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# STABILITY RESULTS FOR NOVEL SERIALY-CONNECTED MAGNETIZABLE PIEZOELECTRIC AND ELASTIC SMART-SYSTEM DESIGNS

MOHAMMAD AKIL<sup>1</sup>, SERGE NICAISE<sup>1</sup>, AHMET ÖZKAN ÖZER<sup>2</sup> AND VIRGINIE RÉGNIER<sup>1</sup>

<sup>1</sup>UNIV. POLYTECHNIQUE HAUTS-DE-FRANCE, INSA HAUTS-DE-FRANCE, CERAMATHS-LABORATOIRE DE MATÉRIAUX CÉRAMIQUES ET DE MATHÉMATIQUES, F-59313 VALENCIENNES, FRANCE.

<sup>2</sup>DEPARTMENT OF MATHEMATICS, WESTERN KENTUCKY UNIVERSITY, BOWLING GREEN, KY 42101, USA.

EMAIL: MOHAMMAD.AKIL@UPHF.FR, SERGE.NICAISE@UPHF.FR, OZKAN.OZER@WKU.EDU, VIRGINIE.REGNIER@UPHF.FR

**ABSTRACT.** In this paper, the stability of longitudinal vibrations for transmission problems of two smart-system designs are studied: (i) a serially-connected Elastic-Piezoelectric-Elastic design with a local damping acting only on the piezoelectric layer and (ii) a serially-connected Piezoelectric-Elastic design with a local damping acting on the elastic part only. Unlike the existing literature, piezoelectric layers are considered magnetizable, and therefore, a fully-dynamic PDE model, retaining interactions of electromagnetic fields (due to Maxwell's equations) with the mechanical vibrations, is considered. The design (i) is shown to have exponentially stable solutions. However, the nature of the stability of solutions of the design (ii), whether it is polynomial or exponential, is dependent entirely upon the arithmetic nature of a quotient involving all physical parameters. Furthermore, a polynomial decay rate is provided in terms of a measure of irrationality of the quotient. Note that this type of result is totally new (see Theorem 3.6 and Condition ( $\mathbf{H}_{\mathbf{P}_{\text{ol}}}$ )). The main tool used throughout the paper is the multipliers technique which requires an adaptive selection of cut-off functions together with a particular attention to the sharpness of the estimates to optimize the results.

**Keywords.** magnetizable piezoelectric beams; serially-connected beams; transmission problems; irrationality measure; partial viscous damping; exponential stability; polynomial stability.

## 1. INTRODUCTION

Piezoelectric materials are multi-functional smart materials (most notably Lead Zirconate Titanate) used to develop electric displacement that is directly proportional to an applied mechanical stress [49]. They can be used as actuators/sensors, and also be integrated to a mother host structure [48]. Due to their small size and high power density, they have become more and more promising in industrial applications such as implantable biomedical devices and sensors [14, 15], wearable human-machine interface sensors [16], and nano-positioners and micro-sensors due to the excellent advantages of the fast response time, large mechanical force, and extremely fine resolution [45].

In deriving a mathematical model for the equations of motion on a piezoelectric beam, actuated by a voltage source, three major effects and their interrelations need to be considered: mechanical, electrical, and magnetic. Mechanical effects are mostly modeled through Kirchhoff, Euler-Bernoulli, or Mindlin-Timoshenko small (linear) [28] or large (nonlinear) [18] displacement assumptions, where the constitutive relations between the nonzero stress and strain tensors are used to model longitudinal displacements of the centerline (stretching/compression), transverse displacements (bending), and rotations of the beam. It is also reported that the small displacement assumptions lead to the bending and rotational motions completely immune the applied voltage [36]. These tensors are coupled to the electric/magnetic displacements and electric/magnetic field tensors. There are mainly three approaches to include electromagnetic effects due the Maxwell's equations: electrostatic, quasi-static, and fully-dynamic [25, p. 336]. Electrostatic and quasi-static approaches are widely used in voltage-controlled piezoelectric beam models - see e.g. [49] and the references therein. These models completely exclude magnetic effects and their coupling with electrical and mechanical effects. Even though the electro-static and quasi-static approaches are sufficient for defining piezoelectricity, electromagnetic waves generated by mechanical fields need to be accounted for in the calculation of radiated electromagnetic power

from a vibrating piezoelectric acoustic device, e.g. see [53] and the references therein. For this reason, the fully dynamic models of piezoelectric beams are needed to be understood well. In fact, the dynamic effects for (acoustic) magnetizable piezoelectric beams are pronounced and must be taken into account in the modeling [36, 50].

Denote  $v(x, t)$  and  $p(x, t)$  by the longitudinal vibrations of the center line of the beam and the total charge accumulated at the electrodes of a single piezoelectric beam. Assuming that the beam is fixed at the left end  $x = 0$  and free at the right end  $x = L$ , the equations of motion is a system of partial differential equations [36] as the following

$$(1.1) \quad \begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma \beta p_{xx} = 0, & (x, t) \in (0, L) \times (0, \infty) \\ \mu p_{tt} - \beta p_{xx} + \gamma \beta v_{xx} = 0, \\ v(0, t) = p(0, t) = 0, \\ \alpha v_x(L, t) - \gamma \beta p_x(L, t) = g(t), \\ \beta p_x(L, t) - \gamma \beta v_x(L, t) = -V(t), & t \in (0, \infty) \\ (v, p, v_t, p_t)(x, 0) = (v_0, p_0, v_1, p_1), & x \in [0, L] \end{cases}$$

where  $\rho$ ,  $\alpha$ ,  $\beta$ ,  $\gamma$ , and  $\mu$  are mass density per unit volume, elastic stiffness, impermeability, piezoelectric constant, and magnetic permeability of the beam, respectively, and  $g(t)$  and  $V(t)$  are strain and voltage actuators, and

$$(1.2) \quad \alpha_1 := \alpha - \gamma^2 \beta > 0.$$

By the electrostatic approach, the model above is simplified to a single wave equation model by taking  $\mu \equiv 0$  and  $p_t = 0$ , considering  $g(t) \equiv 0$ , and (1.2), e.g. see [37],

$$(1.3) \quad \begin{cases} \rho v_{tt} - \alpha_1 v_{xx} = 0, & (x, t) \in (0, L) \times (0, \infty) \\ v(0, t) = 0, \quad \alpha_1 v_x(L, t) = \gamma V(t) & t \in (0, \infty) \\ (v, v_t)(x, 0) = (v_0, v_1), & x \in [0, L]. \end{cases}$$

The model by the quasi-static approach is the same as (1.3) yet  $p_t \neq 0$ .

The exact observability/stabilizability and the type of stability of the solutions (1.1) of the PDE model by each approach differs substantially. For example, the PDE model obtained by electrostatic/quasi-static approach is the boundary damped wave equation in (1.3), and it is known to be exactly observable/exponentially stabilizable with one state measurement  $v_t(x, t)$  on the boundary  $x = L$ , e.g. see [12, 24]. In deep contrast to this result, the PDE model obtained by the fully-dynamic approach in (1.1) can not be exactly observable/exponentially stabilizable for almost all choices of material parameters with only one state measurement,  $v_t(x, t)$  or  $p_t(x, t)$ , on the boundary  $x = L$ , e.g. see [36, 37]. Explicit polynomial decay estimates are obtained for more regular initial data and for a small class of materials satisfying certain number-theoretical conditions [37, 38]. The same model (1.1) is considered in [41] for the open-loop sensor configuration (i.e.  $g(t), V(t) \equiv 0$ ) with a dissipative damping term  $\delta v_t$  with  $\delta > 0$ , acting only in the first equation of (1.1). It is also reported that two nonzero state feedback measurements  $v_t(L, t)$  (tip velocity) and  $p_t(L, t)$  (total current on the electrodes) are necessary to achieve exact observability/exponential stabilizability [42, 52]. This underlines the fact that the two boundary damping terms or one viscous damping term are both able to exponentially dissipate non-stabilizing (high-frequency) magnetic effects. There is also a large literature considering the model (1.1) under thermal effects, fractional-type damping, and distributed or boundary-type memory and delay terms, see [2]-[5],[17]-[22], [47, 54] and the references therein.

