On Germs of Constriction Curves in Model of Overdamped Josephson Junction, Dynamical Isomonodromic Foliation and Painlevé 3 Equation

Alexey Glutsyuk

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Alexey Glutsyuk*†‡

October 9, 2023

To my dear teacher Yu.S.Ilyashenko on the occasion of his 80-th birthday

Abstract

B.Josephson (Nobel Prize, 1973) predicted a *tunnelling effect* for a system of two superconductors separated by a narrow dielectric (such a system is called *Josephson junction*): existence of a supercurrent through it and equations governing it. The *overdamped Josephson junction* is modeled by a family of differential equations on 2-torus depending on 3 parameters: B (abscissa), A (ordinate), ω (frequency). We study its *rotation number* $\rho(B, A; \omega)$ as a function of parameters. The *three-dimensional phase-lock areas* are the level sets $L_r := \{\rho = r\} \subset \mathbb{R}^3$ with non-empty interiors; they exist for $r \in \mathbb{Z}$ (Buchstaber, Karpov, Tertychnyi). For every fixed $\omega > 0$ and $r \in \mathbb{Z}$ the planar slice $L_r \cap (\mathbb{R}^2_{B,A} \times \{\omega\})$ is a garland of domains going vertically to infinity and separated by points; those separating points for which $A \neq 0$ are called *constrictions.* In a joint paper by Yu.Bibilo and the author, it was shown that 1) at each constriction the rescaled abscissa $\ell := \frac{B}{\omega}$ is integer and $\ell = \rho$; 2) the family $Constr_\ell$ of constrictions with given $\ell \in \mathbb{Z}$ is an analytic submanifold in $(\mathbb{R}^2_+)_{a,s}$, $a = \omega^{-1}$, $s = \frac{A}{\omega}$. In the present paper we show that 1) the limit points of $Constr_\ell$ are $\beta_{\ell,k} = (0, s_{\ell,k})$, where $s_{\ell,k}$ are the positive zeros of the ℓ-th Bessel function $J_\ell(s)$; 2) to each $\beta_{\ell,k}$ accumulates exactly one its component $\mathcal{C}_{\ell,k}$ (constriction curve), and it lands at $\beta_{\ell,k}$ regularly. Known numerical phase-lock area pictures show that high components of interior of each phase-lock area L_r look similar. In his paper with Bibilo, the author introduced a candidate to the self-similarity map between neighbor components: the Poincaré map of the dynamical isomonodromic foliation governed by Painlevé 3 equation. Whenever well-defined, it preserves the rotation number function. We show that the Poincaré map is well-defined on a neighborhood of the plane $\{a = 0\} \subset \mathbb{R}^2_{\ell,a} \times (\mathbb{R}_+)_s$, and it sends each constriction curve germ $(\mathcal{C}_{\ell,k}, \beta_{\ell,k})$ to $(\mathcal{C}_{\ell,k+1}, \beta_{\ell,k+1})$.

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Contents

1 Introduction and main results

1.1 Model of Josephson junction. Introduction and brief description of main results

The tunnelling effect predicted by B.Josephson in 1962 [35] (Nobel Prize 1973) deals with a *Josephson junction:* a system of two superconductors separated by a narrow dielectric. It states existence of a supercurrent through it and yields equations governing it (discovered by Josephson). It was confirmed experimentally by P.W.Anderson and J.M.Rowell in 1963 [1].

The model of the so-called *overdamped Josephson junction*, see [54, 46, 39, 51], [7, p. 306], [40, pp. 337–340], [41, p.193], [42, p. 88] is described by the family of nonlinear differential equations

$$\frac{d\phi}{dt} = -\sin\phi + B + A\cos\omega t,\ \omega > 0,\ B \geq 0. \tag{1.1}$$

Here ϕ is the difference of phases (arguments) of the complex-valued wave functions describing the quantum mechanic states of the two superconductors. Its derivative is equal to the voltage up to known constant factor.

Equations (1.1) also arise in several models in physics, mechanics and geometry, e.g., in planimeters, see [23, 24].

The variable and parameter changes

$$\tau := \omega t,\ \theta := \phi + \frac{\pi}{2},\ \ell := \frac{B}{\omega},\ a = \frac{1}{\omega},\ s := \frac{A}{\omega}, \tag{1.2}$$

transform (1.1) to a non-autonomous ordinary differential equation on the two-torus $\mathbb{T}^2 = S^1 \times S^1$ with coordinates $(\theta, \tau) \in \mathbb{R}^2/2\pi\mathbb{Z}^2$:

$$\frac{d\theta}{d\tau} = a\cos\theta + \ell + s\cos\tau. \tag{1.3}$$

The graphs of its solutions are the orbits of the vector field

$$\begin{cases} \dot{\theta} = a\cos\theta + \ell + s\cos\tau \\ \dot{\tau} = 1 \end{cases} \tag{1.4}$$

on $\mathbb{T}^2$. The *rotation number* of its flow, see [3, p. 104], is a function $\rho(B, A)$ of parameters[1]:

$$\rho(B, A; \omega) = \lim_{k\to+\infty} \frac{\theta(2\pi k)}{2\pi k}.$$

Here $\theta(\tau)$ is a general $\mathbb{R}$-valued solution of the first equation in (1.4) whose parameter is the initial condition for $\tau = 0$. Recall that the rotation number exists and is independent on the choice of the initial condition, see [3, p.104]. The parameter B is called *abscissa,* A is called the *ordinate,* ω is called *frequency.* Recall the following well-known definition.

Definition 1.1 (cf. [25, definition 1.1]) The *r-th planar phase-lock area* is the level set

$$L_r(\omega) = \{(B, A) \in \mathbb{R}^2 \mid \rho(B, A; \omega) = r\} \subset \mathbb{R}^2_{B,A},$$

provided that it has a non-empty interior.

[1] There is a misprint, missing 2π in the denominator, in analogous formulas in previous papers of the author with co-authors: [25, formula (2.2)], [13, the formula after (1.16)].

The planar phase-lock areas were studied by V.M.Buchstaber, O.V.Karpov, S.I.Tertychnyi, Yu.P.Bibilo, the author et al, see [8, 9], [12]–[20], [25]–[27], [29, 31, 37, 38], [55, 56] and references therein. The following results are known and proved mathematically:

1) Planar phase-lock areas exist only for integer rotation number values (*the rotation number quantization effect* discovered and proved by V.M.Buchstaber, O.V.Karpov and S.I.Tertychnyi in [17], later also proved in [29, 31]).

2) The boundary of each $L_r(\omega)$ consists of two analytic curves, which are the graphs of two functions $B = G_{r,\alpha}(A)$, $\alpha = 0, \pi$, (see [18]; this fact was later explained by A.V.Klimenko via symmetry, see [38]).

3) The latter functions have Bessel asymptotics

$$\begin{cases} G_{r,0}(A) = r\omega - J_r(-\frac{A}{\omega}) + O(\frac{\ln|A|}{A}) \\ G_{r,\pi}(A) = r\omega + J_r(-\frac{A}{\omega}) + O(\frac{\ln|A|}{A}) \end{cases}, \text{ as } A \to \infty \tag{1.5}$$

(observed and proved on physics level in [52], see also [40, p. 338], [7, section 11.1], [16]; proved mathematically in [38]); J_r is the r-th Bessel function.

4) Each planar phase-lock area is a garland of infinitely many bounded domains going to infinity in the vertical direction. In this chain each two subsequent domains are separated by one point. This was proved in [38] using the above statement 3). Those separation points that lie on the horizontal B-axis, namely $A = 0$, were calculated explicitly, and we call them the *growth points*, see [18, corollary 3]. The other separation points, which lie outside the horizontal B-axis, are called the *constrictions*.

5) For every $r \in \mathbb{Z}$ and $\omega > 0$ the r-th planar phase-lock area $L_r(\omega)$ is symmetric to the $-r$-th one with respect to the vertical A-axis.

6) Every planar phase-lock area is symmetric with respect to the horizontal B-axis. See Figures 1–3 below.

7) In each planar phase-lock area $L_r(\omega)$ all its constrictions lie in the same vertical line $\Lambda_r := \{B = r\omega\}$, see [8, theorem 1.4].

8) Each constriction $X \in L_r(\omega)$ is *positive:* the intersection of the interior of the area $L_r(\omega)$ with the vertical line Λ_r contains a punctured neighborhood of the point X in Λ_r, see [8, theorem 1.7].

9) For every fixed $\ell \in \mathbb{Z}$ the set of constrictions $(B, A; \omega)$ with abscissa $B = \ell\omega$ (i.e., constrictions in $L_\ell(\omega)$), with variable (A, ω), is a *one-dimensional analytic submanifold* in $(\mathbb{R}^2_+)_{A,\omega}$ [8, theorem 1.12]. We will denote it $Constr_\ell$ and represent it in the coordinates (a, s):

$$a := \frac{1}{\omega}, \quad s := \frac{A}{\omega},$$

$$Constr_\ell := \{(a, s) \in \mathbb{R}^2_+ \mid \left(\frac{\ell}{a}, \frac{s}{a}; a^{-1}\right) \text{ is a constriction}\} \subset (\mathbb{R}^2_+)_{a,s}.$$

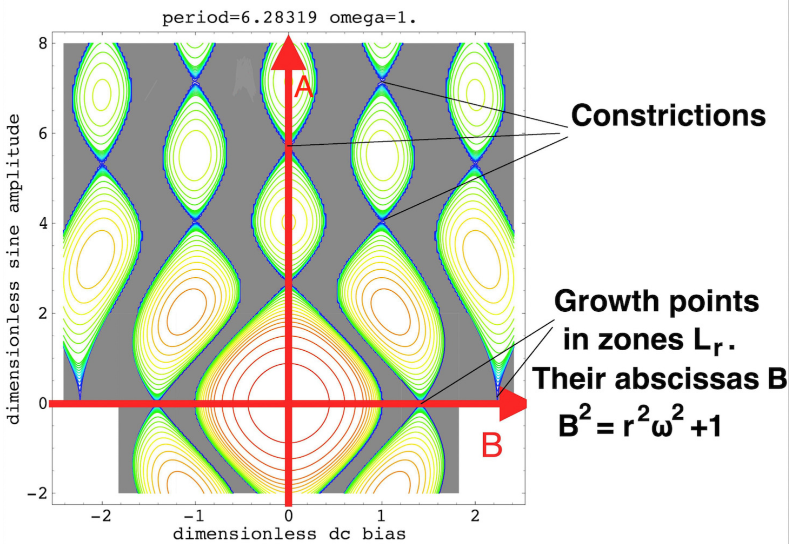

Figure 1: Phase-lock areas and their constrictions for $\omega = 1$. The abscissa is B, the ordinate is A. Figure taken from paper [13, fig. 1b)] with authors' permission, with coordinate axes added.

See Figures 1–3 of planar phase-lock areas and their constrictions for $\omega = 1, 0.5, 0.3$.

For a survey of other results on model (1.1) of overdamped Josephson junction and related topics see paper [8] and its bibliography.

Definition 1.2 A *constriction curve* is a connected component of the above constriction submanifold $Constr_\ell \subset (\mathbb{R}^2_+)_{a,s}$. The corresponding curve in the parameters $(B, A; \omega)$, $B = \ell\omega = \frac{\ell}{a}$, $A = \frac{s}{a}$, $\omega = a^{-1}$) (which is a family of constrictions lying in $L_\ell(\omega)$) will be also called a *constriction curve*.

It would be interesting to study the *three-dimensional phase-lock areas:* those level sets

$$L_r = \{(B, A; \omega) \in \mathbb{R}^2_{B,A} \times (\mathbb{R}_+)_\omega \mid \rho(B, A; \omega) = r\}$$

for which $Int(L_r) \neq \emptyset$.

Remark 1.3 The Buchstaber–Karpov–Tertychnyi rotation number quantization effect also holds for the above three-dimensional areas L_r: they exist only for integer values of the rotation number r. This follows immediately from their analogous result and its proof in two dimensions [17]. One has

$$L_r = \cup_{\omega>0}(L_r(\omega) \times \{\omega\}), \tag{1.6}$$

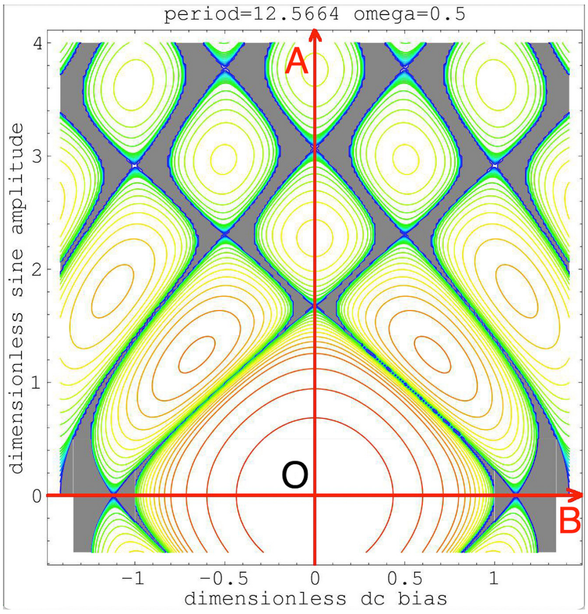

Figure 2: Phase-lock areas and their constrictions for $\omega = 0.5$. Figure taken from papers [13, fig. 1d)], [20, p. 331] with authors' permission, with coordinate axes added.

where $L_r(\omega)$ are the planar phase-lock areas for given ω. The boundary ∂L_r is the union of graphs $\{B = g_{r,\alpha}(A, \omega)\}$, of functions $g_{r,\alpha}$, $\alpha \in \{0, \pi\}$, analytic on $\mathbb{R} \times \mathbb{R}_+$, whose restrictions to each line $\omega = const$ coincide with the corresponding functions $G_{r,\alpha}$. The proof of this fact repeats the proof of the analogous statement for planar phase-lock areas, see [18, 38]. Thus,

$$Int(L_r) = \cup_{\omega>0} Int(L_r(\omega)) \times \{\omega\}. \tag{1.7}$$

This implies that *the interior of each three-dimensional phase-lock area* L_r *is a union of domains separated by constriction curves* or the *growth point curves* $\{A = 0,\ B = \pm\sqrt{r^2\omega^2 + 1}\}$.

Problem 1.4 Describe the asymptotic behavior of the constriction curves.

