^r_i)\right)\right)^{\mathbb{1}_{\{\theta^r_i\leq\tau_i\}}}\prod_{i\in L_\tau\setminus D_{\theta^r}}\left(\beta^i_{\tau_i}g_i\left(X^x_i(\tau_i)\right)\right)^{\mathbb{1}_{\{\tau_i<\theta^r_i\}}}\right].$$

Fix some $\varepsilon\in\left(0,\frac12\right)$. By definition, we can find a stopping line $\tau^\varepsilon$ such that

$$\bar v_r(x) \leq \mathbb{E}\left[\prod_{i\in L_{\theta^r}\setminus D_{\tau^\varepsilon}}\left(\beta^i_{\theta^r_i}v_i(X^x_i(\theta^r_i))\right)^{\mathbb{1}_{\{\theta^r_i\leq\tau^\varepsilon_i\}}}\prod_{i\in L_{\tau^\varepsilon}\setminus D_{\theta^r}}\left(\beta^i_{\tau^\varepsilon_i}g_i(X^x_i(\tau^\varepsilon_i))\right)^{\mathbb{1}_{\{\tau^\varepsilon_i<\theta^r_i\}}}\right]+\varepsilon\ . \quad (4.36)$$

Consider now a partition $\{B_n\}_n$ of the closure of $B(x,r)$ and a sequence $\{x_n\}_n$ such that $x_n\in B_n$ for any $n\geq 0$. It is clear that we can find $\tau^{i,n}\in\mathcal{SL}$ such that

$$v_i(x_n)\leq J_i(x_n,\tau^{i,n})+\varepsilon/3\ , \quad (4.37)$$

for any $i\in\mathcal{I}$. Moreover, the proof of (ii) in Proposition 3.3 shows that $J_i(\cdot,\tau)$ is a continuous function for any $i\in\mathcal{I}$ and any $\tau\in\mathcal{SL}$, and, from (3.23), have that $J_i$ vanishes for $i$ that tends to infinity. Combining this with the continuity of the value functions $v_i$ for $i\in\mathcal{I}$, we have that the couple $(x_n,B_n)$ can be chosen to satisfy the following

$$\max_{i\in\mathcal{I}}\left(|v_i(x)-v_i(x_n)|+|J_i(x,\tau^{i,n})-J_i(x_n,\tau^{i,n})|\right)\leq\varepsilon/3,\qquad \text{for } x\in B_n\ . \quad (4.38)$$

Define the following family of random variables $\hat\tau=\{\hat\tau_i(x)\}_{x\in\mathbb{R}^d,i\in\mathcal{I}}$ by

$$\hat\tau_{i\ell}(x) := \tau^\varepsilon_{i\ell}\ ,$$

for $x\in\mathbb{R}^d$ and $i\in L_{\tau^\varepsilon}\setminus D_{\theta^r}$ such that $\tau^\varepsilon_i<\theta^r_i$ and $\ell\in\mathcal{I}$, and

$$\begin{aligned}\hat\tau_i(x) &:= \theta^r_i+\sum_{n\geq 0}\tau^{i,n}_\varnothing\mathbb{1}_{B_n}(x)\ , \qquad \text{for } x\in\mathbb{R}^d\ ,\\ \hat\tau_{i\ell}(x) &:= \sum_{n\geq 0}\tau^{i,n}_\ell\mathbb{1}_{B_n}(x)\ , \qquad \text{for } x\in\mathbb{R}^d,\ \ell\in\mathcal{I}\setminus\{\varnothing\}\ ,\end{aligned}$$

for $i\in L_{\theta^r}\setminus D_{\tau^\varepsilon}$ such that $\theta^r_i\leq\tau^\varepsilon_i$. We observe that $(\hat\tau_i)_{i\in\mathcal{I}}$, defined by

$$\bar\tau_i := \hat\tau_i(X^x_i(\theta^r_i))\ , \quad i\in\mathcal{I}\ ,$$

is a stopping line and, from (4.37) and (4.38), we get

$$\mathbb{E}\left[\prod_{i\in L_{\theta^r}\setminus D_{\tau^\varepsilon}}\left(\beta^i_{\theta^r_i}v_i\left(X^x_i(\theta^r_i)\right)\right)^{\mathbb{1}_{\{\theta^r_i\le\tau^\varepsilon_i\}}}\prod_{i\in L_{\tau^\varepsilon}\setminus D_{\theta^r}}\left(\beta^i_{\tau^\varepsilon_i}g_i\left(X^x_i(\tau^\varepsilon_i)\right)\right)^{\mathbb{1}_{\{\tau^\varepsilon_i<\theta^r_i\}}}\right] \;\le$$

$$\mathbb{E}\left[\prod_{i\in L_{\theta^r}\setminus D_{\bar\tau}}\left(\beta^i_{\theta^r_i}[J_i\left(X^x_i(\theta^r_i),(\bar\tau_{i\ell})_{\ell\in\mathcal{I}}\right)+\varepsilon]\right)^{\mathbb{1}_{\{\theta^r_i\le\bar\tau_i\}}}\prod_{i\in L_{\bar\tau}\setminus D_{\theta^r}}\left(\beta^i_{\bar\tau_i}g_i\left(X^x_i(\bar\tau_i)\right)\right)^{\mathbb{1}_{\{\bar\tau_i<\theta^r_i\}}}\right] \quad .$$

From Assumption A2 (i), there exists $N\in\mathbb{N}$ such that

$$\sup_{x\in\mathbb{R}^d}\left|g_i(x)\right| \;\le\; \frac{1}{2}\,,$$

for $i\notin\mathcal{I}_N$ where $\mathcal{I}_N:=\{j\in\mathcal{I}\;:\;|j|\le N\}$. We then

$$|J_i(x,\tau)| \;\le\; \frac{1}{2} \text{ if } |i| \;\notin\; \mathcal{I}_N\,,$$

for any $x\in\mathbb{R}^d$ and $\tau\in\mathcal{SL}$, where $\bar N=\#\mathcal{I}_N$. Moreover, from Proposition 3.3 there exist $C>0$ and $p\ge1$ such that

$$|J_i(x,\tau)| \;\le\; C(1+|x|^p) \qquad \text{if } i\;\in\;\mathcal{I}_N\,,$$

for any $x \in \mathbb{R}^d$ and $\tau \in \mathcal{SL}$. We therefore get from the inclusion–exclusion principle