A serially-connected smart system is an elastic structure consisting of longitudinally attached fully-elastic and piezoelectric layers, see Figs. 1 and 2. Use of piezoelectric materials for a serially-connected design in various transmission mechanisms of aerospace, aviation, automobile, ships, and robots has boosted substantially in the last decade, see [13, 30] and the references therein. A rigorous mathematical treatment for a transmission problem of a three serially-connected purely-elastic waves/strings/beams is provided in [20]. Indeed, if the outer wave equations have both viscous damping terms, an exponential stability result is shown to be immediate. Several authors have also studied transmission problems of serially-connected strings/beams with e.g. a thermoelastic material [35] or a viscoelastic material [43].

To the best of our best knowledge, serially-connected transmission systems involving elastic and magnetizable piezoelectric systems are not treated mathematically in the literature, especially with Condition ( $\mathbf{H}_{\mathbf{P}01}$ ), which appears in Section 3.3. The goal of this paper is to fix this gap by considering two particular designs, for which we obtain novel decay rates of the energy, see Theorems 2.6, 3.5 and 3.6. The first design, whose PDE model is described below in ( $\mathbf{E}/\mathbf{P}/\mathbf{E}$ ), is the transmission problem of an Elastic-Piezoelectric-Elastic system, as in Fig.

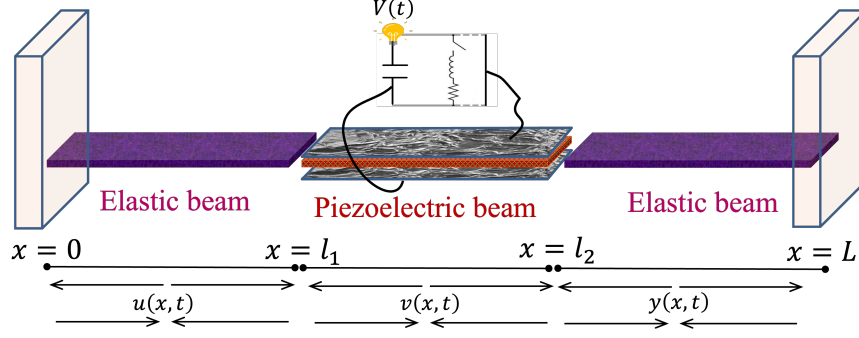


FIGURE 1. Serially-connected Elastic-Piezoelectric-Elastic transmission system clamped at both ends. The piezoelectric material itself is an elastic material covered by electrodes at their top and bottom surfaces, and connected to an external electric circuit. As the elastic layers stretches or shrinks, the piezoelectric beam stretches or shrinks as well, and therefore, charges separate and line up in the vertical direction, and electric field (voltage) is induced in the electrodes. The overall motions on the system are considered to be only longitudinal.

1, with only one local damping acting on the longitudinal displacement of the center line of the piezoelectric material:

$$\begin{cases}
 u_{tt} - c_1 u_{xx} = 0, & (x, t) \in (0, l_1) \times (0, \infty), \\
 \rho v_{tt} - \alpha v_{xx} + \gamma \beta p_{xx} + d_2(x) v_t = 0, & (x, t) \in (l_1, l_2) \times (0, \infty), \\
 \mu p_{tt} - \beta p_{xx} + \gamma \beta v_{xx} = 0, & (x, t) \in (l_1, l_2) \times (0, \infty), \\
 y_{tt} - c_2 y_{xx} = 0, & (x, t) \in (l_2, L) \times (0, \infty), \\
 u(0, t) = y(L, t) = 0, \\
 v(l_1, t) = u(l_1, t), \\
 v(l_2, t) = y(l_2, t), \\
 \alpha v_x(l_1, t) - \gamma \beta p_x(l_1, t) = c_1 u_x(l_1, t), \\
 \alpha v_x(l_2, t) - \gamma \beta p_x(l_2, t) = c_2 y_x(l_2, t), \\
 \beta p_x(l_1, t) = \gamma \beta v_x(l_1, t), \\
 \beta p_x(l_2, t) = \gamma \beta v_x(l_2, t), \\
 (u, v, p, y, u_t, v_t, p_t, y_t)(\cdot, 0) = (u_0, v_0, p_0, y_0, u_1, v_1, p_1, y_1)(\cdot)
 \end{cases} \quad t \in (0, \infty),$$

(E/P/E)

where  $0 < l_1 < l_2 < L$ ,  $c_1, c_2 > 0$  and  $d_2 \in L^\infty(l_1, l_2)$ , such that

$$\text{(LD - P)} \quad d_2(x) \geq d_{2,0} > 0 \quad \text{in } (a_2, b_2) \subset (l_1, l_2), \text{ and } d_2(x) \geq 0 \quad \text{in } (l_1, l_2) \setminus (a_2, b_2).$$

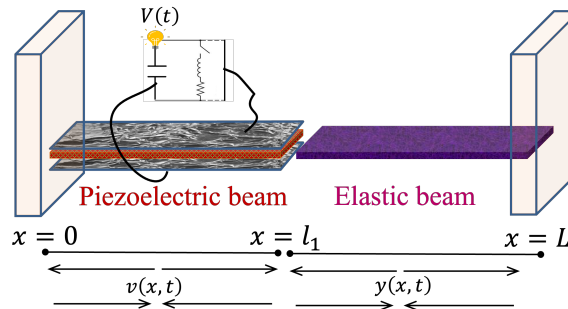


FIGURE 2. Serially connected Elastic-Piezoelectric transmission line clamped at both ends.

The second design, whose PDE model is described below in (P/E), is for the transmission problem of a Piezoelectric-Elastic system, as in Fig. 2, with only one local damping acting on the elastic part:

$$(P/E) \quad \begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma \beta p_{xx} = 0, & (x, t) \in (0, l_1) \times (0, \infty), \\ \mu p_{tt} - \beta p_{xx} + \gamma \beta v_{xx} = 0, & (x, t) \in (0, l_1) \times (0, \infty), \\ y_{tt} - c_2 y_{xx} + d_1(x) y_t = 0, & (x, t) \in (l_1, L) \times (0, \infty), \\ v(0, t) = p(0, t) = y(L) = 0, \\ v(l_1, t) = y(l_1, t), \\ \alpha v_x(l_1, t) - \gamma \beta p_x(l_1, t) = c_2 y_x(l_1, t), \\ \beta p_x(l_1, t) = \gamma \beta v_x(l_1, t), \\ (v, p, y, v_t, p_t, y_t)(\cdot, 0) = (v_0, p_0, y_0, v_1, p_1, y_1)(\cdot), \end{cases} \quad t \in (0, \infty),$$

where  $0 < l_1 < L$ ,  $c_2 > 0$  and  $d_1 \in L^\infty(l_1, L)$  such that

$$(LD - E) \quad d_1(x) \geq d_{1,0} > 0 \quad \text{in } (a_1, b_1) \subset (l_1, L), \text{ and } d_1(x) \geq 0 \quad \text{in } (l_1, L) \setminus (a_1, b_1).$$

The paper is organized as follows. In Section 2, first, well-posedness and the exponential stability of the model (E/P/E) are studied under the conditions (LD - P) on the damping function  $d_2$ . In Section 3, the well-posedness and the strong stability of (P/E) system are analyzed under the conditions (LD - E) on the damping function  $d_1$ . Moreover, the decay rate of the energy depends on the arithmetic nature of a quotient involving all physical parameters of the system. More precisely, if the quotient is a rational number different from  $\frac{2n_+ - 1}{2n_- - 1}$ , for all  $n_+, n_- \in \mathbb{N}$  (see (SC) in Theorem 3.3), the energy is proved to decay exponentially. If the quotient is an irrational number, the energy is proved to decrease polynomially if the irrationality measure of this quotient is finite. The proof of the main results are all based on the multipliers technique, requiring an adapted choice of cut-off functions, and a particular attention to the sharpness of the estimates to optimize the results.