It follows from [8, lemma 4.14 and arguments on pp. 5459–5460] that *the constriction submanifold* $Constr_\ell$ *accumulates to no point of the a-axis.*

One of the main results of the present paper is Theorem 1.17 (Subsection 1.2). It states that the limit set of the constriction submanifold $Constr_\ell$ is the infinite sequence of the points $\beta_{\ell,k} = (0, s_{\ell,k})$ of the s-axis, where $s_{\ell,1}, s_{\ell,2}, \dots$ are the positive zeros of the Bessel function J_ℓ; exactly one constriction curve, denoted by $\mathcal{C}_{\ell,k}$, accumulates to each $\beta_{\ell,k}$ (which is its

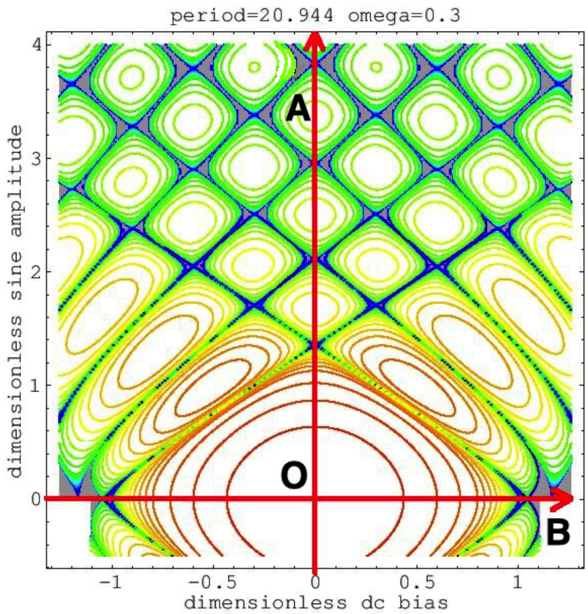

Figure 3: Phase-lock areas and their constrictions for $\omega = 0.3$. Figure taken from paper [13, fig. 1e)] with authors' permission, with coordinate axes added.

unique limit point); each $\mathcal{C}_{\ell,k}$ has unique limit point $\beta_{\ell,k}$, and it lands at $\beta_{\ell,k}$ regularly. Using Theorem 1.17, we deduce the following theorem.

Theorem 1.5 *The interior of each three-dimensional phase-lock area has infinitely many components.*

Conjecture 1.6 Each constriction curve lands at some $\beta_{\ell,k}$, i.e., coincides with some $\mathcal{C}_{\ell,k}$.

Conjecture 1.7 [8, conjecture 6.8] Each constriction curve is bijectively analytically projected onto the a-axis $(\mathbb{R}_+)_a$ (or equivalently, onto $(\mathbb{R}_+)_\omega$). The corresponding conjectural arrangement of constriction curves and components of the phase-lock areas L_ℓ is presented at Fig. 4a).

Definition 1.8 A constriction curve different from $\mathcal{C}_{\ell,k}$ (if any) is called a *queer constriction curve.*

We prove Proposition 1.18 (Subsection 1.2), which states that a constriction curve is queer, if and only if the restriction to it of the coordinate a is unbounded on both its sides. This will show that Conjecture 1.7 would imply Conjecture 1.6.

We deduce Corollary 1.20 (of Proposition 1.18), which states that for every queer constriction curve $\mathcal{C}$ in L_ℓ (if any) there exists a connected component of the interior $Int(L_\ell)$ that is adjacent to $\mathcal{C}$ and is adjancent to no curve $\mathcal{C}_{\ell,k}$. This shows that *Conjecture 1.6 is equivalent to the statement saying that each component of* $Int(L_r)$ *is adjacent to some* $\mathcal{C}_{\ell,k}$.

Remark 1.9 Conjecture 1.7 implies that as ω varies, in the corresponding planar phase-lock areas $L_r(\omega)$ constrictions can neither be born, nor disappear, as ω crosses a value ω_0 for which the boundary curves of $L_r(\omega_0)$ are tangent to each other at a constriction. Conjectural picture of the three-dimensional phase-lock areas and conjecturally impossible picture, with a queer constriction curve and the corresponding component of $Int(L_r)$ (called a *queer component*), are presented at Fig. 4a) and 4b) respectively.

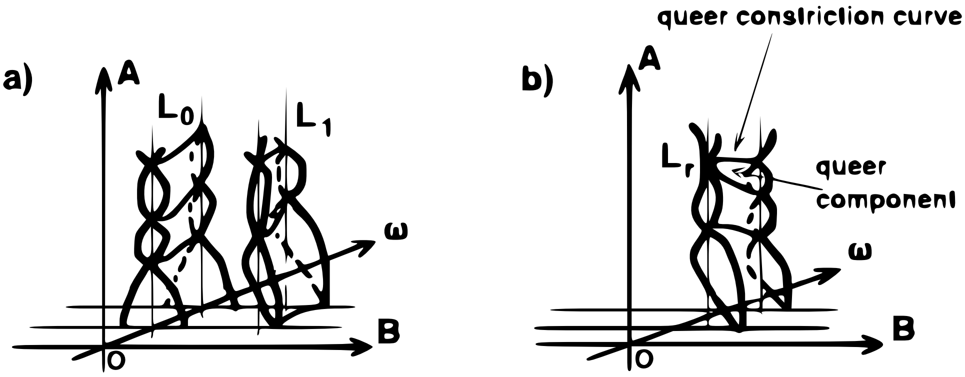

Figure 4: Three-dimensional phase-lock areas and constriction curves:
a) the conjectural picture, without queer constriction curves;
b) a priori possible picture with a queer curve and a queer component.

Each three-dimensional phase-lock area has a kind of *self-similarity structure:* any two its components lying high enough look similarly. It would be interesting to find and study a self-similarity map of the parameter space that sends one component to the next one, adjacent to it from above. In his paper with Yu.Bibilo, see [8, subsection 6.2], the author suggested a candidate to such a self-similarity map: the *Poincaré map of the so-called dynamical isomonodromic foliation* $\mathcal{G}$. Namely, he introduced the following four-dimensional extension of the three-dimensional family (1.4) of dynamical systems on torus modeling overdamped Josephson junction:

$$\frac{d\theta}{d\tau} = \nu + a\cos\theta + s\cos\tau + \psi\cos(\theta-\tau); \quad \nu, a, \psi \in \mathbb{R}, \quad s > 0, \ (a,\psi) \neq (0,0). \tag{1.8}$$

Family (1.3) is embedded to (1.8) by the mapping

$$(\ell, s, \omega) \mapsto (\nu, a, s, \psi) = (\ell, \omega^{-1}, s, 0). \tag{1.9}$$

In the new coordinates (ℓ, χ, a, s) on the parameter space of family (1.8),

$$\chi = \frac{\psi}{2s}, \ \ell := \nu - \frac{\psi a}{s} = \nu - 2\chi a,$$

consider the line field given by the following system of non-autonomous differential equations introduced in [8, formula (6.4)]:

$$\begin{cases} \ell'_s = 0 \\ \chi'_s = \frac{a - 2\chi(\ell + 2\chi a)}{2s} \\ a'_s = -2s\chi + \frac{a}{s}(\ell + 2\chi a) \end{cases}. \tag{1.10}$$

The foliation of the four-dimensional parameter space by its phase curves, which are graphs of solutions of system (1.10), is called the *dynamical isomonodromic foliation* and denoted by $\mathcal{G}$. (This name comes from relation to isomonodromic families of linear systems; see explanation below, in Remark 1.16.) The author has shown in [8, subsection 6.2] that each its leaf is a family of flows on $\mathbb{T}^2$ that are conjugated to each other by diffeomorphisms isotopic to the identity. Thus, *the rotation number and the number ℓ are constant along each leaf of the foliation $\mathcal{G}$.* For every $\ell \in \mathbb{R}$ the function

$$w(s) := \frac{a(s)}{2s\chi(s)} = \frac{a(s)}{\psi(s)} \tag{1.11}$$

satisfies Painlevé 3 equation

$$w'' = \frac{(w')^2}{w} - \frac{w'}{s} - 2\ell\frac{w^2}{s} + (2\ell - 2)\frac{1}{s} + w^3 - \frac{1}{w}. \tag{1.12}$$

along solutions of (1.10), see [8, theorem 6.6].

Consider family (1.3) modeling Josephson junction as the subfamily in (1.8): the hyperplane $\{\chi = 0\}$, see (1.9). A solution $(\ell, \chi(s), a(s))$ with initial condition $(\ell, 0, a_0)$ in the latter hyperplane at $s = s_0$ may return back to the hyperplane at its another point $(\ell, 0, a_1)$ at some next moment $s_1 > s_0$. If it happens, then this induces the *Poincaré first return map* $\mathcal{P} : (\ell, a_0, s_0) \mapsto (\ell, a_1, s_1)$ defined on some subset of the hyperplane $\{\chi = 0\}$.

Problem 1.10 (see [8, problems 6.11–6.13]) Study the action of the Poincaré map $\mathcal{P}$ on the three-dimensional phase-lock area portrait of family (1.3). Is it true that it sends each interior component of every phase-lock area (starting from the second component) to its next component, adjacent to it from above? Is it true that it sends each constriction curve $\mathcal{C}_{\ell,k}$ to $\mathcal{C}_{\ell,k+1}$?

Problem 1.11 Describe the subset where the Poincaré map $\mathcal{P}$ is well-defined. Describe the subset where all its iterates are well-defined.

Remark 1.12 The second part of the latter problem is closely related to the description of those solutions of Painlevé 3 equation (1.12) that have an infinite lattice of residue one poles in $\mathbb{R}_+$. See Subsection 1.5, where relation to poles of solutions of Painlevé 3 equation will be presented together with a brief survey of results on poles.

Our second main result is Theorem 1.23 (Subsection 1.3). It states that

- the Poincaré map $\mathcal{P}$ is well-defined on a neighborhood of the plane $\{a = 0\} \subset \mathbb{R}_\ell \times (\mathbb{R}^2_+)_{a,s}$;

- its restriction to the latter plane sends a point $(\ell, 0, s)$ to $(\ell, 0, s^*)$, $s^* > 0$, where s, s^* are neighbor zeros of a solution of the ℓ-th Bessel equation.

To study the domain of the map $\mathcal{P}$, it is important to study analytic extensions of solutions of system (1.10) and of the Painlevé 3 equation (1.12).

Remark 1.13 It is well-known that solutions $w(s)$ of Painlevé 3 equation (1.12) are meromorphic on the universal cover of the punctured line $\mathbb{C}^*_s = \mathbb{C} \setminus \{0\}$ (classical result coming from the Painlevé property), and *all their poles are simple with residues* ± 1 [28, theorem 31.1]. But a generic solution of (1.12) has a nontrivial monodromy and is not single-valued on $\mathbb{C}^*$.

Our third main result is Theorem 1.24 (Subsection 1.4) classifying singularities $s_0 \in \mathbb{C}_s$ of solutions $(\chi(s), a(s))$ of system (1.10). It states that each solution is meromorphic on the universal cover over $\mathbb{C}^*_s$, and each its pole s_0 is either a zero of the corresponding solution $w(s)$ of the Painlevé 3 equation (1.12) with unit derivative, or its pole with residue -1. Theorem 1.24 also states that for each initial condition lying in the constriction submanifold $Constr_\ell$ the corresponding solution $(\chi(s), a(s))$ is meromorphic either on $\mathbb{C}^*$, or on its double cover, and $w(s)$ is meromorphic on $\mathbb{C}^*$. In the case, when the initial condition lies on a constriction curve $\mathcal{C}_{\ell,k}$, Theorem 1.24 states that the solution $(\chi(s), a(s))$ is meromorphic single-valued on $\mathbb{C}^*_s$.

Let us note the Hamiltonian nature of non-autonomous system (1.10).

Proposition 1.14 *For every fixed $\ell \in \mathbb{R}$ the corresponding differential equation (1.10) on vector function $(\chi(s), a(s))$ is Hamiltonian with time s-depending Hamiltonian function*

$$H(\chi, a, s) := -\frac{\chi^2 a^2}{s} + \frac{a^2}{4s} + s\chi^2 - \frac{\ell\chi a}{s}. \tag{1.13}$$

In particular, for every positive s_1, s_2, $s_1 < s_2$, the non-autonomous flow map of equation (1.10) from time s_1 to time s_2 (on a domain in $\mathbb{C}^2_{\chi,a}$ where it is well-defined) preserves the standard symplectic form $d\chi \wedge da$.

The proposition follows by straightforward calculation.

Remark 1.15 Representations of Painlevé equations as a Hamiltonian systems were found by J.Malmquist [45] (for all of them except for Painlevé 3 equations). For Painlevé 3 equations this was done by K.Okamoto [50, p. 265]. His result implies that our Painlevé 3 equation (1.12) is equivalent to the Hamiltonian system with the Okamoto Hamiltonian

$$H_{Ok}(w,p;s) := \frac{1}{s}(w^2p^2 - (w^2s - (2\ell - 1)w + s)p).$$

It appears that system (1.10) is equivalent to the Hamiltonian equation with the Okamoto Hamiltonian function H_{Ok} via the variable change

$$(w,p) = G(\chi, a) := (\frac{a}{2\chi s},\ 4s\chi^2).$$

For fixed s, the G-pullback of the symplectic form $dw \wedge dp$ is $-4d\chi \wedge da$, and one has

$$H(\chi, a, s) = -\frac{1}{4}\left(H_{Ok}(w,p) + \frac{wp}{s}\right) = -\frac{1}{4}\left(H_{Ok} \circ G(\chi, a) + \frac{2\chi a}{s}\right).$$

Family of dynamical systems (1.4) modeling Josephson junction can be equivalently described by a family of two-dimensional linear systems of differential equations on the Riemann sphere, see [15, 17, 20, 23, 29, 31], [12, subsection 3.2]. This is also true for the extended family (1.8). Namely, in the complex variables

$$\Phi = e^{i\theta}, \quad z = e^{i\tau}$$

equations (1.8) can be equivalently written as Riccati equations

$$\frac{d\Phi}{dz} = \frac{1}{z^2}\left(\frac{s}{2}\Phi + \frac{\psi}{2}\Phi^2\right) + \frac{1}{z}\left(\nu\Phi + \frac{a}{2}(\Phi^2 + 1)\right) + \left(\frac{s}{2}\Phi + \frac{\psi}{2}\right). \tag{1.14}$$

A function $\Phi(z)$ is a solution of the latter Riccati equation, if and only if $\Phi(z) = \frac{Y_2(z)}{Y_1(z)}$, where $Y = (Y_1, Y_2)(z)$ is a solution of the linear system

$$Y' = \left(-s\frac{\mathbf{K}}{z^2} + \frac{\mathbf{R}}{z} + s\mathbf{N}\right)Y, \tag{1.15}$$

$$\mathbf{K} = \begin{pmatrix} \frac{1}{2} & \chi \\ 0 & 0 \end{pmatrix}, \ \mathbf{R} = \begin{pmatrix} -(\ell + \chi a) & -\frac{a}{2} \\ \frac{a}{2} & \chi a \end{pmatrix}, \ \mathbf{N} = \begin{pmatrix} -\frac{1}{2} & 0 \\ \chi & 0 \end{pmatrix}; \quad \chi = \frac{\psi}{2s}.$$

Given a $z_0 \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, say, $z_0 = 1$, the space of germs of solutions of linear system (1.15) at z_0 is identified with the vector space $\mathbb{C}^2$ of initial conditions at z_0. Analytic extension along a counterclockwise circuit around the origin is a linear operator acting on the latter local solution space, called the *monodromy operator*.