$$\begin{aligned}
\prod_{i\in L_{\theta^r}} \beta^i_{\theta^r_i}[J_i\left(X^x_i(\theta^r_i),\bar\tau_i\right)+\varepsilon] &\leq \prod_{i\in L_{\theta^r}} \beta^i_{\theta^r_i} J_i\left(X^x_i(\theta^r_i),\bar\tau_i\right) + \sum_{A\subsetneq L_{\theta^r}} \left(\prod_{i\in A} \beta^i_{\theta^r_i} J_i\left(X^x_i(\theta^r_i),\bar\tau_i\right)\right) \varepsilon^{\#(L_{\theta^r}\setminus A)} \\
&\leq \prod_{i\in L_{\theta^r}} \beta^i_{\theta^r_i} J_i\left(X^x_i(\theta^r_i),\bar\tau_i\right) \\
&\quad + \sum_{\substack{A\subset L_{\theta^r}\cap\mathcal{I}_N\,,\\ B\subset L_{\theta^r}\cap\mathcal{I}_N^c\,,\, A\cup B\neq L_{\theta^r}}} \prod_{i\in A}\left(1+|X^x_i(\theta^r_i)|^p\right) C^{\#A}\left(\frac12\right)^{\#B} \varepsilon^{\#L_{\theta^r}\setminus(A\cup B)} \\
&\leq \prod_{i\in L_{\theta^r}} \beta^i_{\theta^r_i} J_i\left(X^x_i(\theta^r_i),\bar\tau_i\right) \\
&\quad + \prod_{i\in\mathcal{I}_N}\left(1+|X^x_i(\theta^r_i)|^p\right)\Bigg( \left(C+\varepsilon\right)^{\#L_{\theta^r}\cap\mathcal{I}_N}\left(\frac12+\varepsilon\right)^{\#L_{\theta^r}\cap\mathcal{I}_N^c} \\
&\quad - C^{\#L_{\theta^r}\cap\mathcal{I}_N}\left(\frac12\right)^{\#L_{\theta^r}\cap\mathcal{I}_N^c}\Bigg) \\
&\leq \prod_{i\in L_{\theta^r}} \beta^i_{\theta^r_i} J_i\left(X^x_i(\theta^r_i),\bar\tau_i\right) \\
&\quad + \prod_{i\in\mathcal{I}_N}\left(1+|X^x_i(\theta^r_i)|^p\right)\Bigg( \left(C+\varepsilon\right)^{\#L_{\theta^r}\cap\mathcal{I}_N}\left(\left(\frac12+\varepsilon\right)^{\#L_{\theta^r}\cap\mathcal{I}_N^c} - \left(\frac12\right)^{\#L_{\theta^r}\cap\mathcal{I}_N^c}\right) \\
&\quad + \left(\frac12\right)^{\#L_{\theta^r}\cap\mathcal{I}_N^c}\left(\left(C+\varepsilon\right)^{\#L_{\theta^r}\cap\mathcal{I}_N} - C^{\#L_{\theta^r}\cap\mathcal{I}_N}\right)\Bigg).
\end{aligned}$$

Since $\varepsilon \in \left(0,\frac12\right)$ and $\#\mathcal{I}_N = \bar N$, we get

$$\prod_{i\in L_{\theta^r}} \beta^i_{\theta^r_i}\Big(J_i\left(X^x_i(\theta^r_i),\bar\tau_i\right)+\varepsilon\Big) \leq \prod_{i\in L_{\theta^r}} \beta^i_{\theta^r_i} J_i\left(X^x_i(\theta^r_i),\bar\tau_i\right) + \Big((C+1)^{\bar N}\varepsilon + \bar N(C+1)^{\bar N}\varepsilon\Big)\prod_{i\in\mathcal{I}_N}\left(1+|X^x_i(\theta^r_i)|^p\right).$$

This means that the previous computation, together with (4.36), (4.37), and (4.38), implies that

$$\begin{aligned}
&\mathbb{E}\left[\prod_{i\in L_{\theta^r}}\Big(\beta^i_{\theta^r_i} v_i\left(X^x_i(\theta^r_i)\right)\Big)^{\mathbb{1}_{\{\theta^r_i\leq\tau^\varepsilon_i\}}}\prod_{i\in L_\tau}\Big(\beta^i_{\tau^\varepsilon_i} g_i\left(X^x_i(\tau^\varepsilon_i)\right)\Big)^{\mathbb{1}_{\{\tau^\varepsilon_i<\theta^r_i\}}}\right] \\
\leq\ &\mathbb{E}\left[\prod_{i\in L_{\theta^r}}\Big(\beta^i_{\theta^r_i} J_i\left(X^x_i(\theta^r_i),\bar\tau_i\right)\Big)^{\mathbb{1}_{\{\theta^r_i\leq\bar\tau_i\}}}\prod_{i\in L_{\bar\tau}}\left(\beta^i_{\bar\tau_i} g_i(X^x_i(\bar\tau_i))\right)^{\mathbb{1}_{\{\bar\tau_i<\theta^r_i\}}}\right] \\
&+\Big((C+1)^{\bar N}\varepsilon + \bar N(C+1)^{\bar N}\varepsilon\Big)\mathbb{E}\left[\prod_{i\in\mathcal{I}_N}\left(1+|X^x_i(\theta^r_i)|^p\right)\right].
\end{aligned}$$

Using Theorem 2.1, we obtain

$$\mathbb{E}\left[\prod_{i\in L_{\theta^r}}\Big(\beta^i_{\theta^r_i} J_i\left(X^x_i(\theta^r_i),\bar\tau\right)\Big)^{\mathbb{1}_{\{\theta^r_i\leq\bar\tau_i\}}}\prod_{i\in L_{\bar\tau}}\left(\beta^i_{\bar\tau_i} g_i\left(X^x_i(\bar\tau_i)\right)\right)^{\mathbb{1}_{\{\bar\tau_i<\theta^r_i\}}}\right] = \mathbb{E}\left[\prod_{i\in L_{\bar\tau}}\beta^i_{\bar\tau_i} g_i\left(X^x_i(\bar\tau_i)\right)\right].$$

From Proposition 2.2 (ii) and the definition of the stopping line $\theta^r$, there exists a constant still denoted by $C$ and an integer $q$ such that

$$\mathbb{E}\left[\prod_{i\in\mathcal{I}_N}(1+|X_i^x(\theta_i^r)|^p)\right] \;\leq\; C(1+|x|^q)\;,$$

for all $x\in\mathbb{R}^d$. Therefore, we achieve

$$\mathbb{E}\left[\prod_{i\in L_{\theta^r}}\left(\beta^i_{\theta_i^r}v_i\left(X_i^x(\theta_i^r)\right)\right)^{\mathbb{1}_{\{\theta_i^r\leq\tau_i^\varepsilon\}}}\prod_{i\in L_\tau}\left(\beta^i_{\tau_i^\varepsilon}g_i\left(X_i^x(\tau_i^\varepsilon)\right)\right)^{\mathbb{1}_{\{\tau_i^\varepsilon<\theta_i^r\}}}\right] \;\leq$$
$$\mathbb{E}\left[\prod_{i\in L_{\bar\tau}}\beta^i_{\bar\tau_i}g_i\left(X_i^x(\bar\tau_i)\right)\right]+\Big((C+1)^{\bar N}\varepsilon+\bar N(C+1)^{\bar N}\varepsilon\Big)C(1+|x|^q)\;,$$

and

$$\bar v_r(x) \;\leq\; v_\varnothing(x)+\varepsilon+\Big((C+1)^{\bar N}\varepsilon+\bar N(C+1)^{\bar N}\varepsilon\Big)C(1+|x|^q)\;.$$

Since $\varepsilon$ is arbitrarily chosen in $(0,1/2)$, we have $\bar v_r(x)\leq v_\varnothing(x)$. Letting $r$ go to infinity, from Assumption A2, Proposition 2.2 and (3.27), we deduce that $\bar v(x)\leq v_\varnothing(x)$.

□

# 5 Dynamic programming equation

The value function associated with an optimal stopping problem is known to be the solution to an obstacle problem (see, *e.g.*, [29, Theorem 4.5] and [26, Lemma 5.2.2]). We prove in this section a similar result for our value function. To this purpose, we consider the following operator $\mathcal{L}$

$$\begin{aligned}\mathcal{L}:\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d\times\mathbf{S}^d\times\mathbb{R}^{\mathbb{N}} &\to \mathbb{R}\\ \big(x,r,p,M,(r_\ell)_{\ell\in\mathbb{N}}\big) &\mapsto \frac{1}{2}\mathrm{Tr}\Big(\sigma\sigma^\top(x)M\Big)+b(x)^\top p+\alpha\sum_{k\geq0}p_k\prod_{\ell=0}^{k-1}r_\ell-(\alpha+\gamma)r\;,\end{aligned}$$

with $\mathbf{S}^d$ being the set of symmetric matrices of dimension $d\times d$. We show in this section that the problem of stopping lines can be characterized by the following PDE

$$\min\Big\{-\mathcal{L}\left(x,v_i(x),Dv_i(x),D^2v_i(x),\big(v_{i\ell}(x)\big)_{\ell\in\mathbb{N}}\right)\;;\;v_i(x)-g_i(x)\Big\}=0\;,\tag{5.39}$$

for $i\in\mathcal{I}$ and $x\in\mathbb{R}^d$. To simplify the notation, we write $\mathcal{L}(i,v)(x)$ for $\mathcal{L}\Big(x,v_i(x),Dv_i(x),D^2v_i(x),\big(v_{i\ell}(x)\big)_{\ell\in\mathbb{N}}\Big)$.