It is crucial to note that, as a consequence of our arguments developed in section 2, the electrostatic/quasi-static design, identical to the design in [20], with a local damping only acting in the middle layer can be shown to be exponentially stable, see Remark 2.15. This is a major improvement of the results in [20] since the exponential stability result is only achieved by fully-distributed viscous damping terms for the outer layers.

## 2. STABILITY RESULTS FOR THE (E/P/E) SYSTEM

Note that the assumption (LD - P) applies to all results in this section. For simplicity, the repetition is avoided unless it is necessary to state.

**2.1. Well-Posedness.** This section is devoted to establish the well-posedness of the system (E/P/E) by a semigroup approach. The natural energy of system (E/P/E) is defined by

$$E(t) = \frac{1}{2} \int_0^{l_1} (|u_t|^2 + c_1 |u_x|^2) dx + \frac{1}{2} \int_{l_1}^{l_2} (\rho |v_t|^2 + \alpha_1 |v_x|^2 + \mu |p_t|^2 + \beta |\gamma v_x - p_x|^2) dx + \frac{1}{2} \int_{l_2}^L (|y_t|^2 + c_2 |y_x|^2) dx.$$

**Lemma 2.1.** *The energy  $E(t)$  is dissipative along the regular solutions  $(u, v, p, y)$  of the system (E/P/E), i.e.*

$$(2.1) \quad \frac{d}{dt} E(t) = - \int_{l_1}^{l_2} d_2 |v_t|^2 dx.$$

**Proof.** First, multiplying (E/P/E)<sub>1</sub> by  $\bar{u}_t$ , integrate by parts over  $(0, l_1)$ , and take the real part to get

$$(2.2) \quad \frac{1}{2} \frac{d}{dt} \int_0^{l_1} |u_t|^2 dx + \frac{c_1}{2} \frac{d}{dt} \int_0^{l_1} |u_x|^2 dx - c_1 \Re \left( u_x(l_1, t) \overline{u_t(l_1, t)} \right) = 0.$$

Next, multiply (E/P/E)<sub>2</sub> by  $\bar{v}_t$ , integrate by parts over  $(l_1, l_2)$ , and take the real part to get

$$(2.3) \quad \begin{aligned} & \frac{\rho}{2} \frac{d}{dt} \int_{l_1}^{l_2} |v_t|^2 dx + \frac{\alpha}{2} \frac{d}{dt} \int_{l_1}^{l_2} |v_x|^2 dx - \alpha \Re \left( v_x(l_2, t) \overline{v_t(l_2, t)} \right) + \alpha \Re \left( v_x(l_1, t) \overline{v_t(l_1, t)} \right) \\ & - \gamma \beta \Re \left( \int_{l_1}^{l_2} p_x \overline{v_{xt}} dx \right) + \gamma \beta \Re \left( p_x(l_2, t) \overline{v_t(l_2, t)} \right) - \gamma \beta \Re \left( p_x(l_1, t) \overline{v_t(l_1, t)} \right) = - \int_{l_1}^{l_2} d_2 |v_t|^2 dx. \end{aligned}$$

Now, multiply  $(\mathbf{E/P/E})_3$  by  $\bar{p}_t$ , integrate by parts over  $(l_1, l_2)$ , and take the real part to get

$$\begin{aligned} & \frac{\mu}{2} \frac{d}{dt} \int_{l_1}^{l_2} |p_t|^2 dx + \frac{\beta}{2} \frac{d}{dt} \int_{l_1}^{l_2} |p_x|^2 dx - \beta \Re \left( p_x(l_2, t) \overline{p_t(l_2, t)} \right) + \beta \Re \left( p_x(l_1, t) \overline{p_t(l_1, t)} \right) \\ & - \gamma \beta \Re \left( \int_{l_1}^{l_2} v_x \overline{p_{xt}} dx \right) + \gamma \beta \Re \left( v_x(l_2, t) \overline{p_t(l_2, t)} \right) - \gamma \beta \Re \left( v_x(l_1, t) \overline{p_t(l_1, t)} \right) = 0. \end{aligned}$$

By implementing  $(\mathbf{E/P/E})_{10}$  and  $(\mathbf{E/P/E})_{11}$ ,

$$(2.4) \quad \frac{\mu}{2} \frac{d}{dt} \int_{l_1}^{l_2} |p_t|^2 dx + \frac{\beta}{2} \frac{d}{dt} \int_{l_1}^{l_2} |p_x|^2 dx - \gamma \beta \Re \left( \int_{l_1}^{l_2} v_x \overline{p_{xt}} dx \right) = 0,$$

and multiplying  $(\mathbf{E/P/E})_4$  by  $\bar{y}$  and integrating by parts over  $(l_2, L)$  lead to

$$(2.5) \quad \frac{1}{2} \frac{d}{dt} \int_{l_2}^L |y_t|^2 dx + \frac{c_2}{2} \frac{d}{dt} \int_{l_2}^L |y_x|^2 dx + c_2 \Re \left( y_x(l_2, t) \overline{y_t(l_2, t)} \right) = 0.$$

Thus, by adding (2.3) and (2.4) and noting (1.2),

$$(2.6) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \left( \int_{l_1}^{l_2} (\rho |v_t|^2 + \alpha_1 |v_x|^2 + \mu |p_t|^2 + \beta |\gamma v_x - p_x|^2) dx \right) + \Re \left( (\alpha v_x(l_1, t) - \gamma \beta p_x(l_1, t)) \overline{v_t(l_1, t)} \right) \\ & - \Re \left( (\alpha v_x(l_2, t) - \gamma \beta p_x(l_2, t)) \overline{v_t(l_2, t)} \right) = - \int_{l_1}^{l_2} d_2 |v_t|^2 dx. \end{aligned}$$

In the final step of the proof, add (2.2), (2.5) and (2.6), use the continuity conditions  $(\mathbf{E/P/E})_6$  and  $(\mathbf{E/P/E})_7$  and the transmission conditions  $(\mathbf{E/P/E})_8$  and  $(\mathbf{E/P/E})_9$ . Hence, (2.1) follows.  $\square$

In order to have a unique solution to  $(\mathbf{E/P/E})$ , the following Hilbert spaces are introduced. For any real numbers  $a, b$  such that  $a < b$ ,

$$\begin{aligned} L_*^2(a, b) &= \left\{ f \in L^2(a, b); \int_a^b f dx = 0 \right\}, & H_L^1(a, b) &= \{ f \in H^1(a, b); f(a) = 0 \}, \\ H_*^1(a, b) &= H^1(a, b) \cap L_*^2(a, b), & H_R^1(a, b) &= \{ f \in H^1(a, b); f(b) = 0 \}. \end{aligned}$$

The energy space  $\mathcal{H}$  is now defined by

$$\mathcal{H} = \left\{ (u, u^1, v, z, p, q, y, y^1) \in H_L^1(0, l_1) \times L^2(0, l_1) \times H^1(l_1, l_2) \times L^2(l_1, l_2) \times H_*^1(0, l_1) \times L_*^2(l_1, l_2) \times H_R^1(l_2, L) \times L^2(l_2, L) : u(l_1) = v(l_1), y(l_2) = v(l_2) \right\},$$

and for  $U = (u, u^1, v, z, p, q, y, y^1) \in \mathcal{H}$ , a norm on  $\mathcal{H}$  can be chosen of the following form

$$(2.7) \quad \begin{aligned} \|U\|_{\mathcal{H}}^2 &= \int_0^{l_1} (c_1 |u_x|^2 + |u^1|^2) dx + \int_{l_1}^{l_2} (\alpha_1 |v_x|^2 + \rho |z|^2 + \beta |\gamma v_x - p_x|^2 + \mu |q|^2) dx \\ &+ \int_{l_2}^L (c_2 |y_x|^2 + |y^1|^2) dx. \end{aligned}$$

noting that the standard norm on  $\mathcal{H}$  is

$$(2.8) \quad \begin{aligned} \|U\|_s^2 &= \|u_x\|_{L^2(0, l_1)}^2 + \|u^1\|_{L^2(0, l_1)}^2 + \|v_x\|_{L^2(l_1, l_2)}^2 + \|v\|_{L^2(l_1, l_2)}^2 + \|z\|_{L^2(l_1, l_2)}^2 + \|p_x\|_{L^2(l_1, l_2)}^2 \\ &+ \|q\|_{L^2(l_1, l_2)}^2 + \|y_x\|_{L^2(l_2, L)}^2 + \|y^1\|_{L^2(l_2, L)}^2. \end{aligned}$$