Remark 1.16 Solutions of (1.8) correspond to isomonodromic families of linear systems (1.15) [8, theorem 6.6]. (This explains the name "isomonodromic foliation" for the foliation by graphs of solutions of (1.10).) The latter isomonodromic families are induced from classical Jimbo isomonodromic deformations given in [33], governed by Painlevé 3 equation [33, pp. 1156–1157], which implies equation (1.12). It is known that *linear systems (1.15) corresponding to constrictions in subfamily (1.3) of (1.8) have trivial monodromy,* see [8, proposition 4.1]. For a necessary background on linear systems with irregular singularities and isomonodromic families see, for example, [8, sections 2 and 3.1] and Subsection 2.7 below.

The plan of proofs of main results is presented in Subsection 1.6.

1.2 The constriction manifold: limit points and landing components

Theorem 1.17 *1) The limit points of the constriction submanifold $Constr_\ell \subset (\mathbb{R}^2_+)_{a,s}$ are the points*

$$\beta_{\ell,k} = (0, s_{\ell,k}), \quad s_{\ell,k} > 0 \ \ \textit{is the} \ \ k-th \ \textit{positive zero of the Bessel function}$$

$$J_\ell(s) = \frac{1}{\pi}\int_0^\pi \cos(\ell\tau - s\sin\tau)d\tau. \tag{1.16}$$

2) For every $\beta_{\ell,k}$ there exists a unique connected component $\mathcal{C}_{\ell,k}$ of the manifold $Constr_\ell$ landing at $\beta_{\ell,k}$, and $\mathcal{C}_{\ell,k}$ accumulates to no other $\beta_{\ell,j}$.

3) The union of the closure $\overline{Constr_\ell}$ in $\mathbb{R}_{\geq 0} \times \mathbb{R}_s$ and its image under the symmetry with respect to the s-axis is a regular analytic submanifold $\widehat{Constr_\ell} \subset \mathbb{R}^2_{a,s}$. It intersects the s-axis orthogonally at the points $\beta_{\ell,k}$.

4) The image of each curve $\mathcal{C}_{\ell,k}$ under the projection to the a-coordinate is the whole a-semiaxis $\mathbb{R}_+$.

Proposition 1.18 *Let $\mathcal{C} \subset Constr_\ell$ be a constriction curve. Let the restriction to $\mathcal{C}$ of the coordinate function a be bounded on at least one its side: we split $\mathcal{C}$ into two semi-curves separated by a point, and we require that a be bounded on at least one of them. Then $\mathcal{C}$ coincides with some of $\mathcal{C}_{\ell,k}$.*

Corollary 1.19 *Conjecture 1.7 implies Conjecture 1.6.*

For a given $\ell \in \mathbb{R}$ and a subset $V \subset (\mathbb{R}^2_+)_{a,s}$, by $\widetilde{V}_\ell$ we denote the subset of those parameters $(B, A; \omega)$ for which $(a, s) = (\omega^{-1}, \frac{A}{\omega}) \in V$ and $\frac{B}{\omega} = \ell$:

$$\widetilde{V}_\ell := \{(B, A; \omega) = (\frac{\ell}{a}, \frac{s}{a}; a^{-1}) \mid (a, s) \in V\} \subset \mathbb{R}_B \times (\mathbb{R}^2_+)_{A,\omega}. \tag{1.17}$$

Corollary 1.20 *Let $\ell \in \mathbb{Z}$. Every queer constriction curve $\mathcal{C} \subset Constr_\ell$ (a component different from $\mathcal{C}_{\ell,k}$'s, if any) satisfies the following statements:*

(i) The coordinate function a is unbounded from above on both sides of $\mathcal{C}$ (i.e. on both semicurves in $\mathcal{C}$ separated by one point);

(ii) Let $U_{\mathcal{C}} \subset (\mathbb{R}^2_+)_{a,s}$ denote the connected component of the complement $\mathbb{R}^2_+ \setminus \mathcal{C}$ lying on the right from $\mathcal{C}$. There exists a domain $V \subset U_{\mathcal{C}}$ adjacent to $\mathcal{C}$ such that the two-dimensional domain $\widetilde{V}_\ell \subset \{B = \ell\omega\} \subset \mathbb{R}^3$, see (1.17), lies in a connected component of the interior of the three-dimensional phase-lock area L_ℓ. The latter component of $Int(L_\ell)$ is called a **queer component** *adjacent to $\mathcal{C}$ from the right. This queer component is adjacent to no curve $\mathcal{C}_{\ell,k}$ (but a priori, may be adjacent to some other queer constriction curves). See Fig. 4b).*

1.3 The Poincaré map of the isomonodromic foliation $\mathcal{G}$

Recall that the foliation $\mathcal{G}$ is the foliation by graphs of solutions of system (1.10). The initial 3-parametric family (1.4) of dynamical systems on $\mathbb{T}^2$ modeling Josephson junction is a cross-section to $\mathcal{G}$ (see (1.9)). In the parameters (ℓ, χ, a, s), this is the hyperplane $\{\chi = 0\}$ with the plane $\{a = 0\}$ deleted. We denote it by $\mathbf{Jos}^o$ and its closure by $\mathbf{Jos}$:

$$\mathbf{Jos} := \{\chi = 0, s > 0\} \subset \mathbb{R}^4_{\ell,\chi,a,s}, \quad \mathbf{Jos}^o := \mathbf{Jos} \setminus \{a = 0\}.$$

The foliation $\mathcal{G}$ is tangent to $\mathbf{Jos}$ at points of its proper hyperplane $\{a = 0\}$.

Definition 1.21 Consider the solution $(\ell, \chi(s), a(s))$ of equation (1.10) whose graph passes through a given point $(\ell, 0, a_0, s_0) \in \mathbf{Jos}^o$ (initial condition). Let $s_1 > s_0$ be the next point where $\chi(s_1) = 0$, and let the vector function $(\chi(s), a(s))$ be well-defined on the segment $[s_0, s_1]$; $a(s_0) = a_0$. The map $\mathcal{P} : (\ell, a_0, s_0) \mapsto (\ell, a(s_1), s_1)$ of the first return to $\mathbf{Jos}^o$ will be called the *Poincaré map* of the isomonodromic foliation $\mathcal{G}$.

Remark 1.22 The Poincaré map $\mathcal{P}$, wherever defined, preserves the rotation number function, and hence, the phase-lock area portrait. This follows from constance of the rotation number along leaves of the foliation $\mathcal{G}$.

The next theorem is our second main result.

Theorem 1.23 *1) The Poincaré map $\mathcal{P}$ is well-defined and analytic on a neighborhood of the hyperplane $\{a = 0\}$ in* **Jos**, *including this hyperplane.*

2) Given an $\ell \in \mathbb{R}$ and an $s > 0$, let $s^* > s$ be the next zero after s of the solution of the ℓ-th Bessel equation (2.11) vanishing at s. The Poincaré map $\mathcal{P}$ sends the point $(\ell, 0, 0, s)$ to the point $(\ell, 0, 0, s^*)$. In particular, it *sends each $\beta_{\ell,k} = (\ell, 0, 0, s_{\ell,k}) \in$* **Jos** *to $\beta_{\ell,k+1}$. Here $s_{\ell,k}$ is the k-th positive zero of the Bessel function $J_\ell(s)$.*

3) For $\ell \in \mathbb{Z}$ let $\mathcal{C}_{\ell,k}$ and $\beta_{\ell,k}$ be the constriction curves and their landing points from Theorem 1.17. The Poincaré map $\mathcal{P}$ sends each constriction curve germ $(\mathcal{C}_{\ell,k}, \beta_{\ell,k})$ to the next germ $(\mathcal{C}_{\ell,k+1}, \beta_{\ell,k+1})$.

1.4 Analytic extension and singularities of solutions of (1.10) and the Painlevé 3 equation (1.12)

It is well-known that each solution of Painlevé 3 equation (1.12) extends analytically to a meromorphic function on the universal cover over $\mathbb{C}^*$ (Painlevé property), and its singular points in the universal cover are simple poles with residues ± 1 [28, theorem 31.1]. Our third main result is the following slightly stronger theorem, on analytic extension of vector solutions $(\chi(s), a(s))$ of system (1.10).

Theorem 1.24 *1) For every given $\ell \in \mathbb{C}$ each solution $(\chi(s), a(s))$ of system (1.10) extends to a meromorphic vector function on the universal cover of punctured complex line $\mathbb{C}^*_s = \mathbb{C} \setminus \{0\}$.*

2) Each its pole $s_0 \in \mathbb{C}^*$ (i.e., a point s_0 that is a pole of some its component and either a pole, or a regular point of the other component) is

- either a zero with derivative one of the corresponding solution $w(s) = \frac{a(s)}{2s\chi(s)}$ of Painlevé 3 equation (1.12); in this case $a(s)$ is holomorphic at s_0, $a(s_0) = \pm s_0$, and $\chi(s)$ has a simple pole at s_0 with residue $\pm\frac{1}{2}$;

- or a pole of $w(s)$ with residue -1; in this case $a(s)$ has simple pole with residue $\pm s_0$ at s_0, and $\chi(s)$ is holomorphic at s_0 and $\chi(s_0) = \mp\frac{1}{2}$.

3) For every initial condition (χ^*, a^*, s^*), $s^* \neq 0$, corresponding to a linear system (1.15) with trivial monodromy the corresponding solution $(\chi(s), a(s))$ of system (1.10) is meromorphic either on $\mathbb{C}^*$, or on its double cover $\mathbb{C}^*_t$ given by $t \mapsto s = t^2$. More precisely, it remains unchanged up to sign after analytic extension along a simple circuit around the origin. The corresponding solution $w(s)$ of Painlevé 3 equation (1.12) is always meromorphic on $\mathbb{C}^*$.

4) Statement 3) hold for every initial condition $(0, a^*, s^*)$, where (a^*, s^*) lies in the constriction submanifold $Constr_\ell$.

5) For every $(0, a^*, s^*)$ such that (a^*, s^*) lies in a constriction curve $\mathcal{C}_{\ell,k}$ the corresponding solution $(\chi(s), a(s))$ of (1.10) is meromorphic on $\mathbb{C}^*$.

1.5 The Poincaré map $\mathcal{P}$ and poles of Painlevé 3 solutions

Lemma 1.25 *[8, lemma 6.7, p.5475]. The graph of a non-identically-zero solution $(\chi(s), a(s))$ of system (1.10) crosses the hyperplane $\{\chi = 0\}$ exactly at points corresponding to those $s = s_0 \neq 0$ where the corresponding solution $w(s) = \frac{a(s)}{2s\chi(s)}$ of Painlevé 3 equation (1.12) has a pole with residue one.*

Our main results imply the following new result.

Proposition 1.26 *The Poincaré map $\mathcal{P}$: $\mathbf{Jos}^o \to \mathbf{Jos}^o$ is well-defined at a point $(\ell, 0, a_0, s_0)$, if and only if the corresponding solution $\frac{a(s)}{2s\chi(s)}$ of Painlevé equation (1.12) defined by the solution $(\chi(s), a(s))$ of (1.10) with $(\chi(s_0), a(s_0)) = (0, a_0)$ is holomorphic on a finite interval (s_0, s_1), has pole with residue one at s_1 and has no zeros with unit derivative in the interval (s_0, s_1).*

Proof The proposition follows from Lemma 1.25 and Theorem 1.24. □

Corollary 1.27 *All the iterates of the Poincaré map $\mathcal{P}$ are defined at a given point $(\ell, 0, a_0, s_0)$, if and only if the corresponding solution $w(s)$ of Painlevé 3 equation (1.12) has an infinite sequence of real positive poles with residue 1 going to $+\infty$, no pole $\hat{s} > s_0$ with residue -1 and no zero $s^* > s_0$ with unit derivative.*

Problem 1.28 Describe those $(\ell, 0, a_0, s_0) \in \mathbf{Jos}^o$ for which $w(s)$ has an infinite sequence of real positive poles with residue 1 going to $+\infty$ and no zero $s^* > s_0$ with unit derivative.

Remark 1.29 Existence of an open subset of initial conditions for Painlevé 3 equation (1.12) for which $w(s)$ has an infinite sequence of real positive poles with residue 1 going to $+\infty$ follows from [2, theorem 3.1] (implying an analogous statement on zeros of solutions of Painlevé 5 equations) and Novokshenov's Backlund transformation argument [49]. This holds, for example, for the real Bessel type solutions, which have the type $w(s) = \frac{\frac{d(s^\ell v(s))}{ds}}{s^\ell v(s)}$, where $v(s)$ is a solution of the ℓ-th Bessel equation (2.11); see Subsection 2.2 and statements 1), 2) of Theorem 1.23. Indeed, each solution $v(s)$ of the Bessel equation has an infinite sequence of real positive zeros, and they are poles of $w(s)$ with residue 1. Novokshenov's numerical experience (2020) has shown that small real deformations of the real Bessel type solutions also have infinite sequence of positive poles with residue 1. Each transcendental meromorphic on $\mathbb{C}^*$ solution of Painlevé 3 equation has infinite number of *complex* poles with residue one; under an additional genericity assumption, it has infinite number of *complex* poles with both residues ± 1 [28, theorem 31.1]. Existence of one-dimensional family of tronquée solutions, which have no poles in a sector containing the real positive semiaxis, was proved in [43].

1.6 Plan of the proof of main results

The proof of main results is based on the following characterization of the constrictions.

Proposition 1.30 *[25, proposition 2.2] Consider the period 2π flow map $h = h^{2\pi}$ of system (1.4) acting on the transversal coordinate θ-circle $\{\tau = 0\}$. (It is also the Poincaré map of the latter cross-section.) A point $(B, A; \omega)$ is a constriction, if and only if $\omega, A \neq 0$ and $h = Id$.*

Proposition 1.31 *In the coordinates (ℓ, a, s) on the parameter space of family (1.4) the Poincaré map $h = h_{\ell,a,s} : S^1_\theta \to S^1_\theta$ is analytic as function on $\mathbb{R}^3_{\ell,a,s} \times S^1_\theta$.*

The proposition follows from the classical theorem on analytic dependence of solution of analytic differential equation on parameters and initial condition.