The previous PDE shows a close connection to the underlying tree structure, having a coupling between the value function valued in $i$ and in its direct offspring $i\ell$ for $\ell\geq0$. Furthermore, when $r_\ell=r$ holds for all $\ell\in\mathbb{N}$, the convergence of the operator $\mathcal{L}$ is connected with the radius of convergence of the power series $\sum_{k\geq0}p_k|x|^k$. This consideration leads us to introduce the following

assumption.

**Assumption A3.** *The series $\sum_{k\geq 0} p_k|x|^k$ has infinite radius of convergence.*

We will prove that the value function (3.26) is a viscosity solution for (5.39). We therefore introduce a definition of viscosity solution adapted to our framework, as in [17, Definition 4.2].

**Definition 5.1.** *Let $u=(u_i)_{i\in\mathcal{I}}$ such that $u_i:\mathbb{R}^d\to\mathbb{R}_+$ is a continuous function for $i\in\mathcal{I}$.*
*(i) $u$ is a* viscosity supersolution *to* (5.39) *if, for $(i_0,x_0)\in\mathcal{I}\times\mathbb{R}^d$, $\varphi_i\in C^2(\mathbb{R}^d)$ for $i\in\mathcal{I}$, and $\bar\varphi\in C^0(\mathbb{R}^d)$ such that $\varphi_i$ is nonnegative for $i\in\mathcal{I}$,*

$$\sup_{i\in\mathcal{I}}\varphi_i(x) \;\leq\; \bar\varphi(x), \qquad \text{for } x\in\mathbb{R}^d\,, \tag{5.40}$$

*and*

$$0 \;=\; \left(u_{i_0}-\varphi_{i_0}\right)(x_0) \;=\; \min_{\mathcal{I}\times\mathbb{R}^d}\left(u_\cdot-\varphi_\cdot\right)\,,$$

*we have*

$$\min\left\{-\mathcal{L}(i_0,\varphi_\cdot)(x_0)\;;\;\varphi_{i_0}(x_0)-g_{i_0}(x_0)\right\}\geq 0\,.$$

*(ii) $u$ is a* viscosity subsolution *to* (5.39) *if, for $(i_0,x_0)\in\mathcal{I}\times\mathbb{R}^d$, $\varphi_i\in C^2(\mathbb{R}^d)$ for $i\in\mathcal{I}$, and $\bar\varphi\in C^0(\mathbb{R}^d)$ such that $\varphi_i$ is nonnegative for $i\in\mathcal{I}$,* (5.40) *is satisfied, and*

$$0 \;=\; \left(u_{i_0}-\varphi_{i_0}\right)(x_0) \;=\; \max_{\mathcal{I}\times\mathbb{R}^d}\left(u_\cdot-\varphi_\cdot\right)\,,$$

*we have*

$$\min\left\{-\mathcal{L}(i_0,\varphi_\cdot)(x_0)\;;\;\varphi_{i_0}(x_0)-g_{i_0}(x_0)\right\}\leq 0\,.$$

*(iii) $u$ is a* viscosity solution *to* (5.39) *if it is both a viscosity sub and supersolution to* (5.39).

**Theorem 5.3.** *Under Assumptions A1, A2, and A3, the value function $v$ is a viscosity solution to* (5.39).

*Proof.* We begin by proving the supersolution property. Fix $(i_0,x_0)\in\mathcal{I}\times\mathbb{R}^d$ and let $\varphi\in C^0(\mathbb{R}^d)$ and $\varphi_i\in C^2(\mathbb{R}^d)$ for $i\in\mathcal{I}$ be such that

$$\sup_i|\varphi_i|\leq\varphi \tag{5.41}$$

and

$$0=\left(v_{i_0}-\varphi_{i_0}\right)(x_0)=\min_{(i,x)\in\mathcal{I}\times\mathbb{R}^d}\left(v_i-\varphi_i\right)(x)\,. \tag{5.42}$$

Without loss of generality, we can assume this minimum to be strict in $x$ once fixed $i_0$.

Consider, first, the following (trivial) stopping line $\tau^{\text{triv}}$

$$\tau_{\varnothing}^{\text{triv}} := 0, \quad \text{and} \quad \tau_j^{\text{triv}} := \rho_j, \quad \text{for } j \in \mathcal{I}\backslash\{\varnothing\} .$$

Combining it with (5.42), we get the inequality $v_{i_0}(x_0) = \varphi_{i_0}(x_0) \geq g_{i_0}(x_0)$.

Consider, now, the following stopping time

$$\bar{\theta}^h := \inf\{t > 0 : X_{\varnothing}^{x_0}(t) \notin B_1(x_0)\} \wedge h ,$$

where $B_1(x_0)$ is the unit ball of $\mathbb{R}^d$ centred at $x_0$. Fix $h > 0$ and define the following stopping line $\theta^h$ by

$$\begin{aligned}
\theta_{\varnothing}^h &:= \bar{\theta}^h \wedge \rho_{\varnothing}, \\
\theta_{\ell}^h &:= \begin{cases} \rho_\ell & \text{if } \bar{\theta}^h < \rho_{\varnothing} \\ 0 & \text{else} \end{cases}, \quad \text{for } \ell \in \mathbb{N}, \\
\theta_j^h &:= \rho_j, \quad \text{for } j \in \mathcal{I}\backslash(\{\varnothing\} \cup \mathbb{N}) .
\end{aligned}$$

This stopping line stops at the exit time $\bar{\theta}^h$ or at the branching time $\rho_{\varnothing}$ if it arrives before $\bar{\theta}^h$. It follows from (4.34), applied with the stopping lines $\theta = \theta^h$ and $\tau = \theta^h$, that

$$v_{i_0}(x_0) \geq \mathbb{E}\left[\left(\beta_{\bar{\theta}^h}^{\varnothing} v_{i_0}\left(X_{\varnothing}^{x_0}\left(\bar{\theta}^h\right)\right)\right)\mathbb{1}_{\bar{\theta}^h < \rho_{\varnothing}} + \prod_{\ell=0}^{\nu_{\varnothing}-1}\left(\beta_{\rho_{\varnothing}}^{\varnothing} v_{i_0\ell}\left(X_{\ell}^{x_0}(0)\right)\right)\mathbb{1}_{\bar{\theta}^h \geq \rho_{\varnothing}}\right] .$$

From the definition of $X_{\ell}^{x_0}(0)$ we get

$$v_{i_0}(x_0) \geq \mathbb{E}\left[\left(\beta_{\bar{\theta}^h}^{\varnothing} v_{i_0}\left(X_{\varnothing}^{x_0}\left(\bar{\theta}^h\right)\right)\right)\mathbb{1}_{\bar{\theta}^h < \rho_{\varnothing}} + \prod_{\ell=0}^{\nu_{\varnothing}-1}\left(\beta_{\rho_{\varnothing}}^{\varnothing} v_{i_0\ell}\left(X_{\varnothing}^{x_0}(\rho_{\varnothing})\right)\right)\mathbb{1}_{\bar{\theta}^h \geq \rho_{\varnothing}}\right] .$$

Using (5.42), since the functions $v_j$ and $\varphi_j$ are positive for $j \in \mathcal{I}$, we have

$$\varphi_{i_0}(x_0) \geq \mathbb{E}\left[\left(\beta_{\bar{\theta}^h}^{\varnothing} \varphi_{i_0}\left(X_{\varnothing}^{x_0}(\bar{\theta}^h)\right)\right)\mathbb{1}_{\bar{\theta}^h < \rho_{\varnothing}} + \prod_{\ell=0}^{\nu_{\varnothing}-1}\left(\beta_{\rho_{\varnothing}-1}^{\varnothing} \varphi_{i_0\ell}\left(X_{\varnothing}^{x_0}(\rho_{\varnothing})\right)\right)\mathbb{1}_{\bar{\theta}^h \geq \rho_{\varnothing}}\right] .$$