**Lemma 2.2.** *The norm defined by (2.7) is equivalent to the standard norm (2.8) on  $\mathcal{H}$ , i.e. for all  $U = (u, u^1, v, z, p, q, y, y^1) \in \mathcal{H}$ , there exist two positive constants  $C_1, C_2$ , independent of  $U$ , such that*

$$(2.9) \quad C_1 \|U\|_s^2 \leq \|U\|_{\mathcal{H}}^2 \leq C_2 \|U\|_s^2.$$

**Proof.** The inequality on the right with  $C_2 = \max(c_1, 1, \alpha_1 + 2\beta \max(\gamma^2, 1), \mu, \rho, c_2)$  is immediate by Young's inequality since

$$\beta \|\gamma v_x - p_x\|_{L^2(l_1, l_2)}^2 \leq 2\beta \gamma^2 \|v_x\|_{L^2(l_1, l_2)}^2 + 2\beta \|p_x\|_{L^2(l_1, l_2)}^2 \leq 2\beta \max(\gamma^2, 1) (\|v_x\|_{L^2(l_1, l_2)}^2 + \|p_x\|_{L^2(l_1, l_2)}^2).$$

We have  $u(l_1) = \int_0^{l_1} u_x dx$ , and by the transmission condition  $u(l_1) = v(l_1)$ ,

$$v(x) = u(l_1) + \int_{l_1}^x v_t(t) dt.$$

Applying Young's and Cauchy-Schwarz inequalities leads to

$$(2.10) \quad |u(l_1)|^2 \leq l_1 \|u_x\|_{L^2(0,l_1)}^2,$$

$$(2.11) \quad |v(x)|^2 \leq 2|u(l_1)|^2 + 2(l_2 - l_1) \|v_x\|_{L^2(l_1,l_2)}^2.$$

As (2.10) and (2.11) are considered together

$$(2.12) \quad \int_{l_1}^{l_2} |v(x)|^2 dx \leq \underbrace{2(l_2 - l_1) \max(l_1, l_2 - l_1)}_{:=c_3} \left( \|u_x\|_{L^2(0,l_1)}^2 + \|v_x\|_{L^2(l_1,l_2)}^2 \right).$$

Next, Young's inequality is applied to get

$$(2.13) \quad \|p_x\|_{L^2(l_1,l_2)}^2 \leq 2\|p_x - \gamma v_x\|_{L^2(l_1,l_2)}^2 + 2\gamma^2 \|v_x\|^2.$$

By combining (2.12) and (2.13)

$$\begin{aligned} \|U\|_s^2 &\leq (1 + c_3) \|u_x\|_{L^2(0,l_1)}^2 + \|u^1\|_{L^2(0,l_1)}^2 + (1 + 2\gamma^2 + c_3) \|v_x\|_{L^2(l_1,l_2)}^2 + \|z\|_{L^2(l_1,l_2)}^2 \\ &\quad + 2\|p_x - v_x\|_{L^2(l_1,l_2)}^2 + \|q\|_{L^2(l_1,l_2)}^2 + \|y_x\|_{L^2(l_2,L)}^2 + \|y^1\|_{L^2(l_2,L)}^2. \end{aligned}$$

Hence, this leads to the left inequality of (2.9) with

$$C_1 = \frac{1}{\max(1, (1 + c_3)c_1^{-1}, (1 + 2\gamma^2 + c_3)\alpha_1^{-1}, \rho^{-1}, 2\beta^{-1}, \mu^{-1}, c_2^{-1})}.$$

□

Define the unbounded linear operator  $\mathcal{A}_{EPE} : D(\mathcal{A}_{EPE}) \subset \mathcal{H} \rightarrow \mathcal{H}$  by

$$\mathcal{A}_{EPE} \begin{pmatrix} u \\ u^1 \\ v \\ z \\ p \\ q \\ y \\ y^1 \end{pmatrix} = \begin{pmatrix} u^1 \\ c_1 u_{xx} - d_1 u^1 \\ z \\ \frac{1}{\rho} (\alpha v_{xx} - \gamma \beta p_{xx} - d_2 z) \\ q \\ \frac{1}{\mu} (\beta p_{xx} - \gamma \beta v_{xx}) \\ y^1 \\ c_2 y_{xx} - d_3 y^1 \end{pmatrix}, \quad \forall U = (u, u^1, v, z, p, q, y, y^1) \in D(\mathcal{A}_{EPE})$$

with the domain

$$D(\mathcal{A}_{EPE}) = \left\{ \begin{aligned} &U = (u, u^1, v, z, p, q, y, y^1) \in \mathcal{H}; \quad u^1 \in H_L^1(0, l_1), \quad z \in H^1(l_1, l_2), \quad q \in H_*^1(l_1, l_2), \\ &y^1 \in H_R^1(l_2, L), \quad u \in H^2(0, l_1) \cap H_L^1(0, l_1), \quad v \in H^2(l_1, l_2), \quad p \in H^2(l_2, l_1) \cap H_*^1(l_1, l_2), \\ &y \in H^2(l_2, L) \cap H_R^1(l_2, L), \quad \alpha v_x(l_1) - \gamma \beta p_x(l_1) = c_1 u_x(l_1), \quad \alpha v_x(l_2) - \gamma \beta p_x(l_2) = c_2 y_x(l_2), \\ &\beta p_x(l_1) = \gamma \beta v_x(l_1), \quad \beta p_x(l_2) = \gamma \beta v_x(l_2), \quad u^1(l_1) = z(l_1), \quad \text{and} \quad y^1(l_2) = z(l_2) \end{aligned} \right\}.$$

**Remark 2.3.** Using (1.2), direct calculations show that the transmission conditions

$$\alpha v_x(l_1) - \gamma \beta p_x(l_1) = c_1 u_x(l_1) \quad \text{and} \quad \beta p_x(l_1) = \gamma \beta v_x(l_1)$$

are equivalent to the transmission conditions

$$\alpha_1 v_x(l_1) = c_1 u_x(l_1) \quad \text{and} \quad \alpha_1 p_x(l_1) = c_1 \gamma u_x(l_1),$$

while the transmission conditions

$$\alpha v_x(l_2) - \gamma \beta p_x(l_2) = c_2 y_x(l_2) \quad \text{and} \quad \beta p_x(l_2) = \gamma \beta v_x(l_2),$$

are equivalent to the transmission conditions

$$\alpha_1 v_x(l_2) = c_2 y_x(l_2) \quad \text{and} \quad \alpha_1 p_x(l_2) = c_2 \gamma y_x(l_2).$$

If  $(u, v, p, y)$  is a sufficiently regular solution of the system (E/P/E), it can be transformed into a first-order evolution equation on the Hilbert space  $\mathcal{H}$  as the following

$$(2.14) \quad U_t = \mathcal{A}_{EPE} U, \quad U(0) = U_0,$$

where  $U = (u, u_1, v, v_t, p, p_t, y, y_t)$  and  $U_0 = (u_0, u_1, v_0, v_1, p_0, p_1, y_0, y_1)$ . By the arguments of Lemma 2.1, for all  $U = (u, u^1, v, z, p, q, y, y^1) \in D(\mathcal{A}_{EPE})$ ,

$$(2.15) \quad \Re(\langle \mathcal{A}_{EPE}U, U \rangle_{\mathcal{H}}) = - \int_{l_1}^{l_2} d_2 |z|^2 dx,$$

which implies that  $\mathcal{A}_{EPE}$  is dissipative. Now, let  $F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8) \in \mathcal{H}$ . By the Lax-Milgram Theorem, one can prove the existence of  $U \in D(\mathcal{A}_{EPE})$  solving the equation

$$-\mathcal{A}_{EPE}U = F.$$

Therefore, the unbounded linear operator  $\mathcal{A}_{EPE}$  is  $m$ -dissipative in  $\mathcal{H}$ , and consequently,  $0 \in \rho(\mathcal{A}_{EPE})$ . Moreover,  $\mathcal{A}_{EPE}$  generates a  $C_0$ -semigroup of contractions  $(e^{t\mathcal{A}_{EPE}})_{t \geq 0}$ , by the Lumer-Phillips theorem. Therefore, the solution of the Cauchy problem (2.14) admits the following representation

$$U(t) = e^{t\mathcal{A}_{EPE}}U_0, \quad t \geq 0,$$

which leads to the well-posedness of (2.14). The following result is immediate.