First in Subsection 2.1 we prove a part of Theorem 1.17. Namely, we show that accumulation points in the s-axis (if any) of the manifold $Constr_\ell$ have ordinates that are zeros of the Bessel function J_ℓ. We show that the germ of $Constr_\ell$ at each intersection point lies in a regular germ of analytic curve. Then we deduce that the union of the closure $\overline{Constr_\ell} \subset \mathbb{R}_a \times (\mathbb{R}_+)_s$ and its image under the symmetry with respect to the s-axis is an analytic submanifold.

In Subsection 2.3 we prove Statements 1) and 2) of Theorem 1.23. To do this, we blow up the space $\mathbb{R}^4_{\ell,\chi,a,s} \setminus \{s = 0\}$ along the plane $\{a = \chi = 0\}$. The complexified blown up manifold will be denoted by $\widehat{\mathbb{C}}^4$. Its exceptional divisor is $\Sigma := \mathbb{C}^2_{\ell,s} \times \mathbb{CP}^2_{[\chi:a]} \setminus \{s = 0\}$. By $\Pi \subset \widehat{\mathbb{C}}^4$ we denote the strict transform of the hyperplane $\{\chi = 0\}$, which corresponds to model of Josephson junction. The line field given by (1.10) induces a holomorphic line field on the complexified blown-up manifold $\widehat{\mathbb{C}}^4$ that is tangent to the exceptional divisor Σ. The hypersurface Π appears to be its global cross-section. The restriction to Σ of the line field appears to be given by the Riccati equation equivalent to the projectivization of the ℓ-th Bessel equation. This yields a geometric interpretation of the well-known fact that Painlevé equation (1.12) has a one-dimensional family of the so-called Bessel type solutions $w(s) = \frac{\frac{du(s)}{ds}}{u(s)}$, where $u(s)$ is s^ℓ times a solution of the ℓ-th Bessel equation. See [21, Section 10 in Chapter 12] and Theorem 2.5 in Subsection 2.2 below.

It will be deduced from the above statements that the first return Poincaré map $\mathcal{P} : \Pi \to \Pi$ is well-defined on a neighborhood of the intersection $\Pi \cap \Sigma$, and its restriction to $\Pi \cap \Sigma$ is the map $(\ell, s) \mapsto (\ell, s^*)$, where $s^* > s$ is the next (after s) zero of a solution of the Bessel equation vanishing at s.

Next, in Subsection 2.4 we show that each zero $s_{\ell,k} > 0$ of Bessel function J_ℓ corresponds to an accumulation point $\beta_{\ell,k} = (0, s_{\ell,k})$ of some constriction curve: a component of the manifold $Constr_\ell$. To do this, we first show that at least some $\beta_{\ell,m}$ is an accumulation point. This is done by using the above-mentioned part of Theorem 1.17 (already proved in Subsection 2.1) and results of Klimenko and Romaskevich [38] on Bessel asymptotics of boundaries of the phase-lock areas. Then we deduce that each $s_{\ell,k}$ corresponds to a constriction curve. This is done by using Statements 1) and 2) of Theorem 1.23 (already proved in Subsection 2.3) and applying iterates of the Poincaré map $\mathcal{P}$ (and its inverse) sending the above accumulation point $\beta_{\ell,m}$ to any other $\beta_{\ell,k}$. This will finish the proofs of Theorems 1.17 and 1.23.

In Subsection 2.5 we prove Proposition 1.18.

In Subsection 2.6 we prove Theorem 1.5 and Corollary 1.20.

In Subsection 2.8 we prove Theorem 1.24. The proof uses Stokes phenomena theory and isomonodromicity of family of linear systems (1.15) along solutions of system (1.10). The corresponding background material on Stokes phenomena and isomonodromic families is briefly recalled in Subsection 2.7.

1.7 Brief historical remarks

Hereby we present a very brief survey of results that were not mentioned in the introduction. See [8] for a more complete survey.

I.A.Bizyaev, A.V.Borisov, and I.S.Mamaev [10] noticed that a big family of dynamical systems on torus in which the rotation number quantization effect realizes was introduced by W.Hess (1890). Such systems were studied in classical mechanics in problems on rigid body movement with fixed point in works by W.Hess, P.A.Nekrassov, A.M.Lyapunov, B.K.Mlodzejewski, N.E.Zhukovsky and others. See [10, 44, 47, 48, 58] and references therein. P.A.Nekrasov observed in [48] that the above-mentioned big family of systems considered by Hess can be equivalently described by a Riccati equation (or by a linear second order differential equation).

Numerical experiences of V.M.Buchstaber, O.V.Karpov, S.I.Tertychnyi, D.A.Filimonov, V.A.Kleptsyn, I.V.Schurov have shown that as $\omega \to 0$, the "upper" part of the phase-lock area portrait converges to a kind of parquet in the renormalized coordinates (ℓ, μ): the renormalized phase-lock areas tend to unions of pieces of parquet, and gaps between the phase-lock areas tend to zero. See Fig. 3 and the paper [37]. This is an open problem. In [37] V.A.Kleptsyn, O.L.Romaskevich, and I.V.Schurov proved some results on smallness of gaps and their rate of convergence to zero, as $\omega \to 0$, using methods of slow-fast systems.

A subfamily of family (1.4) of dynamical systems on 2-torus was studied by J.Guckenheimer and Yu.S.Ilyashenko in [30] from the slow-fast system point of view. They obtained results on its limit cycles, as $\omega \to 0$.

2 Proof of main results

2.1 Intersection points of $\overline{Constr_\ell}$ with the s-axis: their regularity and landing components

Here we prove the following lemma.

Lemma 2.1 *1) For every $\ell \in \mathbb{Z}$ each finite accumulation point of the submanifold $Constr_\ell \subset \mathbb{R}^2_+$ (if any) lies in the s-axis and has the type $\beta = (0, y)$, where $y > 0$ is a zero of the Bessel function J_ℓ.*

2) The subset $\widehat{Constr_\ell} \subset \mathbb{R}^2$, which is the union of the closure $\overline{Constr_\ell} \subset \mathbb{R}^2$ and its symmetric with respect to the s-axis, is a one-dimensional analytic submanifold intersecting the s-axis orthogonally at the above points β.

3) Each connected component of the submanifold $Constr_\ell$ landing at some point β of the s-axis accumulates to no other point of the s-axis.

4) Each above landing component is projected onto the whole a-axis $\mathbb{R}_+$.

The first step of the proof of Lemma 2.1 is the following proposition.

Proposition 2.2 *Let $\ell \in \mathbb{Z}$. Let a sequence of points in $Constr_\ell$ accumulate to a point $(0, y)$ of the s-axis. Then $J_\ell(y) = 0$.*

Proof The submanifold $Constr_\ell$ is the intersection of the positive quadrant $\mathbb{R}^2_+$ with the analytic subset

$$\mathcal{M}_\ell := \{h_{\ell,a,s} = Id\} \subset \mathbb{R}^2_{a,s}, \tag{2.1}$$

see Propositions 1.30 and 1.31. The germ at $(0, y)$ of the analytic subset $\mathcal{M}_\ell$ is one-dimensional, by assumption. Therefore, it is a finite union of germs of analytically parametrized curves. Let us calculate the derivative of the time τ flow maps $h^\tau : S^1 \to S^1$ (and in particular, of the map $h = h^{2\pi}$) in the parameters (a, s) from the equation in variations corresponding to (1.3). Set $\theta(\tau, \theta_0, a, s) = h^\tau(\theta_0)$ to be the solution of differential equation (1.3) with the initial condition $\theta = \theta_0$ at $\tau = 0$. One has

$$\theta|_{a=0} = \theta(\tau, \theta_0, 0, s) = f(\tau) := \theta_0 + \ell\tau + s \sin\tau, \ h|_{a=0} = Id, \tag{2.2}$$

$$\dot{\theta}'_a = \frac{d}{d\tau}\frac{\partial\theta}{\partial a} = \cos\theta - a \sin\theta\theta'_a. \tag{2.3}$$

Substituting $a = 0$ and $\theta = f(\tau)$, see (2.2), to (2.3) and integrating (2.3) with the initial condition $\theta'_a|_{\tau=0} = 0$ yields

$$\theta'_a(\tau) = \int_0^\tau \cos f(\tau) d\tau,$$

$$\frac{\partial h}{\partial a}(\theta_0)|_{a=0} = \int_0^{2\pi} \cos(\theta_0 + \ell\tau + s \sin\tau) d\tau. \tag{2.4}$$

Claim 1. *If $(0, y)$ is an accumulation point of the set $Constr_\ell$, then*

$$\frac{\partial h_{\ell,a,y}}{\partial a}(\theta_0)|_{a=0} = 0 \text{ for every } \theta_0 \in S^1. \tag{2.5}$$

Proof Suppose the contrary: there exists a sequence of points $(a_k, s_k) \in Constr_\ell$ converging to $(0, y)$, as $k \to \infty$, but the derivative (2.5) is non-zero at some θ_0. One has

$$h_{\ell,a_k,s_k}(\theta_0) = \theta_0 + a_k \frac{\partial h_{\ell,a,s_k}}{\partial a}(\theta_0)|_{a=0} + O(a_k^2),$$

since $h_{\ell,0,s} \equiv Id$, see (2.2). Therefore, the latter right-hand side cannot be equal to θ_0 for all k, since $a_k > 0$. But $h_{\ell,a,s} = Id$ for all $(a, s) \in Constr_\ell$, by Proposition 1.30. The contradiction thus obtained proves the claim. □

System of identities (2.5) for all θ_0 is equivalent to the system of equalities

$$\int_0^{2\pi} \cos(\ell\tau + y \sin\tau)d\tau = 0; \int_0^{2\pi} \sin(\ell\tau + y \sin\tau)d\tau = 0, \tag{2.6}$$

by (2.4) and cosine addition formula. The second identity in (2.6) holds automatically, since the function $\sin(x)$ is odd and 2π-periodic. The first integral in (2.6) is equal to

$$\int_0^{2\pi} \cos(\ell\tau + y \sin\tau)d\tau = 2\int_0^{\pi} \cos(\ell\tau + y \sin\tau)d\tau = 2\pi J_\ell(-y),$$

see (1.16), since the function $\cos(x)$ is even and 2π-periodic. One has $J_\ell(-x) = (-1)^\ell J(x)$, see [57, section 2.1, formula (2)]. Thus, (2.5) is equivalent to the equality $J_\ell(y) = 0$. This together with Claim 1 proves Proposition 2.2. □

Proposition 2.3 *Let $y \in \mathbb{R}_+$ be such that $\beta = (0, y)$ is an accumulation point of the submanifold $Constr_\ell$. Then the germ at β of the subset $\widehat{Constr_\ell}$ is a germ of regular curve orthogonal to the s-axis.*

Proof The submanifold $Constr_\ell$ consists of those $(a, s) \in \mathbb{R}_+^2$ for which the difference $\Delta(a, s; \theta) := h_{\ell,a,s}(\theta) - \theta$ vanishes identically in $\theta \in \mathbb{R}$. The Taylor series in (a, s) of the latter difference is divisible by a, since it vanishes for $a = 0$. Therefore, the function

$$\xi(a, s; \theta) = a^{-1}\Delta(a, s; \theta)$$

is analytic. One has $J_\ell(y) = 0$ (Claim 1) and

$$\xi(0, y; \theta_0) = 0 \text{ for every } \theta_0, \tag{2.7}$$

by the equality $\frac{\partial}{\partial a}\Delta(0, y; \theta_0) = 0$, see (2.5).

Proposition 2.4 *For every $y > 0$ such that $J_\ell(y) = 0$ one has*

$$\frac{\partial \xi}{\partial s}(0, y; \theta_0) \neq 0 \quad \textit{for every} \quad \theta_0 \notin \frac{\pi}{2} + \pi\mathbb{Z}. \tag{2.8}$$

Proof Inequality (2.8) is equivalent to the inequality

$$\frac{\partial^2 \Delta}{\partial a \partial s}(0, y; \theta_0) \neq 0 \quad \text{for} \quad \theta_0 \notin \frac{\pi}{2} + \pi\mathbb{Z}. \tag{2.9}$$

Let us prove (2.9). Equations in variations on the similar derivative of a solution $\theta(\tau, \theta_0, a, s)$ is obtained by differentiation of equation (2.3) in s and subsequent substitution $a = 0$ and $\theta'_s|_{a=0} = \sin\tau$ (which is also found from the equation in variation on θ'_s):

$$\frac{d}{d\tau}\frac{\partial^2 \theta}{\partial a \partial s} = -\sin\theta\theta'_s = -\sin(\theta_0 + \ell\tau + s\sin\tau)\sin\tau$$

$$= \frac{1}{2}(\cos(\theta_0 + (\ell+1)\tau + s\sin\tau) - \cos(\theta_0 + (\ell-1)\tau + s\sin\tau)).$$

Integrating the latter differential equation with zero initial condition at $\tau = 0$ and substituting $\tau = 2\pi$ yields

$$\frac{\partial^2 \theta}{\partial a \partial s}(2\pi, \theta_0, 0, y) = \frac{\partial^2 \Delta}{\partial a \partial s}$$

$$= \frac{1}{2}\int_0^{2\pi}(\cos(\theta_0 + (\ell+1)\tau + y\sin\tau) - \cos(\theta_0 + (\ell-1)\tau + y\sin\tau))d\tau$$

$$= \pi\cos\theta_0(J_{\ell+1}(-y) - J_{\ell-1}(-y)), \tag{2.10}$$

by (1.16), the addition formula for cosine and the second equality in (2.6) (which holds for every integer number ℓ). The Bessel function difference in the latter right-hand side is equal to $-2J'_\ell(-y)$, by well-known formula, see [57, p. 17, formula (2)]. The latter derivative is non-zero, since J_ℓ is a solution of the second order differential equation (Bessel equation) and $J_\ell(-y) = (-1)^\ell J_\ell(y) = 0$. Finally, the right-hand side in (2.10) is non-zero. This together with (2.10) implies (2.9). Proposition 2.4 is proved. □