From Proposition 2.1, we get

$$\begin{aligned}
\varphi_{i_0}(x_0) &\geq \mathbb{E}\left[\left(\beta_{\bar{\theta}^h}^{\varnothing} \varphi_{i_0}\left(X_{\varnothing}^{x_0}(\bar{\theta}^h)\right)\right)\mathbb{1}_{\bar{\theta}^h < \rho_{\varnothing}} + \sum_{k\geq 1} p_k \prod_{\ell=0}^{k-1}\left(\beta_{\rho_{\varnothing}}^{\varnothing} \varphi_{i_0\ell}\left(X_{\varnothing}^{x_0}(\rho_{\varnothing})\right)\right)\mathbb{1}_{\bar{\theta}^h \geq \rho_{\varnothing}}\right] \quad (5.43) \\
&= \mathbb{E}\Bigg[\left(\beta_{\bar{\theta}^h \wedge \rho_{\varnothing}}^{\varnothing} \varphi_{i_0}\left(X_{\varnothing}^{x_0}(\bar{\theta}^h \wedge \rho_{\varnothing})\right)\right) \\
&\qquad + \sum_{k\geq 1} p_k \left(\prod_{\ell=0}^{k-1} \beta_{\rho_{\varnothing}}^{\varnothing} \varphi_{i_0\ell}\left(X_{\varnothing}^{x_0}(\rho_{\varnothing})\right) - \beta_{\rho_{\varnothing}}^{\varnothing} \varphi_{i_0}\left(X_{\varnothing}^{x_0}(\rho_{\varnothing})\right)\right)\mathbb{1}_{\bar{\theta}^h \geq \rho_{\varnothing}}\Bigg] .
\end{aligned}$$

Still using Proposition 2.1, we have

$$\begin{aligned}\varphi_{i_0}(x_0) &\geq \mathbb{E}\Bigg[\left(\beta^{\varnothing}_{\bar\theta^h\wedge\rho_\varnothing}\varphi_{i_0}\left(X^{x_0}_{\varnothing}(\bar\theta^h\wedge\rho_\varnothing)\right)\right)\\ &\quad+\sum_{k\geq1}p_k\mathbb{E}\left[\int_0^{\bar\theta^h}\left(\prod_{\ell=0}^{k-1}\beta^{\varnothing}_s\varphi_{i_0\ell}\left(X^{x_0}_{\varnothing}(s)\right)-\beta^{\varnothing}_s\varphi_{i_0}\left(X^{x_0}_{\varnothing}(s)\right)\right)\alpha e^{-\alpha s}ds\right].\end{aligned}$$

Applying Ito's formula and Proposition 2.1, we get

$$\begin{aligned}0 &\geq \mathbb{E}\left[\int_0^{\bar\theta^h\wedge\rho_\varnothing}\beta^{\varnothing}_s\left(\frac{1}{2}\mathrm{Tr}\left(\sigma\sigma^\top D^2\varphi_{i_0}\right)+\left(b^\top D\varphi_{i_0}\right)-\gamma\varphi_{i_0}\right)\left(X^{x_0}_{\varnothing}(s)\right)ds\right]\\ &\quad+\sum_{k\geq1}p_k\mathbb{E}\left[\int_0^{\bar\theta^h}\left(\prod_{\ell=0}^{k-1}\beta^{\varnothing}_s\varphi_{i_0\ell}\left(X^{x_0}_{\varnothing}(s)\right)-\beta^{\varnothing}_s\varphi_{i_0}\left(X^{x_0}_{\varnothing}(s)\right)\right)\alpha e^{-\alpha s}ds\right]. \qquad (5.44)\end{aligned}$$

We divide by $h>0$ both sides of (5.44) and we get from the mean value theorem and the dominated convergence theorem that $-\mathcal{L}\left(i_0,\varphi_\cdot\right)(x_0)\geq 0$.

We turn now to the proof of the subsolution property. Fix $(i_0,x_0)\in\mathcal{I}\times\mathbb{R}^d$ and let $\varphi\in C^0(\mathbb{R}^d)$ and $\varphi_i\in C^2(\mathbb{R}^d)$ for $i\in\mathcal{I}$ be such that $\sup_i|\varphi_i|\leq\varphi$ and

$$0=\left(v_{i_0}-\varphi_{i_0}\right)(x_0)=\max_{(i,x)\in\mathcal{I}\times\mathbb{R}^d}\left(v_i-\varphi_i\right)(x)\ . \qquad (5.45)$$

Without loss of generality, we suppose that $i_0=\varnothing$ and we take the maximum to be strict in $x$ and that

$$\max_{(\ell,x)\in\mathbb{N}\times\mathbb{R}^d}\left(v_\ell-\varphi_\ell\right)(x) \;=\; -\delta \;<\; 0\,. \qquad (5.46)$$

We argue by contradiction and assume that

$$2\eta:=\min\left\{-\mathcal{L}(\varnothing,\varphi_\cdot)(x_0)\ ;\ \varphi_\varnothing(x_0)-g_\varnothing(x_0)\right\}>0\,.$$

Since all the functions in the previous inequality are continuous, we may find $\varepsilon>0$ such that

$$-\mathcal{L}(\varnothing,e^{-\gamma s}(\varphi_\cdot-y))(x) \;>\; \eta\,, \qquad (5.47)$$

$$\left(\varphi_\varnothing-g_\varnothing\right)(x) \;>\; \eta\,, \qquad (5.48)$$

for all $s,y\in[0,\varepsilon)$ and $x\in B_\varepsilon(x_0)$, with $B_\varepsilon(x_0)$ the open ball centred at $x_0$ with radius $\varepsilon$. Observe that, since $x_0$ is a strict maximizer, we have

$$-\zeta \;=\; \max_{\partial B_\varepsilon(x_0)}(v_\varnothing-\varphi_\varnothing)(x)<0\,, \qquad (5.49)$$

where $\partial B_\varepsilon(x_0)$ denotes the boundary of $B_\varepsilon(x_0)$. We now show that (5.47), (5.48), and (5.49) lead

to a contradiction with (4.34). Define the stopping time $\bar{\theta}^\varepsilon$ by

$$\bar{\theta}^\varepsilon := \inf\left\{t > 0 : (t, X^{x_0}_\varnothing(t)) \notin [0,\varepsilon) \times B_\varepsilon(x_0)\right\} \ .$$

As for the supersolution property, we consider the stopping line $\theta^\varepsilon$ defined by

$$\begin{array}{rcl} \theta^\varepsilon_\varnothing & := & \bar{\theta}^\varepsilon \wedge \rho_\varnothing, \\ \theta^\varepsilon_\ell & := & \begin{cases} \rho_\ell & \text{if } \bar{\theta}^\varepsilon < \rho_\varnothing \\ 0 & \text{else} \end{cases} \quad , \quad \text{for } \ell \in \mathbb{N}, \\ \theta^\varepsilon_j & := & \rho_j, \quad \text{for } j \in \mathcal{I} \backslash \left(\{\varnothing\} \cup \mathbb{N}\right) \ . \end{array}$$