**Theorem 2.4.** *Letting  $U_0 \in \mathcal{H}$ , the system (2.14) admits a unique weak solution  $U$  satisfying*

$$U \in C^0(\mathbb{R}^+, \mathcal{H}).$$

*Moreover, if  $U_0 \in D(\mathcal{A}_{EPE})$ , the system (2.14) admits a unique strong solution  $U$  satisfying*

$$U \in C^1(\mathbb{R}^+, \mathcal{H}) \cap C^0(\mathbb{R}^+, D(\mathcal{A}_{EPE})).$$

**2.2. Strong Stability.** Now the following result is about the strong stability of (E/P/E).

**Theorem 2.5.** *The  $C_0$ -semigroup of contraction  $(e^{t\mathcal{A}_{EPE}})$  is strongly stable in  $\mathcal{H}$ ; i.e., for all  $U_0 \in \mathcal{H}$ , the solution of (2.14) satisfies*

$$\lim_{t \rightarrow \infty} \|e^{t\mathcal{A}_{EPE}}U_0\|_{\mathcal{H}} = 0.$$

**Proof.** Since the resolvent of  $\mathcal{A}_{EPE}$  is compact in  $\mathcal{H}$ , it follows from the Arendt-Batty's theorem (see page 837 in [6]) that the system (E/P/E) is strongly stable if and only if  $\mathcal{A}_{EPE}$  does not have pure imaginary eigenvalues, i.e.  $\sigma(\mathcal{A}_{EPE}) \cap i\mathbb{R} = \emptyset$ . From Section 2.1, it is already known that  $0 \in \rho(\mathcal{A}_{EPE})$ . Therefore, only  $\sigma(\mathcal{A}_{EPE}) \cap i\mathbb{R}^* = \emptyset$  must be proved. For this purpose, suppose that there exists a real number  $\lambda \neq 0$  and  $U = (u, u^1, v, z, p, q, y, y^1) \in D(\mathcal{A}_{EPE})$  such that

$$(2.16) \quad \mathcal{A}_{EPE}U = i\lambda U.$$

which is equivalent to the following system

$$(2.17) \quad u^1 = i\lambda u \text{ in } (0, l_1), \quad z = i\lambda v \text{ in } (l_1, l_2), \quad q = i\lambda p \text{ in } (l_1, l_2), \quad y^1 = i\lambda y \text{ in } (l_2, L),$$

and

$$(2.18) \quad \begin{cases} \lambda^2 u + c_1 u_{xx} = 0, & x \in (0, l_1), \\ \rho \lambda^2 v + \alpha v_{xx} - \gamma \beta p_{xx} - d_2 z = 0, & x \in (l_1, l_2), \\ \mu \lambda^2 p + \beta p_{xx} - \gamma \beta v_{xx} = 0, & x \in (l_1, l_2), \\ \lambda^2 y + c_2 y_{xx} = 0, & x \in (l_2, L), \end{cases}$$

From (2.15), (LD - P) and (2.16),

$$(2.19) \quad 0 = \Re(i\lambda U, U)_{\mathcal{H}} = \Re(\mathcal{A}_{EPE}U, U)_{\mathcal{H}} = - \int_{l_1}^{l_2} d_2 |z|^2 dx.$$

On the other hand, from (2.17), (2.19), (LD - P) and the fact that  $\lambda \neq 0$ , we have

$$(2.20) \quad d_2 z = 0 \text{ in } (l_1, l_2) \text{ and consequently } z = v = 0, \quad x \in (a_2, b_2).$$

By  $\alpha = \alpha_1 + \gamma^2 \beta$  and (2.20) in (2.18)<sub>2</sub>,

$$(2.21) \quad \rho \lambda^2 v + \alpha_1 v_{xx} + \gamma(\gamma \beta v_{xx} - \beta p_{xx}) = 0, \quad x \in (l_1, l_2).$$

Combining (2.18)<sub>3</sub> and (2.21) leads to

$$(2.22) \quad \lambda^2(\rho v + \gamma \mu p) + \alpha_1 v_{xx} = 0, \quad x \in (l_1, l_2).$$



Next, by (2.20) in (2.22) and  $\lambda \neq 0$  we get  $p = 0$  in  $(a_2, b_2)$ , the third equation in (2.17) yields

$$(2.23) \quad p = q = 0 \quad \text{in} \quad (a_2, b_2).$$

Since  $v, p \in H^2(l_1, l_2) \subset C^1([a_2, b_2])$ ,

$$(2.24) \quad v(\zeta) = v_x(\zeta) = p(\zeta) = p_x(\zeta) = 0, \quad \zeta \in \{a_2, b_2\}.$$

Now, combining (2.22) and (2.18)<sub>3</sub>, the following reduced system is obtained

$$(2.25) \quad v_{xx} = -\lambda^2 \alpha_1^{-1} (\rho v + \gamma \mu p), \quad x \in (l_1, l_2)$$

$$(2.26) \quad p_{xx} = -\lambda^2 \alpha_1^{-1} (\gamma \rho v + \mu \alpha \beta^{-1} p), \quad x \in (l_1, l_2).$$

Let  $U_{piezo} = (v, v_x, p, p_x)^\top$ . From (2.24),  $U_{piezo}(b_2) = 0$ . Now, the system (2.25)-(2.26) can be written in  $(b_2, l_2)$  as the following

$$(2.27) \quad (U_{piezo})_x = BU_{piezo} \quad \text{in} \quad (b_2, l_2),$$

where

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\rho \alpha_1^{-1} \lambda^2 & 0 & -\gamma \mu \alpha_1^{-1} \lambda^2 & 0 \\ 0 & 0 & 0 & 1 \\ -\rho \gamma \alpha_1^{-1} \lambda^2 & 0 & -\mu \alpha \beta^{-1} \alpha_1^{-1} \lambda^2 & 0 \end{pmatrix}.$$

The solution of the differential equation (2.27) is given by

$$(2.28) \quad U_{piezo}(x) = e^{B(x-b_2)} U_{piezo}(b_2) = 0 \quad \text{in} \quad (b_2, l_2).$$

Analogously, it can be proved that  $U_{piezo} = 0$  in  $(l_1, a_2)$ . Consequently,  $v = p = 0$  in  $(l_1, l_2)$ . Since  $v, p \in H^2(l_1, l_2) \subset C^1([l_1, l_2])$ ,

$$(2.29) \quad v(\zeta) = v_x(\zeta) = p(\zeta) = p_x(\zeta) = 0 \quad \text{where} \quad \zeta \in \{l_1, l_2\}.$$

By  $U \in D(\mathcal{A}_{EPE})$ , the continuity and transmission conditions,

$$(2.30) \quad u(0) = u(l_1) = u_x(l_1) = y(l_2) = y_x(l_2) = y(L) = 0.$$

Finally, by (2.18)<sub>1</sub>, (2.18)<sub>4</sub> and (2.30) it is easy to conclude that  $u = 0$  in  $(0, l_1)$  and  $y = 0$  in  $(l_2, L)$ . Hence,  $U = 0$ . The proof is thus complete.  $\square$

**2.3. Exponential Stability.** The aim of the subsection is to prove the exponential stability of System (E/P/E) under the sole assumption (LD – P). The main result of this section is the following theorem.