The analytic subset $\mathcal{M}_\ell \subset \mathbb{R}^2$ is symmetric with respect to the s-axis. This follows from the fact that changing sing of ω is equivalent to applying the variable change $\theta \mapsto \theta + \pi$ to dynamical system (1.4), which does not change triviality of the Poincaré map (the time 2π flow map of the zero fiber $S^1_\theta \times \{0\}$). The subset $\mathcal{M}_\ell \setminus \{a = 0\} \subset \mathbb{R}^2_{a,s}$ coincides with the union of the subset $Constr_\ell$ and its image under the symmetry with respect to the s-axis. Both latter subsets are contained in the analytic subset

$$\mathcal{A}_{\theta_0} := \{\xi(a, s; \theta_0) = 0\} \subset \mathbb{R}^2_{a,s}$$

for every fixed $\theta_0 \in \mathbb{R}$. For $\theta_0 \notin \frac{\pi}{2} + \pi\mathbb{Z}$, the germ of the subset $\mathcal{A}_{\theta_0}$ at an accumulation point $\beta = (0, y)$ of the set $Constr_\ell$ is a germ (γ, β) of regular analytic curve transversal to the s-axis, since $J_\ell(y) = 0$ (Proposition 2.2) and by Proposition 2.4. It is orthogonal to the s-axis, by symmetry. The punctured germ $\gamma \setminus \{\beta\}$ coincides with the germ at β of the set $\mathcal{M}_\ell \setminus \{a = 0\}$, since the latter is a one-dimensional submanifold in $\mathbb{R}^2_+ \subset \mathbb{R}^2 \setminus \{a = 0\}$ whose germ at β lies in γ. This implies the statements of Proposition 2.3. $\square$

Proof of Lemma 2.1. The submanifold $Constr_\ell$ accumulates to no point of the a-axis. For non-accumulation to the origin see [8, lemma 4.14]. Non-accumulation to points $(a, 0)$ with $a > 0$ is proved in [8, pp. 5459–5460, Case (1)]. (Formally speaking, the argument in loc. cit. proves non-accumulation to $(a, 0)$ of sequence of points lying in the same component in $Constr_\ell$. But in fact, the argument is valid for every sequence in $Constr_\ell$.) The germs of the set $\mathcal{M}_\ell \setminus \{a = 0\}$ at points of its accumulation to the s-axis coincide with those of the analytic subset $\mathcal{A}_{\theta_0}$, $\theta_0 \notin \frac{\pi}{2} + \pi\mathbb{Z}$, punctured at the latter accumulation points: see the above proof of Proposition 2.3. They are regular at the base (accumulation) points, as is $\mathcal{A}_{\theta_0}$, which was proved above. The intersection of the set $\mathcal{M}_\ell \setminus \{a = 0\}$ with the positive quadrant coincides with $Constr_\ell$. This implies that the union of the closure $\overline{Constr_\ell}$ and its symmetric image with respect to the s-axis is an analytic submanifold in $\mathbb{R}^2$ (symmetry of the set $\mathcal{M}_\ell$). Its orthogonality to the s-axis at their intersection points is obvious. The coordinate a is unbounded from above on each connected component of the submanifold $Constr_\ell \subset \mathbb{R}^2_+$, by [8, theorem 1.12]. This implies that each component landing at some point in the s-axis is projected to the whole a-axis $\mathbb{R}_+$ and cannot start and end at points of the s-axis. This implies Statements 3) and 4) of Lemma 2.1 and finishes the proof of the lemma. $\square$

2.2 Bessel type solutions of Painlevé 3 equation

Theorem 2.5 *[21, Section 10, Chapter 12] For every $\ell \in \mathbb{R}$ the corresponding Painlevé 3 equation (1.12) admits a one-parameter family of solutions of type*

$$w(s) = \frac{\frac{du(s)}{ds}}{u(s)}, \quad u(s) = s^\ell (C_1 J_\ell(s) + C_2 Y_\ell(s)).$$

Here $J_\ell(s)$ and $Y_\ell(s)$ are linearly independent solutions of Bessel equation

$$\frac{d^2 v}{ds^2} + \frac{1}{s}\frac{dv}{ds} + (1 - \frac{\ell^2}{s^2})v = 0. \tag{2.11}$$

Namely, J_ℓ is the ℓ-th Bessel function

$$J_\ell(x) = \sum_{p=0}^{\infty} \frac{(-1)^p}{\Gamma(p+1)\Gamma(p+\ell+1)} \left(\frac{x}{2}\right)^{2p+\ell}.$$

(It is well-known that for $\ell \in \mathbb{Z}$, J_ℓ is also given by formula (1.16).) The above solutions $w(s)$ of (1.12) are called the **Bessel type solutions.**

Proposition 2.6 *A function $v(s)$ satisfies Bessel equation (2.11), if and only if the function $u(s) = s^\ell v(s)$ satisfies the equation*

$$u'' + \frac{1-2\ell}{s}u' + u = 0 \tag{2.12}$$

In this case the function

$$w(s) = \frac{\frac{du(s)}{ds}}{u(s)} \tag{2.13}$$

satisfies the Riccati equation

$$w' = \frac{2\ell - 1}{s}w - w^2 - 1. \tag{2.14}$$

Each solution of (2.14) satisfies Painlevé 3 equation (1.12).

Proof Writing $v = us^{-\ell}$ and substituting this to Bessel equation (2.11) yields

$$s^{-\ell}u'' - 2\ell s^{-\ell-1}u' + \ell(\ell+1)s^{-\ell-2}u + \frac{1}{s}(s^{-\ell}u' - \ell s^{-\ell-1}u) + (1 - \frac{\ell^2}{s^2})s^{-\ell}u = 0,$$

which is equivalent to (2.12). Differentiating (2.13) and substituting (2.12) yields

$$w'(s) = -w^2 + \frac{u''(s)}{u(s)} = -w^2 + \frac{2\ell-1}{s}w(s) - 1.$$

Thus, Riccati equation (2.14) is the projectivization of a linear system equivalent to (2.12). This implies that each its solution is of type (2.13) with $u(s) = s^\ell v(s)$, where v satisfies Bessel equation (2.11). Hence, each its solution is a solution of Painlevé 3 equation (1.12), by Theorem 2.5. $\square$

2.3 Blow up and domain of the Poincaré map $\mathcal{P}$

Convention 2.7 We will be dealing with the plane $\mathbb{C}^2_{\chi,a}$ blown up at the origin. The blown up plane will be denoted by $\widehat{\mathbb{C}}^2$. The exceptional divisor (the pasted projective line) is naturally identified with the projectivization of the space $\mathbb{C}^2_{\chi,a}$: the projective line $\mathbb{CP}^1$ equipped with homogeneous coordinates $[\chi : a]$. Since formally speaking, the functions χ and a on $\widehat{\mathbb{C}}^2$ vanish on the exceptional divisor $\mathbb{CP}^1 \subset \widehat{\mathbb{C}}^2$, we rename its canonical homogeneous coordinates $[\chi : a]$ by $[y_1 : y_2]$, in order to avoid confusion. Similarly, the space $\mathbb{C}^4_{\ell,\chi,a,s}$ blown up along the (ℓ, s)-space $\{\chi = a = 0\}$ will be denoted by $\widehat{\mathbb{C}}^4$. Its exceptional divisor will be denoted by

$$\Sigma := \mathbb{CP}^1_{[y_1:y_2]} \times \mathbb{C}^2_{\ell,s}.$$

Proposition 2.8 *Let $\Pi\subset\widehat{\mathbb{C}}^4$ denote the strict transform of the hyperplane $\{\chi=0\}$ (which corresponds to complexified model of Josephson junction) under the above blow up.*

1) System (1.10) lifts to a well-defined line field $\mathcal{V}$ on

$$\widehat{\mathbb{C}}^{4,0}:=\widehat{\mathbb{C}}^4\setminus\{s=0\}:$$

a non-autonomous ordinary differential equation on $\mathbb{C}_\ell\times\widehat{\mathbb{C}}^2$-valued function in s with constant ℓ-component. The exceptional divisor Σ is tangent to the line field $\mathcal{V}$. Its intersection with Π is the plane $\mathbb{C}^2_{\ell,s}=\{[0:1]\}\times\mathbb{C}^2_{\ell,s}$.

2) The restriction to Σ of the line field $\mathcal{V}$ is given by the Riccati equation on the function $\Psi(s)=\frac{y_2(s)}{y_1(s)}$ that is the projectivization of linear system

$$\begin{pmatrix}\dot y_1\\ \dot y_2\end{pmatrix}=\begin{pmatrix}-\frac{\ell}{s} & \frac{1}{2s}\\ -2s & \frac{\ell}{s}\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix}. \tag{2.15}$$

Namely, $\Psi(s)$ is a solution of the above-mentioned Riccati equation, if and only if $\Psi(s)=\frac{y_2(s)}{y_1(s)}$ where $y=(y_1(s),y_2(s))$ is a solution of system (2.15).

3) Along each solution $y(s)$ of system (2.15) the corresponding function

$$w(s):=\frac{y_2(s)}{2sy_1(s)} \tag{2.16}$$

satisfies Riccati equation (2.14) and Painlevé 3 equation (1.12).

4) The line field $\mathcal{V}$ restricted to Σ is transversal to the hypersurface Π at their intersection points.

Proof System (1.10) considered as a differential equation on vector function $(\chi(s),a(s))$ with constant parameter ℓ has right-hand side that vanished for $\chi=a=0$. This implies that system (1.10) lifts to a holomorphic line field $\mathcal{V}$ on $\widehat{\mathbb{C}}^4$ that is tangent to the exceptional divisor $\Sigma=\{\chi=a=0\}=\mathbb{CP}^1_{[y_1:y_2]}\times\mathbb{C}^2_{\ell,s}$. Its restriction to Σ is given by the projectivized linear part in (χ,a) of the right-hand side of (1.10): linear system (2.15), whose matrix in the right-hand side coincides with the Jacobian matrix in (χ,a) at $(0,0,s)$ of the (χ,a)-component of system (1.10). The ratio $w(s)$, see (2.16), satisfies Riccati equation (2.14):

$$w'=-\frac{1}{s}w+\frac{(-2sy_1+\frac{\ell}{s}y_2)y_1-y_2(-\frac{\ell}{s}y_1+\frac{1}{2s}y_2)}{2sy_1^2}=\frac{2\ell-1}{s}w-1-w^2.$$

Painlevé 3 equation (1.12) on $w(s)$ can be proved in two possible ways:

a) It follows from Proposition 2.6.

b) Take a solution $(\ell,\chi(s),a(s))$ of system (1.10) with the initial condition (ℓ,χ_0,a_0) at $s=s_0$. The corresponding solution $\frac{a(s)}{2s\chi(s)}$ of Painlevé 3 equation

(1.12) converges to $w(s)$, as $(\chi_0, a_0) \to (0,0)$ so that $\frac{a_0}{2s_0\chi_0} \to w_0 := w(s_0)$. Therefore, the limit $w(s)$ also satisfies Painlevé 3 equation (1.12).

Statements 1) – 3) of Proposition 2.8 are proved. Transversality statement 3) follows immediately from the fact that linear system (2.15) has matrix with off-diagonal elements being non-zero for every $s \neq 0$. □

Consider the standard vector field directing $\mathcal{V}$:

$$\begin{cases} \ell'_s = 0 \\ \chi' = \frac{a-2\chi(\ell+2\chi a)}{2s} \\ a' = -2s\chi + \frac{a}{s}(\ell + 2\chi a) \\ s' = 1 \end{cases} . \tag{2.17}$$

Set

$$\Pi_+ := \text{ the real part of } \Pi \cap \{s > 0\}, \quad X_+ := \Pi_+ \cap \Sigma.$$

Proposition 2.9 *1) For every initial condition $x_0 = [0:1] \times (\ell, s_0) \in X_+$ the real forward orbit of field (2.17) starting at x_0 meets X_+ at some point $x_1 = [0:1] \times (\ell, s_1)$, $s_1 > s_0$ (take x_1 to be the first point of return to X_+).*

2) The same holds for every $x_0 \in \Pi_+$ lying in some neighborhood of X_+, and the map $x_0 \mapsto x_1$ is a well-defined germ of analytic Poincaré first return map $\mathcal{P} : (\Pi_+, X_+) \mapsto (\Pi_+, X_+)$.

3) In the case, when the above x_0, x_1 lie in X_+, one has $x_1 = \mathcal{P}(x_0)$, if and only if $0 < s_0 < s_1$ and s_0, s_1 are neighbor zeros of a solution of Bessel equation (2.11).

Proof The flow of field (2.17) on Σ is given by solutions $w(s)$ of the Riccati equation (2.14). Intersections of its orbit with Π are poles of the solution $w(s)$. Recall that $w = \frac{u'}{u}$, see (2.13), where $u(s) = s^\ell v(s)$, $v(s)$ is a solution of Bessel equation (2.11). Therefore poles of w are exactly zeros of $v(s)$. The Bessel function $J_\ell(s)$ has infinite sequence of positive zeros. For every other solution $v(s)$ of Bessel equation its zeros are intermittent with those of J_ℓ, by Sturm Separation Theorem. Therefore, each its solution v has also infinite sequence of positive zeros. The map sending one zero of a solution to the next one depends analytically on the parameter of solution, since no two zeros can collide: a non-trivial solution of a second order differential equation cannot have multiple zeros. This proves Statements 1) and 3). Statement 2) follows from transversality statement 4) of Proposition 2.8. □

2.4 The Poincaré map on constriction curves. Proofs of Theorems 1.17 and 1.23

Proposition 2.10 *The submanifold $\widehat{Constr}_\ell \subset \mathbb{R}^2$ intersects the s-axis in at least one point (in fact, at an infinite number of points) with positive* *ordinate.*

Proof The proof is analogous to the discussion in [8, subsection 4.3]. It is based on the Bessel asymptotics of the boundary curves of phase-lock areas proved by A.Klimenko and O.Romaskevich [38]. Let us recall their result. The boundary of the phase-lock area L_r consists of two curves $\partial L_{r,0}$, $\partial L_{\ell,\pi}$, corresponding to those parameter values, for which the Poincaré map of the corresponding dynamical system (1.4) acting on the circle $\{\tau = 0\}$ (i.e., the time 2π flow map) has fixed points 0 and π respectively. These are graphs

$$\partial L_{r,\alpha} = \{B = G_{r,\alpha}(A)\}, \ G_{r,\alpha} \text{ are analytic functions on } \mathbb{R}; \ \alpha = 0, \pi.$$

Theorem 2.11 *[38, theorem 2]. There exist positive constants ξ_1, ξ_2, K_1, K_2, K_3 such that the following statement holds. Let $r \in \mathbb{Z}$, A, $\omega > 0$ be such that*

$$|r\omega| + 1 \leq \xi_1\sqrt{A\omega}, \quad A \geq \xi_2\omega. \tag{2.18}$$

Let J_r denote the r-th Bessel function. Then

$$\left|\frac{1}{\omega}G_{r,0}(A) - r + \frac{1}{\omega}J_r\left(-\frac{A}{\omega}\right)\right| \leq \frac{1}{A}\left(K_1 + \frac{K_2}{\omega^3} + K_3\ln\left(\frac{A}{\omega}\right)\right), \tag{2.19}$$

$$\left|\frac{1}{\omega}G_{r,\pi}(A) - r - \frac{1}{\omega}J_r\left(-\frac{A}{\omega}\right)\right| \leq \frac{1}{A}\left(K_1 + \frac{K_2}{\omega^3} + K_3\ln\left(\frac{A}{\omega}\right)\right). \tag{2.20}$$

Let $u_1 < u_2 < \dots$ denote the sequence of points of local maxima of the modulus $|J_\ell(-u)|$: $u_k \to +\infty$, as $k \to \infty$.