This stopping line stops at the exit time $\bar{\theta}^\varepsilon$ or at the branching event $\rho_\varnothing$ if it arrives before $\bar{\theta}^\varepsilon$. We next have from (5.46) and (5.49) that

$$\begin{aligned} v_\varnothing(x_0) - \mathbb{E}\left[\prod_{j\in L_{\theta^\varepsilon}\backslash D_\tau} \left(\beta^j_{\theta^\varepsilon_j} v_j\left(X^{x_0}_j(\theta^\varepsilon_j)\right)\right)^{\mathbb{1}_{\{\theta^\varepsilon_j \le \tau_j\}}} \prod_{j\in L_\tau \backslash D_{\theta^\varepsilon}} \left(\beta^j_{\tau_j} g_j\left(X^{x_0}_j(\tau_j)\right)\right)^{\mathbb{1}_{\{\tau_j < \theta^\varepsilon_j\}}}\right] &= \\ \varphi_\varnothing(x_0) - \mathbb{E}\left[\prod_{j\in L_{\theta^\varepsilon}\backslash D_\tau} \left(\beta^j_{\theta^\varepsilon_j} v_j\left(X^{x_0}_j(\theta^\varepsilon_j)\right)\right)^{\mathbb{1}_{\{\theta^\varepsilon_j \le \tau_j\}}} \prod_{j\in L_\tau \backslash D_{\theta^\varepsilon}} \left(\beta^j_{\tau_j} g_j\left(X^{x_0}_j(\tau_j)\right)\right)^{\mathbb{1}_{\{\tau_j < \theta^\varepsilon_j\}}}\right] &= \\ \varphi_\varnothing(x_0) - \mathbb{E}\left[\mathbb{1}_{\{\bar{\theta}^\varepsilon < \rho_\varnothing\}} \left(\beta^\varnothing_{\bar{\theta}^\varepsilon} v_\varnothing\left(X^{x_0}_\varnothing(\bar{\theta}^\varepsilon)\right)\right)^{\mathbb{1}_{\{\bar{\theta}^\varepsilon \le \tau_\varnothing\}}} \left(\beta^\varnothing_{\tau_\varnothing} g_\varnothing\left(X^{x_0}_\varnothing(\tau_\varnothing)\right)\right)^{\mathbb{1}_{\{\tau_\varnothing < \theta^\varepsilon_\varnothing\}}}\right] & \\ -\mathbb{E}\left[\mathbb{1}_{\{\bar{\theta}^\varepsilon \ge \rho_\varnothing\}} \mathbb{1}_{\{\tau_\varnothing \ge \rho_\varnothing\}} \left(\prod_{\ell=0}^{\nu_\varnothing - 1} \beta^\varnothing_{\rho_\varnothing} v_\ell\left(X^{x_0}_\varnothing(\rho_\varnothing)\right)\right)\right] - \mathbb{E}\left[\mathbb{1}_{\{\bar{\theta}^\varepsilon \ge \rho_\varnothing\}} \mathbb{1}_{\{\tau_\varnothing < \rho_\varnothing\}} \beta^\varnothing_{\tau_\varnothing} g_\varnothing\left(X^{x_0}_\varnothing(\tau_\varnothing)\right)\right] &\ge \\ \varphi_\varnothing(x_0) - \mathbb{E}\left[\mathbb{1}_{\{\bar{\theta}^\varepsilon < \rho_\varnothing\}} \left(\beta^\varnothing_{\bar{\theta}^\varepsilon} \left(\varphi_\varnothing\left(X^{x_0}_\varnothing(\bar{\theta}^\varepsilon)\right) - \zeta\right) \mathbb{1}_{\{\bar{\theta}^\varepsilon \le \tau_\varnothing\}} + \beta^\varnothing_{\tau_\varnothing} g_\varnothing\left(X^{x_0}_\varnothing(\tau_\varnothing)\right) \mathbb{1}_{\{\tau_\varnothing < \theta^\varepsilon_\varnothing\}}\right)\right] & \\ -\mathbb{E}\left[\mathbb{1}_{\{\bar{\theta}^\varepsilon \ge \rho_\varnothing\}} \mathbb{1}_{\{\tau_\varnothing \ge \rho_\varnothing\}} \left(\prod_{\ell=0}^{\nu_\varnothing - 1} \beta^\varnothing_{\rho_\varnothing} \left(\varphi_\ell\left(X^{x_0}_\varnothing(\rho_\varnothing)\right) - \delta\right)\right)\right] & \\ -\mathbb{E}\left[\mathbb{1}_{\{\bar{\theta}^\varepsilon \ge \rho_\varnothing\}} \mathbb{1}_{\{\tau_\varnothing < \rho_\varnothing\}} \beta^\varnothing_{\tau_\varnothing} g_\varnothing\left(X^x_\varnothing(\tau_\varnothing)\right)\right] &\ge \\ \varphi_\varnothing(x_0) - \mathbb{E}\left[\mathbb{1}_{\{\bar{\theta}^\varepsilon < \rho_\varnothing\}} \beta^\varnothing_{\bar{\theta}^\varepsilon} \left(\varphi_\varnothing\left(X^{x_0}_\varnothing(\bar{\theta}^\varepsilon)\right) - \zeta\right) \mathbb{1}_{\{\bar{\theta}^\varepsilon \le \tau_\varnothing\}}\right] & \\ -\mathbb{E}\left[\mathbb{1}_{\{\bar{\theta}^\varepsilon \ge \rho_\varnothing\}} \mathbb{1}_{\{\tau_\varnothing \ge \rho_\varnothing\}} \prod_{\ell=0}^{\nu_\varnothing - 1} \beta^\varnothing_{\rho_\varnothing} \left(\varphi_\ell\left(X^{x_0}_\varnothing(\rho_\varnothing)\right) - \delta\right)\right] & \\ -\mathbb{E}\left[\mathbb{1}_{\{\tau_\varnothing < \bar{\theta}^\varepsilon \wedge \rho_\varnothing\}} \beta^\varnothing_{\tau_\varnothing} g_\varnothing\left(X^{x_0}_\varnothing(\tau_\varnothing)\right)\right] & \ , \end{aligned}$$

for any $\tau \in \mathcal{SL}$. Using (5.48), we get

$$\begin{aligned} v_\varnothing(x) - \mathbb{E}\left[\prod_{j\in L_{\theta^\varepsilon}\setminus D_\tau}\left(\beta^j_{\theta^\varepsilon_j} v_j\left(X^{x_0}_j(\theta^\varepsilon_j)\right)\right)^{\mathbb{1}_{\{\theta^\varepsilon_j\le\tau_j\}}}\prod_{j\in L_\tau\setminus D_{\theta^\varepsilon}}\left(\beta^j_{\tau_j} g_j\left(X^{x_0}_j(\tau_j)\right)\right)^{\mathbb{1}_{\{\tau_j<\theta^\varepsilon_j\}}}\right] &\ge \\ \varphi_\varnothing(x) - \mathbb{E}\left[\mathbb{1}_{\{\bar\theta^\varepsilon<\rho_\varnothing\}}\beta^\varnothing_{\bar\theta^\varepsilon}\left(\varphi_\varnothing\left(X^{x_0}_\varnothing(\bar\theta^\varepsilon)\right)-\zeta\right)\mathbb{1}_{\{\bar\theta^\varepsilon\le\tau_\varnothing\}}\right] & \\ -\mathbb{E}\left[\mathbb{1}_{\{\bar\theta^\varepsilon\ge\rho_\varnothing\}}\mathbb{1}_{\{\tau_\varnothing\ge\rho_\varnothing\}}\prod_{\ell=0}^{\nu_\varnothing-1}\beta^\varnothing_{\rho_\varnothing}\Big(\varphi_\ell\left(X^{x_0}_\varnothing(\rho_\varnothing)\right)-\delta\Big)\right] & \\ -\mathbb{E}\left[\mathbb{1}_{\{\tau_\varnothing<\bar\theta^\varepsilon\wedge\rho_\varnothing\}}\beta^\varnothing_{\tau_\varnothing}\left(\varphi_\varnothing\left(X^{x_0}_\varnothing(\tau_\varnothing)\right)-\eta\right)\right] &\ge \\ \varphi_\varnothing(x) - \mathbb{E}\left[\mathbb{1}_{\{\bar\theta^\varepsilon\wedge\tau_\varnothing<\rho_\varnothing\}}\beta^\varnothing_{\bar\theta^\varepsilon\wedge\tau_\varnothing}\left(\varphi_\varnothing\left(X^{x_0}_\varnothing(\bar\theta^\varepsilon\wedge\tau_\varnothing)\right)-\zeta\wedge\eta\wedge\delta\wedge\varepsilon\right)\right] & \\ -\mathbb{E}\left[\mathbb{1}_{\{\bar\theta^\varepsilon\wedge\tau_\varnothing\ge\rho_\varnothing\}}\prod_{\ell=0}^{\nu_\varnothing-1}\beta^\varnothing_{\rho_\varnothing}\Big(\varphi_\ell\left(X^{x_0}_\varnothing(\rho_\varnothing)\right)-\zeta\wedge\eta\wedge\delta\wedge\varepsilon\Big)\right] &= \\ \varphi_\varnothing(x) - \mathbb{E}\left[\prod_{i\in\mathcal{V}_{\bar\theta^\varepsilon\wedge\tau_\varnothing\wedge\rho_\varnothing}}\beta^\varnothing_{\bar\theta^\varepsilon\wedge\tau_\varnothing\wedge\rho_\varnothing}\Big(\varphi_i\left(X^{x_0}_\varnothing(\rho_\varnothing)\right)-\zeta\wedge\eta\wedge\delta\wedge\varepsilon\Big)\right] &, \end{aligned}$$