**Theorem 2.6.** *If (LD – P) holds, the  $C_0$ -semigroup of contractions  $(e^{t\mathcal{A}_{EPE}})_{t \geq 0}$  is exponentially stable, i.e. there exists  $M \geq 1$  and  $\omega > 0$  such that*

$$(2.31) \quad \|e^{t\mathcal{A}_{EPE}} U_0\|_{\mathcal{H}} \leq M e^{-\omega t} \|U_0\|_{\mathcal{H}}, \quad \forall U_0 \in \mathcal{H}.$$

Before diving into the technicality of the proof of Theorem E/P/E, recall from, e.g. [27], [40], that a  $C_0$ -semigroup of contractions  $(e^{t\mathcal{A}_{EPE}})_{t \geq 0}$  on  $\mathcal{H}$  must satisfy two conditions (3.17) if

$$(M1) \quad i\mathbb{R} \subset \rho(\mathcal{A}_{EPE})$$

$$(M2) \quad \sup_{\lambda \in \mathbb{R}} \| (i\lambda I - \mathcal{A}_{EPE})^{-1} \|_{\mathcal{L}(\mathcal{H})} < +\infty.$$

Since we already proved in Theorem 2.5 that  $i\mathbb{R} \subset \rho(\mathcal{A}_{EPE})$ , condition (M1) is satisfied. Now only the condition (M2) must be proved. We follow a contradiction argument, for this purpose, suppose that (M2) is false, then there exists  $\{(\lambda^n, U^n)\}_{n \geq 1} \subset \mathbb{R}^* \times D(\mathcal{A}_{EPE})$  with

$$(2.32) \quad |\lambda^n| \rightarrow \infty \quad \text{and} \quad \|U^n\|_{\mathcal{H}} = \|(u^n, u^{1,n}, v^n, z^n, p^n, q^n, y^n, y^{1,n})^\top\|_{\mathcal{H}} = 1,$$

such that

$$(2.33) \quad (i\lambda^n I - \mathcal{A}_{EPE}) U^n = \mathcal{F}^n := (f^{1,n}, f^{2,n}, f^{3,n}, f^{4,n}, f^{5,n}, f^{6,n}, f^{7,n}, f^{8,n})^\top \rightarrow 0 \quad \text{in} \quad \mathcal{H}.$$

For simplicity, let the index  $n$  be dropped. Then, (2.33) is equivalent to

$$(2.34) \quad \begin{cases} i\lambda u - u^1 = f^1 \rightarrow 0 & \text{in } H_L^1(0, l_1), \\ i\lambda v - z = f^3 \rightarrow 0 & \text{in } H^1(l_1, l_2), \\ i\lambda p - q = f^5 \rightarrow 0 & \text{in } H_*^1(l_1, l_2), \\ i\lambda y - y^1 = f^7 \rightarrow 0 & \text{in } H_R^1(l_2, L), \end{cases}$$

and

$$(2.35) \quad \begin{cases} i\lambda u^1 - c_1 u_{xx}^1 = f^2 \rightarrow 0 & \text{in } L^2(0, l_1), \\ i\lambda \rho z - \alpha v_{xx} + \gamma \beta p_{xx} + d_2 z = \rho f^4 \rightarrow 0 & \text{in } L^2(l_1, l_2), \\ i\lambda \mu q - \beta p_{xx} + \gamma \beta v_{xx} = \mu f^6 \rightarrow 0 & \text{in } L^2(l_1, l_2), \\ i\lambda y^1 - c_2 y_{xx} = f^8 \rightarrow 0 & \text{in } L^2(l_2, L). \end{cases}$$

Merging (2.34) and (2.35), a more compact system of equations is obtained

$$(2.36) \quad \begin{cases} \lambda^2 u + c_1 u_{xx} = F^1, \\ \lambda^2 \rho v + \alpha v_{xx} - \gamma \beta p_{xx} - i\lambda d_2 v = F^2, \\ \lambda^2 \mu p + \beta p_{xx} - \gamma \beta v_{xx} = F^3, \\ \lambda^2 y + c_2 y_{xx} = F^4, \end{cases}$$

where

$$(2.37) \quad \begin{cases} F^1 = -(f^2 + i\lambda f^1), \quad F^2 = -(\rho f^4 + d_2 f^3 + i\lambda \rho f^3), \\ F^3 = -(\mu f^6 + i\lambda \mu f^5) \quad \text{and} \quad F^4 = -(f^8 + i\lambda f^7). \end{cases}$$

By  $\alpha = \alpha_1 + \gamma^2 \beta$  in (2.36)<sub>2</sub>,

$$\lambda^2 \rho v + \alpha_1 v_{xx} + \gamma(\gamma \beta v_{xx} - \beta p_{xx}) - i\lambda d_2 v = F^2.$$

Now, combining (2.36)<sub>3</sub> and the above equality lead to

$$\alpha_1 v_{xx} = -\lambda^2 \rho v - \gamma \lambda^2 \mu p + i\lambda d_2 v + F^2 + \gamma F^3.$$

Inserting the above equation in (2.36)<sub>3</sub>, the system is reduced to

$$(2.38) \quad \begin{cases} \lambda^2 u + c_1 u_{xx} = F^1, \\ \lambda^2 \rho v + \alpha_1 v_{xx} + \gamma \mu \lambda^2 p - i\lambda d_2 v = F^5, \\ \lambda^2 \mu \alpha p + \alpha_1 \beta p_{xx} + \rho \gamma \beta \lambda^2 v - i\lambda \gamma \beta d_2 v = F^6, \\ \lambda^2 y + c_2 y_{xx} = F^4, \end{cases}$$

where

$$(2.39) \quad F^5 = F^2 + \gamma F^3 \quad \text{and} \quad F^6 = \alpha F^3 + \gamma \beta F^2.$$

At this moment, the following series of technical lemmas, as consequences of the dissipativity property of the solutions  $(u, u^1, v, z, p, q, y, y^1)$  of the system (2.34)-(2.35), are needed to finish the proof of Theorem E/P/E.

**Lemma 2.7.** *The solution  $(u, u^1, v, z, p, q, y, y^1)$  of the system (2.34)-(2.35) satisfies the following estimates*

$$(2.40) \quad \int_{l_1}^{l_2} d_2 |z|^2 dx = o(1), \quad \int_{l_1}^{l_2} d_2 |\lambda v|^2 dx = o(1) \quad \text{and} \quad \int_{a_2}^{b_2} |\lambda v|^2 dx = o(1).$$

**Proof.** To get the first estimate in (2.40), take the inner product of (2.33) with  $U$  in  $\mathcal{H}$ , and use  $\|U\|_{\mathcal{H}} = 1$  and  $\|\mathcal{F}\|_{\mathcal{H}} = o(1)$  so that

$$(2.41) \quad \int_{l_1}^{l_2} d_2 |z|^2 dx = -\Re(\langle \mathcal{A}_{EPE} U, U \rangle_{\mathcal{H}}) = \Re(\langle (i\lambda I - \mathcal{A}_{EPE})U, U \rangle_{\mathcal{H}}) = \Re\langle \mathcal{F}, U \rangle_{\mathcal{H}} = o(1).$$

Next, by multiplying (2.34)<sub>2</sub> by  $\sqrt{d_2}$ , using the first estimation in (2.40), and  $\|\mathcal{F}\|_{\mathcal{H}} = o(1)$ , the second estimate in (2.40) is immediate. Finally, by (LD - P) and the second estimate in (2.40), the third estimate in (2.40) is obtained.  $\square$

Note that for all  $0 < \varepsilon < \frac{b_2 - a_2}{4}$ , the following cut-off functions are fixed

- $\theta_k \in C^2([l_1, l_2])$ ,  $k \in \{1, 2\}$  such that  $0 \leq \theta_k(x) \leq 1$ , for all  $x \in [l_1, l_2]$  and

$$\theta_k(x) = \begin{cases} 1 & \text{if } x \in [a_2 + k\varepsilon, b_2 - k\varepsilon], \\ 0 & \text{if } x \in [l_1, a_2 + (k-1)\varepsilon] \cup [b_2 + (1-k)\varepsilon, l_2]. \end{cases}$$

Observe that  $\theta_1 \equiv 1$  on the support of  $\theta_2$ .