Proposition 2.12 *For every $\ell \in \mathbb{Z}$, $a_0 > 0$ and every k large enough dependently on ℓ, a_0, for every $a \in (0, a_0]$ there exists a $w_k = w_k(a) \in (u_k, u_{k+1})$ such that $(a, w_k) \in Constr_\ell$.*

Proof In the coordinates (a, s) inequalities (2.18), (2.19) and (2.20) can be rewritten for $r = \ell$ respectively as

$$|\frac{\ell}{a}| + 1 \leq \frac{\xi_1}{a}\sqrt{s}, \ s \geq \xi_2, \tag{2.21}$$

$$\left|aG_{\ell,0}(\frac{s}{a}) - \ell + aJ_\ell(-s)\right| \leq \frac{a}{s}\left(K_1 + K_2a^3 + K_3\ln(s)\right), \tag{2.22}$$

$$\left|aG_{\ell,\pi}(\frac{s}{a}) - \ell - aJ_\ell(-s)\right| \leq \frac{a}{s}\left(K_1 + K_2a^3 + K_3\ln(s)\right). \tag{2.23}$$

For every k large enough the value $s = u_k$ satisfies inequality (2.21) for all $a \in (0, a_0]$. Substituting $s = u_k$ to the right-hand side in (2.22) transforms it to a sequence of functions of $a \in (0, a_0]$ with uniform asymptotics

$a(O(\frac{1}{u_k}) + O(\frac{\ln u_k}{u_k}))$, as $k \to \infty$. The values $|J_\ell(-u_k)|$ are known to behave asymptotically as $\frac{1}{\sqrt{u_k}}$ up to a known constant factor, see asymptotic formula for $J_\ell(s)$, as $s \to +\infty$, in [57, section 7.1, p. 195]. Therefore, they dominate the right-hand sides in (2.22) and (2.23). This together with (2.22), (2.23) implies that for every k large enough, set $A_k = \frac{u_k}{a}$, $\omega = a^{-1}$,

$$G_{\ell,0}(A_k) = \ell\omega - J_\ell(-u_k)(1 + o(1)), \ G_{\ell,\pi}(A_k) = \ell\omega + J_\ell(-u_k)(1 + o(1)),$$

as $k \to \infty$, uniformly in $\omega > a_0^{-1}$, i.e., uniformly in $a \in (0, a_0]$. This implies that the difference

$$\Delta(A) := G_{\ell,\pi}(A) - G_{\ell,0}(A) \text{ has the same sign, as } J_\ell(-u_k), \text{ at } A = A_k,$$

whenever k is large enough, for every $a \in (0, a_0]$. The extrema u_k are intermittent maxima and minima of the function $J_\ell(-u)$, and the values $J_\ell(-u_k)$ have intermittent signs. Therefore, the same holds for the above differences $\Delta(A_k)$. Thus, for every k large enough the function $\Delta(A)$ has at least one zero W_k lying between A_k and A_{k+1}. The latter zero corresponds to an intersection point of the boundaries $\partial L_{\ell,0}$, $\partial L_{\ell,\pi}$. The latter intersection point is a constriction of the phase-lock area L_ℓ. Hence, its abscissa is equal to $\ell\omega$, by [8, theorem 1.4]. Finally, we have shown that for every k large enough for every $a \in (0, a_0]$, set $w_k = w_k(a) := aW_k$, the corresponding point (a, w_k) lies in the constriction submanifold $Constr_\ell$. One has $w_k \in (u_k, u_{k+1})$, by construction. Proposition 2.12 is proved. □

The family of points $(a, w_k(a))$ from Proposition 2.12 with $a \in (0, a_0]$ accumulates to a point $(0, w_k^*)$, $w_k^* \in [u_k, u_{k+1}]$. Thus, the manifold $Constr_\ell$ has infinitely many limit points $(0, w_k^*)$. This proves Proposition 2.10. □

Proposition 2.13 *Let $\beta_{\ell,k} = (0, s_{\ell,k})$ lie in $\widehat{Constr}_\ell \subset \mathbb{R}^2_{a,s}$. Then $\beta_{\ell,k+1} = (0, s_{\ell,k+1})$ also lies in the manifold $\widehat{Constr}_\ell$, and the germ of $\widehat{Constr}_\ell$ at $\beta_{\ell,k+1}$ is the image of its germ at $\beta_{\ell,k}$ under the Poincaré map $\mathcal{P}$. If in addition, $k \geq 2$, then $\beta_{\ell,k-1}$ also lies in $\widehat{Constr}_\ell$, and its germ at $\beta_{\ell,k-1}$ is the image of its germ at $\beta_{\ell,k}$ under the inverse $\mathcal{P}^{-1}$.*

Proof The Poincaré map $\mathcal{P}$ sends $\beta_{\ell,k}$ to $\beta_{\ell,k+1}$, by Proposition 2.9, Statement 3). It sends the germ of the manifold $\widehat{Constr}_\ell$ at $\beta_{\ell,k}$ to its germ at $\beta_{\ell,k+1}$. Indeed, it preserves ℓ and the conjugacy class of the underlying dynamical system on torus, and hence, property of dynamical system to have identity Poincaré map. Therefore, it preserves the property to be a constriction, by Proposition 1.30. This implies the first statement of the proposition, on the map $\mathcal{P}$. Its second statement, on $\mathcal{P}^{-1}$, is proved analogously. □

Proof of Theorems 1.17 and 1.23. An intersection point from Proposition 2.10 is some of the $\beta_{\ell,k} = (0, s_{\ell,k})$ (Proposition 2.2). It is a regular point of the submanifold $\widehat{Constr_\ell}$, and it is unique intersection point of its component passing through $\beta_{\ell,k}$ with the s-axis. Their intersection is orthogonal. The component is projected to the whole a-axis. These statements follow from Lemma 2.1. All the $\beta_{\ell,k}$ are intersection points. This follows from Proposition 2.13. Theorem 1.17 is proved. Theorem 1.23 follows from Theorem 1.17 and Propositions 2.9, 2.13. □

2.5 Conjectures 1.6 and 1.7. Proof of Proposition 1.18

Proof of Proposition 1.18. Let $\mathcal{C}$ be a connected component of the submanifold $Constr_\ell \subset (\mathbb{R}^2_+)_{a,s}$. The coordinate $a = \omega^{-1}$ is unbounded from above on $\mathcal{C}$, see [8, theorem 1.12]. Therefore, its projection to the a-semiaxis is a semi-infinite interval $(a_0, +\infty)$. Let now a be bounded on a semicurve $\mathcal{C}_-$ of the curve $\mathcal{C}$. The semicurve $\mathcal{C}_-$ should tend to "infinity" in $\mathbb{R}^2_+$, since $\mathcal{C}$ is a submanifold in $\mathbb{R}^2_+$: a closed subset in $\mathbb{R}^2_+$ that is locally a submanifold. Therefore, either $\mathcal{C}_-$ has a limit point in the positive s-axis, or s tends to infinity along the semicurve $\mathcal{C}_-$ (Theorem 1.17). In the first case one has $\mathcal{C} = \mathcal{C}_{\ell,k}$ for some k, by Theorem 1.17. The second case is impossible, by [8, proposition 4.13]. Finally, $\mathcal{C} = \mathcal{C}_{\ell,k}$. This proves Proposition 1.18. □

Suppose now that the statement of Conjecture 1.7 is true for a given $\ell \in \mathbb{Z}$: each component $\mathcal{C}$ of the constriction submanifold $Constr_\ell$ is bijectively projected onto the a-semiaxis. Then $\mathcal{C}$ is continuously parametrized by $a \in \mathbb{R}_+$, and hence, a is bounded on at least one side of the curve $\mathcal{C}$. Therefore, $\mathcal{C} = \mathcal{C}_{\ell,k}$ for some k, by Proposition 1.18. Hence, the statement of Conjecture 1.6 holds. Thus, Conjecture 1.7 implies Conjecture 1.6.

2.6 Constriction curves and components of three-dimensional phase-lock areas. Proof of Theorem 1.5 and Corollary 1.20

In the proof of Theorem 1.5 and Corollary 1.20 we use the following theorem and lemma.

Theorem 2.14 *[8, theorem 1.7] For every $\omega > 0$ and $r \in \mathbb{Z}$ each constriction (B_0, A_0) in the corresponding two-dimensional phase-lock area $L_r(\omega)$ is* **positive,** *that is, there exists a neighborhood $U = U(A_0) \subset \mathbb{R}$ such that the punctured interval $B_0 \times (U \setminus \{A_0\})$ lies in $Int(L_r(\omega))$.*

Proof of Theorem 1.5. Fix some $r \in \mathbb{Z}$ and $\omega > 0$. Consider the line $\Lambda := \{a = \omega^{-1}\} \subset \mathbb{R}^2_{a,s}$. We will identify its points with their s-coordinates. For every point $x = (B_0, A_0; \omega) \in Int(L_r)$ consider those constrictions in $\partial L_r(\omega)$ that lie below x, i.e., with $A < A_0$. Let us denote their subset by $Constr_{r,x}$. Let $C_{max} \in Constr_{r,x}$ denote the upper of them; set $s_{max} := s(C_{max}) =$

$\frac{A(C_{max})}{\omega}$. The constrictions in $Constr_{r,x}$ correspond to intersection points of the segment $[0, s_{max}] \subset \Lambda$ with constriction curves (treated as components of the constriction submanifold $Constr_r$). To each constriction $\eta \in Constr_{r,x}$ we assign its *multiplicity,* which is equal to the index of the intersection $Constr_r \cap \Lambda$ at η. For each constriction curve $\mathcal{C}$ let $n(\mathcal{C}, x) \in \{0, 1\}$ denote the sum of multiplicities of those constrictions in $Constr_{r,x}$ (taken modulo two) that lie in $\mathcal{C}$. To each point $x \in Int(L_r)$ we put into correspondence its *code:*

$$\text{code}(x) := \text{ the collection of those constriction curves } \mathcal{C}$$
$$\text{for which } n(\mathcal{C}, x) = 1,$$

i.e., the collection of those constriction curves that contain odd number of constrictions (with multiplicities) lying in $Constr_{r,x}$.

Proposition 2.15 *For every $r \in \mathbb{Z}$ the code is constant on each connected component of the three-dimensional phase-lock area L_r.*

Proof As a point x moves continuously inside $Int(L_r)$, the constrictions of the two-dimensional phase-lock area $L_r(\omega(x))$ lying below x either remain the same, or some new constrictions are born, or some of them disappear[2]. A birth of constrictions comes from a tangency point of the line $\{a = \omega^{-1}(x)\} \subset \mathbb{R}^2_{a,s}$ with a constriction curve $\mathcal{C}$. The number of new constrictions born from a tangency point is obviously even, and all of them lie in $\mathcal{C}$. Therefore, birth of new constrictions does not change the code. The case of disappearance of constrictions is treated analogously. □

Proposition 2.16 *1) For every $r \in \mathbb{Z}$, $N \in \mathbb{N}$ there exists a small $\varepsilon > 0$ such that for every $a_0 \in (0, \varepsilon)$ the line $\Lambda = \{a = a_0\} \subset \mathbb{R}^2_{a,s}$ intersects each one of the curves $\mathcal{C}_{r,k}$, $k = 1, \dots, N$, transversally at a point $x_k = x_k(a_0)$ close to $\beta_{r,k}$ so that $x_1, \dots, x_N$ are the only points of intersection of the segment $I := \{a_0\} \times [0, x_N + \varepsilon]$ with the constriction manifold $Constr_r$.*

2) For every $k = 1, \dots, N$ let us identify x_k with the corresponding constriction of the two-dimensional phase-lock area $L_r(\omega)$, $\omega = a_0^{-1}$. On the component of $Int(L_r(\omega))$ adjacent to x_k from above the code is identically equal to $(\mathcal{C}_1, \dots, \mathcal{C}_k)$.

The proposition follows immediately from Theorem 1.17. It implies that for every $r \in \mathbb{Z}$ infinitely many different codes are realized by points of the interior of the three-dimensional phase-lock area L_r. Therefore, $Int(L_r)$ consists of infinitely many components. Theorem 1.5 is proved. □

[2]Conjecture 1.7 implies that as ω varies continuously, birth (disappearance) of constrictions is impossible.

Proof of Corollary 1.20. Statement (i) of Corollary 1.20 follows from Proposition 1.18. There exists a domain $V \subset U_{\mathcal{C}}$ adjacent to $\mathcal{C}$ such that the corresponding subset $\widetilde{V}_\ell \subset \{B = \ell\omega\}$, see (1.17), lies in a connected component of $Int(L_\ell)$. Indeed, take an arbitrary $a_0 > 0$ such that the line $\Lambda = \{a = a_0\}$ intersects $\mathcal{C}$ transversally in at least one point x. There exists a semi-interval $J(x) \subset \Lambda$ adjacent to x such that $J(x)$ lies both in $U_{\mathcal{C}}$ and in the two-dimensional phase-lock area $L_\ell(\omega(x))$, $\omega(x) = a_0^{-1}$ (positivity of constrictions, see Theorem 2.14, and transversality). We take the above interval J to be the maximal one. The domain V we are looking for is the union of the intervals $J(x)$ for all the points x of transversal intersection $\Lambda \cap \mathcal{C}$. The corresponding subset $\widetilde{V}_\ell \subset \{B = \ell\omega\}$ lies in a connected component of the three-dimensional phase-lock area $Int(L_\ell)$. Let us denote the latter component by $Comp(V)$. Let us show that it is adjacent to no curve $\mathcal{C}_k$.