with $\mathcal{V}$ defined as in (2.4). Applying Ito's formula we have

$$\begin{aligned} \varphi_\varnothing(x_0) - \mathbb{E}\left[\prod_{i\in\mathcal{V}_{\bar\theta^\varepsilon\wedge\tau_\varnothing\wedge\rho_\varnothing}}\beta^\varnothing_{\bar\theta^\varepsilon\wedge\tau_\varnothing\wedge\rho_\varnothing}\Big(\varphi_i\left(X^{x_0}_\varnothing(\rho_\varnothing)\right)-\zeta\wedge\eta\wedge\delta\wedge\varepsilon\Big)\right] &= \\ \zeta\wedge\eta\wedge\delta\wedge\varepsilon + \mathbb{E}\left[\int_0^{\bar\theta^\varepsilon\wedge\tau_\varnothing\wedge\rho_\varnothing} -\mathcal{L}\big(\varnothing,\beta^\varnothing_s\big(\varphi_\cdot-\zeta\wedge\eta\wedge\delta\wedge\varepsilon\big)\big)(X^{x_0}_\varnothing(s))\right] &. \end{aligned}$$

From (5.47) and the definition of $\bar\theta^\varepsilon$, we have

$$\mathbb{E}\left[\int_0^{\bar\theta^\varepsilon\wedge\tau_\varnothing\wedge\rho_\varnothing} -\mathcal{L}(\varnothing,\varphi_\cdot-\zeta\wedge\eta\wedge\delta\wedge\varepsilon)(X^{x_0}_\varnothing(s))\right] \ge 0\,.$$

Therefore,

$$\begin{aligned} v_\varnothing(x_0) - \mathbb{E}\left[\prod_{j\in L_{\theta^\varepsilon}\setminus D_\tau}\left(\beta^j_{\theta^\varepsilon_j} v_j\left(X^{x_0}_j(\theta^\varepsilon_j)\right)\right)^{\mathbb{1}_{\{\theta^\varepsilon_j\le\tau_j\}}}\prod_{j\in L_\tau\setminus D_{\theta^\varepsilon}}\left(\beta^j_{\tau_j} g_j\left(X^{x_0}_j(\tau_j)\right)\right)^{\mathbb{1}_{\{\tau_j<\theta^\varepsilon_j\}}}\right] &\ge \\ \zeta\wedge\eta\wedge\delta\wedge\varepsilon &, \end{aligned}$$

for any $\tau \in \mathcal{SL}$. Since $\zeta\wedge\eta\wedge\delta\wedge\varepsilon > 0$, this contradicts (4.34).

□

We provide a strong comparison principle for the obstacle problem (5.39). The proof of this result is an extension of the usual comparison principle (see, *e.g.*, [26, 29]) with the use of some ideas from [11]. We consider an additional assumption and, for the sake of completeness, provide

the complete proof. We recall that $M$ is the mean of the branching mechanism, and is defined in Assumption A1(ii).

**Assumption A4.** *(i) We have* $\gamma > \alpha(M-1)$.

*(ii) The functions* $g_i$, *for* $i \in \mathcal{I}$, *are uniformly bounded, i.e.,*

$$\sup_{i\in\mathcal{I}} \sup_{x\in\mathbb{R}^d} \left|g_i(x)\right| \;<\; +\infty \,. \tag{5.50}$$

Mimicking the proof of point (i) of Proposition 3.3, we have that the value function is bounded as a consequence of the previous assumption. Therefore, we restrict to prove the following comparison theorem within the set of bounded viscosity solutions.

Prior to establishing the comparison principle, we present the subsequent preliminary lemma. We examine the alterations in the PDE (5.39) when a multiplicative penalization is applied to the viscosity solutions. In particular, take $\kappa > 0$, which will be fixed later, and define $\phi : \mathbb{R}^d \to \mathbb{R}$ as $\phi(x) = (|x|^2+1)^\kappa$, together with the following operator

$$\begin{array}{rcl} \tilde{\mathcal{L}} : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbf{S}^d \times \mathbb{R}^{\mathbb{N}} & \to & \mathbb{R} \\ \big(x, r, p, M, (r_\ell)_{\ell\in\mathbb{N}}\big) & \mapsto & \dfrac{1}{2}\mathrm{Tr}\left(\sigma\sigma^\top(x)M\right) + \tilde{b}(x)^\top p + \alpha \displaystyle\sum_{k\geq 0} p_k \phi^{k-1}(x) \prod_{\ell=0}^{k-1} r_\ell - (\alpha + \tilde{\gamma}(x))r \,, \end{array}$$

with

$$\tilde{b}(x) = b(x) + \left(\frac{\sigma\sigma^\top D\phi}{\phi}\right)(x) \qquad \tilde{\gamma}(x) = \gamma - \left(\frac{b^\top D\phi}{\phi}\right)(x) - \frac{1}{2\phi(x)}\mathrm{Tr}\left(\sigma\sigma^\top D^2\phi\right)(x) \,.$$

**Lemma 5.1.** *Let* $\{u_i\}_{i\in\mathcal{I}}$ *(resp.* $\{v_i\}_{i\in\mathcal{I}}$*) be a nonnegative continuous viscosity supersolution (resp. subsolution) to* (5.39)*, satisfying* (3.27)-(3.29)*. Then, the functions* $\{\tilde{u}_i\}_{i\in\mathcal{I}}$ *(resp.* $\{\tilde{v}_i\}_{i\in\mathcal{I}}$*) defined by*

$$\tilde{u}_i(x) = \frac{u_i(x)}{\phi(x)} \quad \left(\textit{resp. } \tilde{v}_i(x) = \frac{v_i(x)}{\phi(x)}\right) \,, \quad x \in \mathbb{R}^d \,,$$

*are nonnegative lsc (resp. usc) viscosity supersolution (resp. subsolution) to*

$$\min\left\{-\tilde{\mathcal{L}}\left(x, \tilde{v}_i(x), D\tilde{v}_i(x), D^2\tilde{v}_i(x), \big(\tilde{v}_{i\ell}(x)\big)_{\ell\in\mathbb{N}}\right) \;;\; \tilde{v}_i(x) - \tilde{g}_i(x)\right\} = 0 \,, \tag{5.51}$$

*with* $\tilde{g}_i(x) = g_i(x)/\phi(x)$, *for* $(i,x) \in \mathcal{I}\times\mathbb{R}^d$.

*Proof.* We prove the supersolution case, the subsolution case is proven with the same techniques.