**Lemma 2.8.** *The solution  $(u, u^1, v, z, p, q, y, y^1)$  of the system (2.34)-(2.35) satisfies the following estimates*

$$(2.42) \quad \int_{l_1}^{l_2} \theta_1 |\lambda p|^2 dx = o(1), \quad \int_{D_\varepsilon} |\lambda p|^2 dx = o(1), \quad \text{and} \quad \int_{D_\varepsilon} |q|^2 dx = o(1),$$

where  $D_\varepsilon := (a_2 + \varepsilon, b_2 - \varepsilon)$  with a positive real number  $\varepsilon$  small enough such that  $\varepsilon < \frac{b_2 - a_2}{4}$ .

**Proof.** First, multiply (2.38)<sub>2</sub> by  $\beta\theta_1\bar{p}$ , integrate over  $(l_1, l_2)$  by parts, and use definition of  $\theta_1$  to get

$$(2.43) \quad \begin{aligned} & \lambda^2 \rho \beta \int_{l_1}^{l_2} \theta_1 v \bar{p} dx - \alpha_1 \beta \int_{l_1}^{l_2} \theta_1 v_x \bar{p}_x dx - \alpha_1 \beta \int_{l_1}^{l_2} \theta_1' v_x \bar{p} dx \\ & + \gamma \mu \beta \int_{l_1}^{l_2} \theta_1 |\lambda p|^2 dx - i \lambda \beta \int_{l_1}^{l_2} d_2 \theta_1 v \bar{p} dx = \beta \int_{l_1}^{l_2} \theta_1 F^5 \bar{p} dx. \end{aligned}$$

It is known that  $\|U\|_{\mathcal{H}} = 1$  and  $\|\mathcal{F}\|_{\mathcal{H}} = o(1)$ , which implies in particular that  $(\lambda p)$  is uniformly bounded in  $L^2(l_1, l_2)$  due to (2.34)<sub>3</sub>). Therefore, by Cauchy-Schwarz inequality, Lemma 2.7, the definition of  $\theta_1$  the following is deduced

$$\left| \lambda^2 \rho \beta \int_{l_1}^{l_2} \theta_1 v \bar{p} dx \right| = o(1), \quad \left| \int_{l_1}^{l_2} \theta_1' v_x \bar{p} dx \right| = O(\lambda^{-1}) = o(1), \quad \left| i \lambda \beta \int_{l_1}^{l_2} d_2 \theta_1 v \bar{p} dx \right| = o(\lambda^{-1}), \quad \left| \int_{l_1}^{l_2} \theta_1 F^5 \bar{p} dx \right| = o(1).$$

Inserting the above estimates into (2.43) and taking the real part leads to

$$(2.44) \quad \gamma \mu \beta \int_{l_1}^{l_2} \theta_1 |\lambda p|^2 dx - \alpha_1 \beta \Re \left( \int_{l_1}^{l_2} \theta_1 v_x \bar{p}_x dx \right) = o(1).$$

Analogously, multiply (2.38)<sub>3</sub> by  $-\theta_1 \bar{v}$ , integrate over  $(l_1, l_2)$  by parts to get

$$(2.45) \quad \begin{aligned} & -\lambda^2 \mu \alpha \int_{l_1}^{l_2} \theta_1 p \bar{v} dx + \alpha_1 \beta \int_{l_1}^{l_2} \theta_1' p_x \bar{v} dx + \alpha_1 \beta \int_{l_1}^{l_2} \theta_1 p_x \bar{v}_x dx \\ & - \rho \gamma \beta \int_{l_1}^{l_2} \theta_1 |\lambda v|^2 dx + i \lambda \gamma \beta \int_{l_1}^{l_2} \theta_1 d_2 |v|^2 dx = - \int_{l_1}^{l_2} \theta_1 F^6 \bar{v} dx. \end{aligned}$$

By the definition of  $\theta_1$ ,  $\|U\|_{\mathcal{H}} = 1$ ,  $\|\mathcal{F}\|_{\mathcal{H}} = o(1)$ , Cauchy-Schwarz inequality, Lemma 2.7

$$(2.46) \quad \left| \lambda^2 \mu \alpha \int_{l_1}^{l_2} \theta_1 p \bar{v} dx \right| = o(1), \quad \left| \int_{l_1}^{l_2} \theta_1' p_x \bar{v} dx \right| = o(\lambda^{-1}) \quad \text{and} \quad \left| \int_{l_1}^{l_2} \theta_1 F^6 \bar{v} dx \right| = o(1).$$

Inserting the estimates above into (2.45) together with Lemma 2.7 yields

$$(2.47) \quad \Re \left( \int_{l_1}^{l_2} \theta_1 p_x \bar{v}_x dx \right) = o(1).$$

Thus, the combination of (2.44) and (2.47) gets the first estimate in (2.42), and together with which, and the definition of  $\theta_1$ , the second estimate in (2.42) is immediate. Finally, by the second estimate in (2.42), (2.34)<sub>3</sub> and the fact that  $\|\mathcal{F}\|_{\mathcal{H}} = o(1)$ , the third estimate in (2.42) is obtained.  $\square$

**Lemma 2.9.** *The solution  $(u, u^1, v, z, p, q, y, y^1)$  of the system (2.34)-(2.35) satisfies the following estimate*

$$(2.48) \quad \int_{l_1}^{l_2} \theta_1 |v_x|^2 dx = o(1), \quad \text{and consequently} \quad \int_{D_\varepsilon} |v_x|^2 dx = o(1).$$

**Proof.** Multiplying (2.38)<sub>2</sub> by  $-\theta_1 \bar{v}$  and integrating over  $(l_1, l_2)$  by parts yield

$$(2.49) \quad \begin{aligned} & -\rho \int_{l_1}^{l_2} \theta_1 |\lambda v|^2 dx + \alpha_1 \int_{l_1}^{l_2} \theta_1' v_x \bar{v} dx + \alpha_1 \int_{l_1}^{l_2} \theta_1 |v_x|^2 dx - \gamma \mu \int_{l_1}^{l_2} \theta_1 \lambda^2 p \bar{v} dx \\ & + i\lambda \gamma \beta \int_{l_1}^{l_2} \theta_1 d_2 |v|^2 dx = - \int_{l_1}^{l_2} \theta_1 F^5 \bar{v} dx. \end{aligned}$$

By Cauchy-Schwarz inequality, Lemma 2.7, the definition of  $\theta_1$ ,  $\|U\|_{\mathcal{H}} = 1$ , and  $\|\mathcal{F}\|_{\mathcal{H}} = o(1)$ , the following hold

$$\left| \int_{l_1}^{l_2} \theta_1' v_x \bar{v} dx \right| = o(\lambda^{-1}), \quad \left| \int_{l_1}^{l_2} \theta_1 \lambda^2 p \bar{v} dx \right| = o(1) \quad \text{and} \quad \int_{l_1}^{l_2} \theta_1 F^5 \bar{v} dx = o(1).$$

Finally, inserting these estimates in (2.49) and by (2.40), the first estimate, and therefore the second estimate, in (2.48) are obtained.  $\square$

**Lemma 2.10.** *The solution  $(u, u^1, v, z, p, q, y, y^1)$  of the system (2.34)-(2.35) satisfies the following estimates*

$$(2.50) \quad \int_{l_1}^{l_2} \theta_2 |p_x|^2 dx = o(1), \quad \text{and consequently} \quad \int_{D_{2\varepsilon}} |p_x|^2 dx = o(1)$$

where  $D_{2\varepsilon} := (a_2 + 2\varepsilon, b_2 - 2\varepsilon)$  with a positive real number  $\varepsilon$  small enough so that  $\varepsilon < \frac{b_2 - a_2}{4}$ .