Making the intersection $\mathcal{C} \cap \Lambda$ transversal (by choosing a generic a_0) and taking the above x to be the lower point of the latter intersection, we get that $J_\ell(x)$ is a vertical line interval adjacent to x from above. The code of every its point contains the curve $\mathcal{C}$, by construction. Thus, the code of each point in $\widetilde{V}_\ell$ contains $\mathcal{C}$. On the other hand, we have the following

Proposition 2.17 *On each component in $Int(L_\ell)$ adjacent to a constriction curve $\mathcal{C}_{\ell,k}$, the code can contain no queer constriction curve $\mathcal{C}$.*

Proof Fix a component W adjacent to $\mathcal{C}_{\ell,k}$ of the interior $Int(L_\ell)$. There exists a domain $S \subset \mathbb{R}^2_{a,s}$ adjacent to $\mathcal{C}_{\ell,k}$ such that the corresponding subset $\widetilde{S}_\ell \subset \{B = \omega\ell\} \subset \mathbb{R}^3_{B,A;\omega}$, see (1.17), lies in W. This is proved analogously to the above proof of a similar statement on the subset $V \subset U_{\mathcal{C}}$, see the beginning of the proof of Corollary 1.20. For every $x \in S$ the corresponding point $(\ell a^{-1}(x), \frac{s(x)}{a(x)}; a^{-1}(x)) \in \widetilde{S}_\ell \subset W$ will be identified with x and also denoted by x. Fix a queer constriction curve $\mathcal{C}$ (if any). We have to show that the code of every $x \in W$ does not contain $\mathcal{C}$. It suffices to prove this statement just for some $x \in W$. Taking $x \in S$ to be close enough to the endpoint $\beta_{\ell,k}$ of the curve $\mathcal{C}_{\ell,k}$, we can achieve that the segment in the vertical line $\Lambda = \{a = a(x)\}$ connecting x with its projection to the a-axis does not intersect $\mathcal{C}$. This follows from the assumption that $\mathcal{C}$ is queer (hence, accumulates to no point of the s-axis, by Theorem 1.17). Then the code of the point x does not contain $\mathcal{C}$, which follows from definition. Proposition 2.17 is proved. $\square$

The component in $Int(L_\ell)$ containing $\widetilde{V}_\ell$ is adjacent to no curve $\mathcal{C}_{\ell,k}$, since its code contains $\mathcal{C}$ (Proposition 2.17). This proves Corollary 1.20. $\square$

2.7 Background material. Linear systems with irregular singularities: Stokes phenomena and isomonodromic deformations

2.7.1 Irregular singularities and Stokes phenomena

The following material on irregular singularities of linear systems and Stokes phenomena is contained in [4, 5, 6, 32, 36, 53].

Consider a two-dimensional linear system

$$Y' = \left(\frac{K}{z^2} + \frac{R}{z} + O(1)\right) Y, \quad Y = (Y_1, Y_2) \in \mathbb{C}^2, \tag{2.24}$$

on a neighborhood of 0. Here K and R are complex 2×2-matrices, K has distinct eigenvalues $\lambda_1 \neq \lambda_2$, and $O(1)$ is a holomorphic matrix-valued function on a neighborhood of 0. Then we say that the singular point 0 of system (2.24) is *irregular non-resonant of Poincaré rank 1.* The matrix K is conjugate to $\widetilde{K} = \mathrm{diag}(\lambda_1, \lambda_2)$, $\widetilde{K} = \mathbf{H}^{-1} K \mathbf{H}$, $\mathbf{H} \in GL_2(\mathbb{C})$, and one can achieve that $K = \widetilde{K}$ by applying the constant linear change (gauge transformation) $Y = \mathbf{H}\widehat{\mathbf{Y}}$.

Recall that two systems of type (2.24) are *analytically equivalent* near the origin, if one can be transformed to the other by linear space coordinate change $Y = H(z)\widetilde{Y}$, where $H(z)$ is a holomorphic $GL_2(\mathbb{C})$-valued function on a neighborhood of the origin. Two systems (2.24) are *formally equivalent,* if the above $H(z)$ exists in the class of invertible formal power series with matrix coefficients.

System (2.24) is formally equivalent to a unique *formal normal form*

$$\widetilde{Y}' = \left(\frac{\widetilde{K}}{z^2} + \frac{\widetilde{R}}{z}\right) \widetilde{Y}, \ \widetilde{K} = \mathrm{diag}(\lambda_1, \lambda_2), \ \widetilde{R} = \mathrm{diag}(b_1, b_2), \tag{2.25}$$

$$\widetilde{R} \quad \text{is the diagonal part of the matrix} \quad \mathbf{H}^{-1} R \mathbf{H}. \tag{2.26}$$

The matrix coefficient K in system (2.24) and the corresponding matrix $\widetilde{K}$ in (2.25) are called the *main term matrices*, and R, $\widetilde{R}$ are *residue matrices.* However the normalizing series $H(z)$ generically diverges. At the same time, there exists a covering of a punctured neighborhood of zero by two sectors S_0 and S_1 with vertex at 0 in which there exist holomorphic $GL_2(\mathbb{C})$-valued matrix functions $H_j(z)$, $j = 0, 1$, that are C^∞ smooth on $\overline{S}_j \cap D_r$ for some $r > 0$, and such that the variable changes $Y = H_j(z)\widetilde{Y}$ transform (2.24) to (2.25). This Sectorial Normalization Theorem holds for the so-called *Stokes sectors.* Namely, consider the rays issued from 0 and forming the set

$$\{\mathrm{Re}\, \frac{\lambda_1 - \lambda_2}{z} = 0\}. \tag{2.27}$$

They are called *imaginary dividing rays* or *Stokes rays.* A sector S_j is called a *Stokes sector,* if it contains one imaginary dividing ray and its closure does not contain the other one.

Let $W(z) = \mathrm{diag}(e^{-\frac{\lambda_1}{z}} z^{b_1}, e^{-\frac{\lambda_2}{z}} z^{b_2})$ denote the canonical diagonal fundamental matrix solution of the formal normal form (2.25). The matrices $X^j(z) := H_j(z)W(z)$ are fundamental matrix solutions of the initial equation (2.24) defining solution bases in S_j called the *canonical sectorial solution bases.* Here we choose the branches $W(z) = W^j(z)$ of the matrix function $W(z)$ in S_j so that $W^1(z)$ is obtained from $W^0(z)$ by counterclockwise analytic extension from S_0 to S_1. And we define the branch $W^2(z)$ of $W(z)$ in $S_2 := S_0$ obtained from $W^1(z)$ by counterclockwise analytic extension from S_1 to S_0. This yields another canonical matrix solution $X^2(z) := H_0(z)W^2(z)$ of system (2.24) in S_0, which is obtained from $X^0(z)$ by multiplication from the right by the monodromy matrix $\exp(2\pi i\widetilde{R})$ of the formal normal form (2.25). Let $S_{j,j+1}$ denote the connected component of intersection $S_{j+1} \cap S_j$, $j = 0, 1$, that is crossed when one moves from S_j to S_{j+1} counterclockwise, see Fig. 5. The transition matrices C_0, C_1 between thus defined canonical solution bases X^j,

$$X^1(z) = X^0(z)C_0 \text{ on } S_{0,1}; \quad X^2(z) = X^1(z)C_1 \text{ on } S_{1,2}, \tag{2.28}$$

are called the *Stokes matrices.*

Remark 2.18 A priori, the canonical fundamental matrix solution $W(z)$ of the formal normal form does not necessary have a single-valued branch, since the exponents b_k may be non-integer. A diagonal fundamental matrix solution $W(z)$ is uniquely defined up to multiplication by diagonal matrix. This renormalization of the matrix $W(z)$ in the above construction transforms the initial Stokes matrix pair to a new Stokes matrix pair obtained from the initial one by conjugation by one and the same diagonal matrix. It is well-known that *two germs of linear systems of type (2.24) are analytically equivalent, if and only if they have the same formal normal form and their Stokes matrix pairs are conjugated by the same diagonal matrix (i.e., can be normalized to be the same).*

Example 2.19 Let $K = \mathrm{diag}(\lambda_1, \lambda_2)$, and let $\lambda_2 - \lambda_1 > 0$. Then the imaginary dividing rays are the positive and negative imaginary semiaxes. The Stokes sectors S_0 and S_1 covering $\mathbb{C}^*$ satisfy the following conditions:

- the sector S_0 contains the positive imaginary semiaxis, and its closure does not contain the negative one;

- the sector S_1 satisfies the opposite condition. See Fig. 5.

The Stokes matrices C_0 and C_1 are unipotent upper and lower triangular respectively.

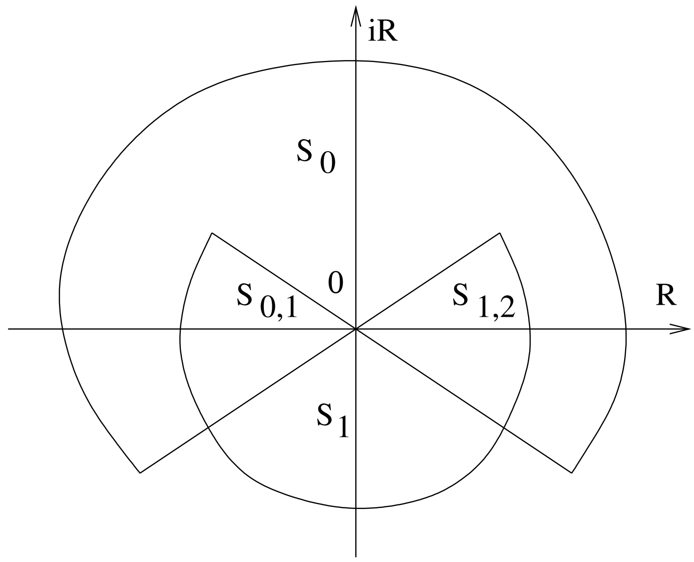

Figure 5: Stokes sectors in the case, when $\lambda_1 - \lambda_2 \in \mathbb{R}$.

Remark 2.20 Let M_f denote the *formal monodromy* of system (2.24): the monodromy matrix $\mathrm{diag}(e^{2\pi i b_1}, e^{2\pi i b_2})$ of its formal normal form (2.25) with respect to a diagonal fundamental matrix solution $W(z)$. Let M denote the monodromy matrix of system (2.24) written in the basis given by the fundamental matrix solution $X^0(z)$. It is well-known, see [32, p.35], that

$$M = M_f C_1^{-1} C_0^{-1}. \tag{2.29}$$

2.7.2 Isomonodromic deformations

We will deal with linear systems on $\overline{\mathbb{C}}$ of the type

$$Y' = \left(\frac{K}{z^2} + \frac{R}{z} + N\right) Y, \quad Y \in \mathbb{C}^2, \tag{2.30}$$

where K, R, N are complex 2×2-matrices such that each one of the matrices K, N has distinct eigenvalues. Let $\lambda_{\infty 1}$, $\lambda_{\infty 2}$ denote the eigenvalues of the matrix N. System (2.30) has two nonresonant irregular singular points of Poincaré rank 1: at the origin and at infinity. Namely, the variable change $\widetilde{z} = \frac{1}{z}$ transforms the infinity in z-coordinate to the origin in $\widetilde{z}$-coordinate, and system (2.30) is transformed to a linear system with irregular singular point at the origin in the $\widetilde{z}$-coordinate. The corresponding Stokes rays, i.e., Stokes rays "at infinity" of initial system (2.30), are given by the equation

$$\mathrm{Re}((\lambda_{\infty 2} - \lambda_{\infty 1})z) = 0 \tag{2.31}$$

For every $p = 0, \infty$ let S_{p0}, S_{p1} be Stokes sectors of system (2.30) at the singular point p. Fix a point $z_0 \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and paths α_p going from the point z_0 to some point in S_{p0}. Let $X^{p0}(z)$ be the canonical sectorial fundamental matrix solutions of system (2.30) in sectors S_{p0}. Consider their analytic extensions to z_0 along the paths α_p^{-1}. Thus obtained germs of fundamental matrix solutions $X^{p0}(z)$ at z_0 depend only on the homotopy classes of the paths α_p in the class of paths with fixed starting point z_0 and free endpoint lying in S_{p0}. Let Q denote the transition matrix between the fundamental matrix solutions X^{p0}, $p = 0, \infty$, near z_0:

$$X^{\infty 0}(z) = X^{00}(z)Q. \tag{2.32}$$

Let C_{p0}, C_{p1} denote the Stokes matrices at p defined by the sectorial matrix solutions $X^{p0}(z)$.

Remark 2.21 When we renormalize $X^{p0}(z)$ by multiplication from the right by diagonal matrix D_p (which means similar renormalization of the corresponding branch of diagonal fundamental matrix solution $W(z) = W^p(z)$ (defining X^{p0}) of formal normal form at p), this transforms the initial Stokes matrix collections and transition matrix Q to

$$\widetilde{C}_{pj} = D_p^{-1} C_{pj} D_p, \ \widetilde{Q} = D_0^{-1} Q D_\infty.$$

Definition 2.22 [22, definition 4.2]. A continuous family of linear systems of type (2.30) is *isomonodromic,* if one can choose a continuous family of canonical fundamental matrix solutions $X^{p0}(z)$ at z_0 defined by continuous families of Stokes sectors S_{p0} and paths α_p so that the monodromy matrix in the basis given by X^{00}, the Stokes matrices C_{pj} and the transition matrix Q remain the same.

Remark 2.23 Isomonodromicity is equivalent to the condition saying that the formal monodromies M_{fp} at singular points p, the Stokes matrices and the transition matrix remain the same: see (2.29). In a continuous family of linear systems (2.30) constance of formal monodromies is equivalent to constance of residue matrices of formal normal forms. It is well-known that if an isomonodromic family of linear systems depends analytically on a parameter lying in a simply connected domain in a complex manifold, then the corresponding fundamental matrix solutions from the above definition can be normalized to depend analytically on the parameter. See [11, theorem 13.1 and its proof] for the Fuchsian case; in the case of irregular singularities the proof is analogous.

2.8 Extension of solutions. Proof of Theorem 1.24

The first step of the proof of Theorem 1.24 is the following proposition.