Fix $(i_0, x_0) \in \mathcal{I}\times\mathbb{R}^d$ and some test functions $\tilde{\varphi}_i \in C^2(\mathbb{R}^d)$, for $i \in \mathcal{I}$, and $\tilde{\varphi} \in C^0(\mathbb{R}^d)$ satisfying (5.40) and

$$0 \;=\; \left(\tilde{u}_{i_0} - \tilde{\varphi}_{i_0}\right)(x_0) \;=\; \min_{\mathcal{I}\times\mathbb{R}^d} \left(\tilde{u}_\cdot - \tilde{\varphi}_\cdot\right) \,.$$

Therefore, for $\varphi_i = \phi\tilde{\varphi}_i$ for $i \in \mathcal{I}$, we have

$$0 \;=\; \left(u_{i_0} - \varphi_{i_0}\right)(x_0) \;=\; \min_{\mathcal{I}\times\mathbb{R}^d} \left(u_\cdot - \varphi_\cdot\right) \,.$$

Moreover, the condition (5.40) is satisfied with respect to the function $\varphi = \phi\tilde{\varphi}$. Therefore, the functions $(\varphi_i)_{i\in\mathcal{I}}$ satisfy (5.39). Dividing this equation by the positive function $\phi$ and applying the product rule, we get that the functions $(\tilde{\varphi}_i)_{i\in\mathcal{I}}$ satisfy (5.51). □

**Theorem 5.4.** *Let $\{u_i\}_{i\in\mathcal{I}}$ (resp. $\{v_i\}_{i\in\mathcal{I}}$) be a bounded nonnegative continuous viscosity supersolution (resp. subsolution) to* (5.39)*, satisfying* (3.27)-(3.29)*. Then, under Assumptions A1, A2, A3 and A4, we have $u_i \le v_i$ for any $i \in \mathcal{I}$ on $\mathbb{R}^d$.*

*Proof.* From (3.29), there exists $N \in \mathbb{N}$ such that

$$\sup_{x\in\mathbb{R}^d} u_i(x) \vee \sup_{x\in\mathbb{R}^d} v_i(x) \;\le\; 1\,,$$

for $i \in \mathcal{I}$ such that $|i| \ge N$. We proceed in two steps. We first show that $u_i \le v_i$ on $\mathbb{R}^d$ for any $i \in \mathcal{I}$ such that $|i| \ge N$. Then, show this result for $|i| < N$.

**Step 1.** Denote by $\mathcal{I}_N$ the set $\{i \in \mathcal{I} \,:\, |i| \ge N\}$. We now prove that $u_i \le v_i$ on $\mathbb{R}^d$ for $i \in \mathcal{I}_N$.

We assume to the contrary that there exists $(z, j) \in \mathbb{R}^d \times \mathcal{I}_N$ such that

$$u_j(z) - v_j(z) \;\ge\; \delta\,, \tag{5.52}$$

for some $\delta > 0$. Take $\tilde{u}_i = u_i/\phi$ (resp. $\tilde{v}_i = v_i/\phi$) for $i \in \mathcal{I}$. Since $u_i$ and $v_i$ are bounded, we have

$$\lim_{(i,x)\to\infty} (\tilde{u}_i + \tilde{v}_i)(x) = 0\,. \tag{5.53}$$

This, together with (5.52), (3.29), and the fact that $\phi > 0$, implies that there exists $(i_0, x_0) \in \mathcal{I}_N \times \mathbb{R}^d$ such that

$$\bar{M}_{0+} := \sup_{(i,x)\in\mathcal{I}_N\times\mathbb{R}^d} \tilde{u}_i(x) - \tilde{v}_i(x) \;=\; \tilde{u}_{i_0}(x_0) - \tilde{v}_{i_0}(x_0) \;\ge\; \frac{\delta}{\phi(z)} \;>\; 0\,. \tag{5.54}$$

For $n \ge 1$, consider the following quantity

$$\bar{M}_n \;=\; \sup_{(i,x,y)\in\mathcal{I}_N\times\mathbb{R}^d\times\mathbb{R}^d} \tilde{u}_i(x) - \tilde{v}_i(y) - \frac{n}{2}|x-y|^2\,.$$

From (5.53), there exists $(i_n, x_n, y_n)$ such that

$$\bar{M}_n \;=\; \tilde{u}_{i_n}(x_n) - \tilde{v}_{i_n}(y_n) - \frac{n}{2}|x_n - y_n|^2\,.$$

From the definition of $N$, taking $x = y$ in the previous supremum, we obtain

$$0 \;<\; \frac{\delta}{\phi(z)} \;\le\; \bar{M}_{0+} \;\le\; \bar{M}_n \;\le\; 2\,. \tag{5.55}$$

This yields

$$\frac{n}{2}|x_n - y_n|^2 \;\le\; 2\,. \tag{5.56}$$

From (5.55) and (5.53), we have, up to a sub-sequence, $i_n = i^*$, for some $i^* \in \mathcal{I}_N$ and all $n$, and, $(x_n, y_n) \to (x^*, y^*)$ as $n \to \infty$. From (5.56), we have

$$\lim_{n\to\infty} |x_n - y_n| \;=\; 0 \quad \text{and} \quad x^* = y^* \,.$$

Moreover, from (5.55), we obtain

$$\lim_{n\to\infty} \frac{n}{2}|x_n - y_n|^2 \;=\; 0 \,.$$

Without loss of generality, we can take the maximization point in (5.54) to be $(i^*, x^*)$, *i.e.*, $(i_0, x_0) = (i^*, x^*)$. Since $(x_n, y_n) \in \mathbb{R}^d \times \mathbb{R}^d$ is a maximizer of $\bar{M}_n$, from Assumption A4, we may apply Ishii's lemma (see, *e.g.*, [12, Theorem 8.3]) and Lemma 5.1. Therefore, there exist $A_n, B_n \in \mathbf{S}^d$ such that

$$\begin{aligned}
\min\left\{-\tilde{\mathcal{L}}\left(x_n, \tilde{u}_{i_0}(x_n), n(x_n - y_n), A_n, \left(\tilde{u}_{i_0\ell}(x_n)\right)_{\ell\in\mathbb{N}}\right) \;;\; \tilde{u}_{i_0}(x_n) - \tilde{g}_{i_0}(x_n)\right\} &\leq 0, \\
\min\left\{-\tilde{\mathcal{L}}\left(y_n, \tilde{v}_{i_0}(y_n), n(x_n - y_n), B_n, \left(\tilde{v}_{i_0\ell}(y_n)\right)_{\ell\in\mathbb{N}}\right) \;;\; \tilde{v}_{i_0}(y_n) - \tilde{g}_{i_0}(y_n)\right\} &\geq 0 \,,
\end{aligned}$$

and

$$-3n\mathbb{I}_{2d} \;\leq\; \begin{pmatrix} A_n & 0 \\ 0 & -B_n \end{pmatrix} \;\leq\; 3n \begin{pmatrix} \mathbb{I}_d & -\mathbb{I}_d \\ -\mathbb{I}_d & \mathbb{I}_d \end{pmatrix} \,.$$

If there exists a subsequence of $\{x_n\}_n$, still denoted $\{x_n\}_n$, such that $\tilde{u}_{i_0}(x_n) - \tilde{g}_{i_0}(x_n) \leq 0$, we get

$$[\tilde{u}_{i_0}(x_n) - \tilde{g}_{i_0}(x_n)] - [\tilde{v}_{i_0}(y_n) - \tilde{g}_{i_0}(y_n)] \;\leq\; 0 \,,$$

for any $n$. This is, however, in contradiction with (5.55), the fact that $(x_n, y_n) \to (x_0, x_0)$ and the definition of $(i_0, x_0)$. Therefore, we must have

$$-\tilde{\mathcal{L}}\left(x_n, \tilde{u}_{i_0}(x_n), n(x_n - y_n), A_n, \left(\tilde{u}_{i_0\ell}(x_n)\right)_{\ell\in\mathbb{N}}\right) \;\leq\; 0, \tag{5.57}$$

$$-\tilde{\mathcal{L}}\left(y_n, \tilde{v}_{i_0}(y_n), n(x_n - y_n), B_n, \left(\tilde{v}_{i_0\ell}(y_n)\right)_{\ell\in\mathbb{N}}\right) \;\geq\; 0 \,, \tag{5.58}$$

for $n$ large enough.

Since $i_0 \in \mathcal{I}_N$, we have

$$\sup_{x\in\mathbb{R}^d} \left(u_{i_0\ell} \vee v_{i_0\ell}\right)(x) \;\leq\; 1 \,,$$

for all $\ell \geq 0$. This implies that

$$\begin{aligned}
&\left| \sum_{k\geq 0} p_k \phi^{k-1}(x) \prod_{\ell=0}^{k-1} \tilde{u}_{i_0\ell}(x_n) - \sum_{k\geq 0} p_k \phi^{k-1}(x) \prod_{\ell=0}^{k-1} \tilde{v}_{i_0\ell}(y_n) \right| \\
&\quad \leq \sum_{k\geq 0} p_k \sum_{\ell=0}^{k-1} \left( \prod_{\bar{\ell}=0}^{\ell-1} u_{i_0\bar{\ell}}(x_n) \right) |\tilde{u}_{i_0\ell}(x_n) - \tilde{v}_{i_0\ell}(y_n)| \left( \prod_{\bar{\ell}=\ell+1}^{k-1} v_{i_0\bar{\ell}}(y_n) \right) \\
&\quad \leq \sum_{k\geq 0} p_k \sum_{\ell=0}^{k-1} |\tilde{u}_{i_0\ell}(x_n) - \tilde{v}_{i_0\ell}(y_n)| \leq M \left( \tilde{u}_{i_0}(x_n) - \tilde{v}_{i_0}(y_n) \right) ,
\end{aligned}$$

where in the last inequality we used that $(x_n, y_n)$ is a maximizer of $\bar{M}_n$, and (5.55). Since

$$\frac{D\phi(x)}{\phi(x)} = \frac{2\kappa x}{|x|^2+1} ,$$

$\tilde{b}$ is locally Lipschitz. Moreover, since

$$\frac{D^2\phi(x)}{\phi(x)} = 4\kappa(\kappa-1)\frac{xx^\top}{(|x|^2+1)^2} + 2\kappa\frac{\mathbb{I}_d}{|x|^2+1} ,$$

we get that $\tilde{\gamma} - \gamma$ is equal to a bounded function in $\mathbb{R}^d$ multiplied by $\kappa$. Then, there exists $\kappa$ small enough such that

$$\tilde{\gamma}(x) - \alpha(M-1) \quad \geq \quad \frac{\gamma - \alpha(M-1)}{2} \quad > \quad 0 ,$$

for all $x \in \mathbb{R}^d$. This means that, from (5.57)-(5.58), we get

$$\begin{aligned}
&(\tilde{\gamma}(x_n) - \alpha(M-1))\tilde{u}_{i_0}(x_n) - (\tilde{\gamma}(y_n) - \alpha(M-1))\tilde{v}_{i_0}(y_n) \quad \leq \\
&\left(\tilde{b}(x_n) - \tilde{b}(y_n)\right)^\top n\,(x_n - y_n) + \frac{1}{2}\mathrm{Tr}\left(\sigma\sigma^\top(x_n)A_n - \sigma\sigma^\top(y_n)B_n\right) \qquad .
\end{aligned}$$

Sending $n$ to infinity we obtain

$$0 \geq (\tilde{\gamma}(x_0) - \alpha(M-1))\,(\tilde{u}_{i_0}(x_0) - \tilde{v}_{i_0}(x_0)) .$$

From Assumption A4 and for $\kappa$ small enough, the previous equation is in contradiction to (5.52).

**Step 2.** We now prove that $u_i \leq v_i$ on $\mathbb{R}^d$ for $|i| \leq N$, by a backward induction on $|i|$. From Step 1, the result holds for $i \in \mathcal{I}$ such that $|i| \leq N$. Fix $q \in \{0, \ldots, N-1\}$ and suppose that $u_i \leq v_i$ on $\mathbb{R}^d$ for $|i| = q+1$.

Fix $i_0 \in \mathcal{I}$ such that $|i_0| = q$. As in Step 1, we argue by contradiction and suppose that there exists $z \in \mathbb{R}^d$ such that

$$u_{i_0}(z) - v_{i_0}(z) \quad \geq \quad \delta , \tag{5.59}$$

for some $\delta > 0$. Consider $\tilde{u}_i$ and $\tilde{v}_i$ as before, which still satisfy (5.53) from Assumption A4(ii).

This assumption also entails that there exists a constant $C > 0$ such that $\tilde{u}_i(x) + \tilde{v}_i(x) \leq C$ for any $i \in \mathcal{I}, x \in \mathbb{R}^d$. As in (5.55), we get

$$0 < \delta\phi(z) \leq \bar{M}_{0+} \leq \bar{M}_n \leq C \ , \tag{5.60}$$

with $\bar{M}_{0+}$ and $\bar{M}_n$ defined as in Step 1. This yields

$$\frac{n}{2}|x_n - y_n|^2 \leq C \ . \tag{5.61}$$

Proceeding as in Step 1, we get (5.57)-(5.58) for $n$ large enough. Since $|i_0\ell| \geq q+1$, we have from the inductive hypothesis $u_{i_0\ell} \leq v_{i_0\ell}$ (therefore $\tilde{u}_{i_0\ell} \leq \tilde{v}_{i_0\ell}$) for any $\ell \geq 0$. Combining (5.57)-(5.58) with the nonnegativity of the functions $\tilde{u}_i$ and $\tilde{v}_i$, and the previous inequalities, we have

$$\begin{aligned}
(\tilde{\gamma}(x_n)+\alpha)\tilde{u}_{i_0}(x_n) - (\tilde{\gamma}(y_n)+\alpha)\tilde{v}_{i_0}(y_n) \quad &\leq \\
\left(\tilde{b}(x_n) - \tilde{b}(y_n)\right)^\top n\,(x_n - y_n) + \frac{1}{2}\mathrm{Tr}\left(\sigma\sigma^\top(x_n)A_n - \sigma\sigma^\top(y_n)B_n\right) + & \\
\alpha\sum_{k\geq 0} p_k \left(\phi^{-(k-1)}(x_n)\prod_{\ell=0}^{k-1}\tilde{u}_{i_0\ell}(x_n) - \phi^{-(k-1)}(y_n)\prod_{\ell=0}^{k-1}\tilde{v}_{i_0\ell}(y_n)\right) \quad &\leq \\
\left(\tilde{b}(x_n) - \tilde{b}(y_n)\right)^\top n\,(x_n - y_n) + \frac{1}{2}\mathrm{Tr}\left(\sigma\sigma^\top(x_n)A_n - \sigma\sigma^\top(y_n)B_n\right) + & \\
\alpha\sum_{k\geq 0} p_k \left(\phi^{-(k-1)}(y_n)\prod_{\ell=0}^{k-1}\tilde{u}_{i_0\ell}(y_n) - \phi^{-(k-1)}(y_n)\prod_{\ell=0}^{k-1}\tilde{v}_{i_0\ell}(y_n)\right) \quad &.
\end{aligned}$$

Therefore, we get a contradiction to (5.59), as in Step 1, sending $n$ to infinity. The results hold for $i_0$ and by induction, the results hold for all $i \in \mathcal{I}$.

□

As an immediate consequence of Theorems 5.3 and 5.4, we have the following characterization of the value function $v$.

**Corollary 5.1.** *Under Assumptions A1, A2, A3 and A4, $v$ is the unique nonnegative bounded viscosity solution to* (5.39)*, satisfying* (3.29)*.*

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