Proposition 2.24 *Fix an $\ell \in \mathbb{C}$ and a path $\alpha : [0,1] \to \mathbb{C}^*$ from a point s^* to a point s_0; $s^*, s_0 \in \mathbb{C}^*$. Let a solution $(\chi(s), a(s))$ of system (1.10) extend holomorphically to every $\alpha(t)$, $t \in [0,1)$, along the path $\alpha : [0,t] \to \mathbb{C}^*$, but do not extend to $s_0 = \alpha(1)$ along the path α. Let $w(s) = \frac{a(s)}{2s\chi(s)}$ be the corresponding solution of Painlevé 3 equation (1.12). Then one of the two following statements holds:*

(i) either $w(s_0) = 0$ and $w'(s_0) = 1$; in this case $a(s)$ is holomorphic at s_0, $a(s_0) = \pm s_0$, and $\chi(s)$ has a simple pole at s_0 with residue $\pm\frac{1}{2}$;

(ii) or $w(s)$ has a simple pole at s_0 with residue -1; in this case $a(s)$ is meromorphic near s_0 and has a simple pole with residue $\pm s_0$ at s_0; $\chi(s)$ is holomorphic near s_0 and $\chi(s_0) = \mp\frac{1}{2}$.

Proof The variable change $u = \chi^{-1}$ transforms system (1.10) to

$$\begin{cases} u'_s = \frac{1}{2s}(-au^2 + 2\ell u + 4a) \\ a'_s = \frac{2}{us}(a^2 - s^2) + \frac{a\ell}{s} \end{cases} \tag{2.33}$$

Therefore, differentiating the formula

$$w(s) = \frac{a(s)}{2s\chi(s)} = \frac{a(s)u(s)}{2s} \tag{2.34}$$

yields

$$\begin{aligned} w' &= \frac{1}{2s}\Big(\frac{1}{2s}(-a^2u^2 + 2\ell au + 4a^2) + \frac{2}{s}(a^2 - s^2) + \frac{\ell au}{s}\Big) - \frac{au}{2s^2} \\ &= \frac{2a^2}{s^2} - 1 + \frac{(2\ell - 1)w}{s} - w^2. \end{aligned} \tag{2.35}$$

Proposition 2.25 *Let α, s_0 and $(\chi(s), a(s))$ be the same, as in the conditions of Proposition 2.24. Then both $\chi^2(s)$ and $a^2(s)$ are meromorphic on a neighborhood of the point s_0.*

Proof Meromorphicity of the function $a^2(s)$ near s_0 follows from equation (2.35) and meromorphicity of the function $w(s)$ on the universal cover over $\mathbb{C}^*_s$ (Painlevé property). This together with (2.34) implies meromorphicity of the function $\chi^2(s) = \frac{a^2(s)}{4s^2w^2(s)}$. □

At least one of the values $\chi(s_0)$, $a(s_0)$ should be infinite: otherwise, the solution $(\chi(s), a(s))$ would be holomorphic at s_0, by Existence and Uniqueness Theorem for ODEs.

Case 1): $w(s)$ is holomorphic at s_0. This together with (2.35) implies that $a^2(s)$ is holomorphic at s_0. Thus, $a(s_0)$ is finite, hence $\chi(s_0) = \infty$, see the above discussion. Therefore, $u(s)$ and $w(s) = \frac{a(s)u(s)}{2s}$ vanish at $s = s_0$. Every solution of Painlevé 3 equation (1.12) has derivative ± 1 at each its zero. This is well-known and follows directly from (1.12): the polar part of its right-hand side at a zero of $w(s)$ comes from $\frac{(w')^2}{w} - \frac{1}{w}$, and it should cancel out. Now equalities $w(s_0) = 0$, $w'(s_0) = \pm 1$ together with (2.35) imply that either $a(s_0) = \pm s_0$ if $w'(s_0) = 1$, or $a(s_0) = 0$ if $w'(s_0) = -1$. In the second case the functions $a^2(s)$, $u^2(s)$ are both holomorphic (Proposition 2.25) and both vanish at s_0. One has $a(s) = c\sqrt{s - s_0}(1+o(1))$, $u(s) = -2sc^{-1}\sqrt{s - s_0}(1 + o(1))$, as $s \to s_0$, due to the latter statement and the asymptotics $w(s) = \frac{a(s)u(s)}{2s} = -(s - s_0)(1 + o(1))$. But the above asymptotics of the function $u(s)$ contradicts the first equation in (2.33): the left-hand side should tend to infinity, as $s \to s_0$, while the right-hand side doesn't. Thus, the second case is impossible. Hence, $w(s_0) = 0$, $w'(s_0) = 1$, $a(s_0) = \pm s_0 \neq 0$, and $a^2(s)$ is holomorphic at s_0 (Proposition 2.25). Hence, $a(s)$ is holomorphic at s_0. This together with holomorphicity of $w(s)$ and equality $w(s) = \frac{a(s)u(s)}{2s} = (s - s_0)(1 + o(1))$ implies holomorphicity of the function $u(s)$, the equalities $u(s_0) = 0$, $u'(s_0) = \pm 2$ and Statement (i) of Proposition 2.24.

Case 2): $w(s)$ has a pole at s_0. Then this pole is simple with residue ± 1, see [28, p.158]. In the case, when the sign is $+$, one has

$$w(s) = \frac{1}{s - s_0} + \frac{2\ell - 1}{2s_0} + O(s - s_0), \tag{2.36}$$

see [28, p.158]. This together with equation (2.35) implies that the Laurent series polar parts at s_0 of the w-terms in its left- and right-hand sides are equal, and hence, $a^2(s)$ is holomorphic at s_0. This together with (2.34) implies that $u(s) \to \infty$, hence, $\chi(s) \to 0$, as $s \to s_0$. Finally, the point $(\chi(s_0), a(s_0))$ is finite, – a contradiction. Thus, the residue of the function $w(s)$ at s_0 is equal to -1. This together with (2.35) implies that the function $a^2(s)$ is meromorphic near s_0 with order two pole at s_0 and asymptotics

$$a^2(s) = \frac{s_0^2}{(s - s_0)^2}(1 + o(1)), \text{ as } s \to s_0.$$

Hence, $a(s)$ is meromorphic near s_0 and has simple pole with residue $\pm s_0$ at s_0. Together with (2.34), this implies that $\chi(s)$ is holomorphic near s_0 and $\chi(s_0) = \mp\frac{1}{2}$. Statement (ii) of Proposition 2.24 is proved. □

Proof of Theorem 1.24. Statements 1) and 2) of Theorem 1.24 follow from Proposition 2.24. Let us prove its Statement 3). A solution $(\chi(s), a(s))$ of (1.10) yields an isomonodromic family of linear systems (1.15), see [8, theorem 6.6]. Therefore, there exist analytic families $X^{00}(z,s)$, $X^{\infty 0}(z,s)$ of sectorial matrix solutions in Stokes sectors S_{00}, $S_{\infty 0}$ of systems (1.15) with $\chi = \chi(s)$, $a = a(s)$ ($S_{p0} = S_{p0}(s)$ depend on s for $p = 0, \infty$) such that

- the Stokes matrices at the origin and at infinity defined by $X^{p0}(z,s)$, $p = 0, \infty$, remain constant (independent on s);

- the transition matrix Q, $X^{\infty 0}(z,s) = X^{00}(z,s)Q$, see (2.32), between analytic extensions to $z_0 = 1$ of the fundamental matrix solutions X^{p0} along paths $\alpha_p^{-1} = \alpha_{ps}^{-1} \subset \mathbb{C}^*$ (depending continuously on s) is also constant.

As s is changed, the Stokes sectors turn and the paths α_{ps} defining the above analytic extension are deformed continuously (homotopied with fixed starting point z_0 and free endpoints lying in variable sectors S_{00}, $S_{\infty 0}$). As s makes complete tour around the origin, both sectors S_{00}, $S_{\infty 0}$ make complete tours, but in opposite directions: see (2.27), (2.31). The system corresponding to the analytic extension of the solution $(\chi(s), a(s))$ along the circuit in question has the same formal normal forms and Stokes matrices at zero and at infinity (for appropriately normalized canonical sectorial solutions in S_{p0}). But the homotopy classes of the paths defining the (unchanged) transition matrix are changed.

Proposition 2.26 *Let a system (1.15) have trivial monodromy. Then the corresponding transition matrix Q given by (2.32) is path-independent: it depends only on the normalization of canonical sectorial fundamental matrix solutions in S_{p0} by multiplication by diagonal matrices.*

Proof Each solution of system (1.15) in question is holomorphic on $\mathbb{C}^*$, by triviality of monodromy. This implies that the result of its analytic continuation along a path α from some point to z_0 is independent on its homotopy class in the class of paths in $\mathbb{C}^*$ with fixed endpoints. Therefore analytic extension of each matrix solution X^{p0} to z_0, and thus, the transition matrix Q, are path-independent. □

Remark 2.27 As was shown in [8, proposition 4.6], triviality of monodromy of a system (2.30) (in particular, of a system (1.15)) implies triviality of its Stokes matrices and formal monodromies at both singular points 0 and ∞.

Let the initial condition (χ^*, a^*, s^*) correspond to a system (1.15) with trivial monodromy. Consider the corresponding solution $(\chi(s), a(s))$ of system (1.10), which yields an isomonodromic deformation of linear system in question. As s makes a circuit around the origin, the initial condition (χ^*, a^*, s^*) is transformed to another triple $(\widetilde{\chi}, \widetilde{a}, s^*)$ corresponding to a system (1.15) with trivial monodromy, the same Stokes sectors S_{pj}, $p = 0, \infty$, $j = 0, 1$ (they are completely defined by s^*, which remains the same) and the same transition matrix Q, which is path-independent (Proposition 2.26).

Proposition 2.28 *Systems (1.15) corresponding to the above* (χ^*, a^*, s^*) *and* $(\widetilde{\chi}, \widetilde{a}, s^*)$ *either coincide, or differ by sign of the two first cordinates:* $\widetilde{\chi} = -\chi^*$, $\widetilde{a} = -a^*$. *Or equivalently, the corresponding systems (1.15) either are the same, or are transformed one to the other by the constant diagonal gauge transformation* $Y \mapsto \mathrm{diag}(1, -1)Y$.

Proof The paths defining the transition matrix Q for the new system can be chosen the same, as for the initial system (Proposition 2.26). Therefore, taking appropriate normalization of the sectorial matrix solutions X^{00}, $X^{\infty 0}$ for each system one can achieve equality of the transition matrices. The Stokes matrices are trivial (Remark 2.27). The formal normal forms of both systems at each singular point are the same. Indeed, they are completely defined by s and the residue matrix eigenvalue ℓ, which are the same for both systems, since ℓ is independent on s (see also Remark 2.23). Therefore, the formal normal forms, the Stokes and transition matrices of both systems are the same for appropriate normalizations of the sectorial matrix solutions $X^{p0}(z)$, $p = 0, \infty$. This together with [34, proposition 2.5, p.319] implies that the systems are gauge equivalent: this means that there exists a matrix $H \in GL_2(\mathbb{C})$ such that the variable change $Y \mapsto HY$ transforms one linear system to the other. Conjugation by the matrix H should preserve the opposite triangular forms of the matrices K and N. Therefore, H is diagonal, since each one of the latter matrices has distinct eigenvalues. Moreover, conjugation by H should preserve equality of opposite off-diagonal terms χ of the matrices K and N. This implies that the ratio of its eigenvalues is a square root of unity, and $H = \mathrm{diag}(1, \pm 1)$ up to scalar factor. Then one has $(\widetilde{\chi}, \widetilde{a}) = (\pm\chi^*, \pm a^*)$. This proves Proposition 2.28. □

Proposition 2.28 implies (together with Statement 1) of Theorem 1.24) that the solution $(\chi(s), a(s))$ of (1.10) with initial conditions corresponding to a system (1.15) with trivial monodromy either is meromorphic on $\mathbb{C}^*$, or changes sign after analytic extension along a simple circuit around the origin. Therefore, the latter analytic extension does not change the ratio $w = \frac{a}{2s\chi}$. Thus, $w(s)$ is meromorphic on $\mathbb{C}^*$. Statement 3) of Theorem 1.24 is proved.

Statement 3) holds for every initial condition $(0, a^*, s^*)$ such that $(a^*, s^*) \in Constr_\ell$, since it corresponds to a linear system with trivial monodromy: for every constriction $(B, A; \omega)$ the corresponding linear system (1.15) with $\chi = 0$, $\ell = \frac{B}{\omega}$, $s = \frac{A}{\omega}$, $a = \omega^{-1}$ has trivial monodromy, see [8, proposition 4.1]. This proves Statement 4).

Let us now prove Statement 5). Let $(0, a^*, s^*)$ be an initial condition with $(a^*, s^*) \in \mathcal{C}_{\ell,k}$ for some $k \in \mathbb{N}$. The corresponding solution $(\chi(s), a(s))$ is either meromorphic single-valued on $\mathbb{C}^*$, or meromorphic double-valued with sign-reversing monodromy. This means that it changes sign after analytic extension along a simple counterclockwise circuit in $\mathbb{C}_s^*$ around the origin. This follows from Statement 4). It remains to show that the solution is single-valued. Indeed, suppose the contrary: it has sign-reversing monodromy for certain $(a^*, s^*) \in \mathcal{C}_{\ell,k}$. Then it has sign-reversing monodromy for every $(a^*, s^*) \in \mathcal{C}_{\ell,k}$, by continuity and connectivity of the curve $\mathcal{C}_{\ell,k}$. In particular, this holds for $(a^*, s^*) \in \mathcal{C}_{\ell,k}$ arbitrarily close to $\beta_k = (0, s_{\ell,k})$. Passing to the rescaled coordinates $(\widetilde{y}_1, \widetilde{y}_2)$, $(\chi, a) := (a^*\widetilde{y}_1, a^*\widetilde{y}_2)$, and taking the limit of thus rescaled system (1.10) and the solution in question with $\chi(s^*) = 0$, $a(s^*) = a^*$, as $(a^*, s^*) \to (0, s_{\ell,k})$, we get linear system (2.15) and its solution $y = (y_1, y_2)(s)$ with $y(s_{\ell,k}) = (0, 1)$. The limit solution $y(s)$ should also have sign-reversing monodromy, by construction. Hence, -1 is an eigenvalue of the monodromy operator of system (2.15). But its monodromy operator is unipotent: its eigenvalues are equal to exponent of ($2\pi i$ times the eigenvalues of its residue matrix at the origin); the residue eigenvalues are $\pm\ell \in \mathbb{Z}$. The contradiction thus obtained finishes the proof of Theorem 1.24. $\Box$

3 Acknowledgements

I am grateful to V.M.Buchstaber and Yu.S.Ilyashenko for attracting my attention to problems on model of Josephson effect and helpful discussions. I am grateful to Yu.P.Bibilo, R.R.Gontsov, V.I.Gromak, V.Yu.Novokshenov, V.V.Sokolov, I.V.Vyugin for helpful discussions and to Yu.P.Bibilo and I.V.Vyugin for reading the paper and helpful remarks.

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