**Proof.** Multiplying (2.38)<sub>3</sub> by  $-\theta_2 \bar{p}$  integrating over  $(l_1, l_2)$  by parts lead to

$$(2.51) \quad \begin{aligned} & -\mu \alpha \int_{l_1}^{l_2} \theta_2 |\lambda p|^2 dx + \alpha_1 \beta \int_{l_1}^{l_2} \theta_2' p_x \bar{p} dx + \alpha_1 \beta \int_{l_1}^{l_2} \theta_2 |p_x|^2 dx - \rho \gamma \beta \int_{l_1}^{l_2} \lambda^2 \theta_2 v \bar{p} dx \\ & + i\lambda \gamma \beta \int_{l_1}^{l_2} d_2 \theta_2 v \bar{p} dx = - \int_{l_1}^{l_2} F^6 \theta_2 \bar{p} dx. \end{aligned}$$

By Cauchy-Schwarz inequality, Lemmas 2.7, 2.9, the definition of  $\theta_2$ ,  $\|U\|_{\mathcal{H}} = 1$ , and  $\|\mathcal{F}\|_{\mathcal{H}} = o(1)$ ,

$$\left| \int_{l_1}^{l_2} \theta_2' p_x \bar{p} dx \right| = o(\lambda^{-1}), \quad \left| \int_{l_1}^{l_2} \lambda^2 \theta_2 v \bar{p} dx \right| = o(1), \quad \left| i\lambda \int_{l_1}^{l_2} d_2 \theta_2 v \bar{p} dx \right| = o(\lambda^{-1}) \quad \text{and} \quad \left| \int_{l_1}^{l_2} F^6 \theta_2 \bar{p} dx \right| = o(1).$$

Finally, inserting the above estimates into (2.51) and by Lemma 2.8, the first estimate, and therefore the second estimate, in (2.50) are obtained.  $\square$

**Lemma 2.11.** *Let  $g \in C^1([l_1, l_2])$ . The solution  $(u, u^1, v, z, p, q, y, y^1)$  of the system (2.34)-(2.35) satisfies the following estimate*

$$(2.52) \quad \begin{aligned} & \int_{l_1}^{l_2} g' (\rho |\lambda v|^2 + \alpha |v_x|^2 + \mu |\lambda p|^2 + \beta |p_x|^2) dx - 2\gamma \beta \Re \left( \int_{l_1}^{l_2} g' p_x \bar{v}_x dx \right) + \mathcal{J}_1(l_1) - \mathcal{J}_1(l_2) \\ & = \mathcal{J}_2(l_2) - \mathcal{J}_2(l_1) + o(1) \end{aligned}$$

where  $\zeta \in \{l_1, l_2\}$  and

$$(2.53) \quad \begin{cases} \mathcal{J}_1(\zeta) = g(\zeta) (\rho |\lambda v(\zeta)|^2 + \alpha |v_x(\zeta)|^2 + \mu |\lambda p(\zeta)|^2 + \beta |p_x(\zeta)|^2) - 2\gamma \beta \Re (g(\zeta) p_x(\zeta) \bar{v}_x(\zeta)), \\ \mathcal{J}_2(\zeta) = 2\Re (i\lambda \rho g(\zeta) f^3(\zeta) \bar{v}(\zeta)) + 2\Re (i\lambda \mu g(\zeta) f^5(\zeta) \bar{p}(\zeta)). \end{cases}$$

**Proof.** First, multiply (2.36)<sub>2</sub> by  $-2g \bar{v}_x$ , and integrate over  $(l_1, l_2)$  to get

$$\begin{aligned} & -\rho \int_{l_1}^{l_2} g (|\lambda v|^2)_x dx - \alpha \int_{l_1}^{l_2} g (|v_x|^2)_x dx + 2\gamma \beta \Re \left( \int_{l_1}^{l_2} g p_{xx} \bar{v}_x dx \right) \\ & + 2\Re \left( i\lambda \int_{l_1}^{l_2} g d_2 v \bar{v}_x dx \right) = -2\Re \left( \int_{l_1}^{l_2} g F^2 \bar{v}_x dx \right). \end{aligned}$$

By several integration by parts and the definition of  $F^2$  in (2.37), the following is obtained

$$\begin{aligned}
(2.54) \quad & \int_{l_1}^{l_2} g' (\rho|\lambda v|^2 + \alpha|v_x|^2) dx - g(l_2) [\rho|\lambda v(l_2)|^2 + \alpha|v_x(l_2)|^2] + g(l_1) [\rho|\lambda v(l_1)|^2 + \alpha|v_x(l_1)|^2] \\
& - 2\gamma\beta\Re \left( \int_{l_1}^{l_2} g' p_x \bar{v}_x dx \right) - 2\gamma\beta\Re \left( \int_{l_1}^{l_2} g p_x \bar{v}_{xx} dx \right) + 2\gamma\beta\Re (g(l_2) p_x(l_2) \bar{v}_x(l_2)) \\
& - 2\gamma\beta\Re (g(l_1) p_x(l_1) \bar{v}_x(l_1)) - 2\Re \left( i\lambda \int_{l_1}^{l_2} g d_2 v \bar{v}_x dx \right) = 2\Re \left( \int_{l_1}^{l_2} g (f^4 + d_2 f^3) \bar{v}_x dx \right) \\
& - 2\Re \left( i\lambda \rho \int_{l_1}^{l_2} (f^3 g)_x \bar{v} dx \right) + 2\Re (i\lambda \rho g(l_2) f^3(l_2) \bar{v}(l_2)) - 2\Re (i\lambda \rho g(l_1) f^3(l_1) \bar{v}(l_1)).
\end{aligned}$$

Since  $v_x$  is uniformly bounded in  $L^2(l_1, l_2)$  and  $\|\mathcal{F}\|_{\mathcal{H}} = o(1)$ , by Cauchy-Schwarz inequality and Lemma 2.7

$$\left| \Re \left( i\lambda \int_{l_1}^{l_2} g d_2 v \bar{v}_x dx \right) \right| = o(1), \quad \left| \Re \left( \int_{l_1}^{l_2} g (f^2 + d_2 f^3) \bar{v}_x dx \right) \right| = o(1), \quad \left| \Re \left( i\lambda \rho \int_{l_1}^{l_2} (f^3 g)_x \bar{v} dx \right) \right| = o(1).$$

Substituting the estimation above into (2.54) leads to

$$\begin{aligned}
(2.55) \quad & \int_{l_1}^{l_2} g' (\rho|\lambda v|^2 + \alpha|v_x|^2) dx - g(l_2) [\rho|\lambda v(l_2)|^2 + \alpha|v_x(l_2)|^2] + g(l_1) [\rho|\lambda v(l_1)|^2 + \alpha|v_x(l_1)|^2] \\
& - 2\gamma\beta\Re \left( \int_{l_1}^{l_2} g' p_x \bar{v}_x dx \right) - 2\gamma\beta\Re \left( \int_{l_1}^{l_2} g p_x \bar{v}_{xx} dx \right) + 2\gamma\beta\Re (g(l_2) p_x(l_2) \bar{v}_x(l_2)) \\
& - 2\gamma\beta\Re (g(l_1) p_x(l_1) \bar{v}_x(l_1)) = 2\Re (i\lambda \rho g(l_2) f^3(l_2) \bar{v}(l_2)) - 2\Re (i\lambda \rho g(l_1) f^3(l_1) \bar{v}(l_1)) + o(1).
\end{aligned}$$

Analogously, multiply (2.36)<sub>3</sub> by  $-2g\bar{p}_x$  and integrate over  $(l_1, l_2)$  to obtain and taking the real part, we get

$$-\mu \int_{l_1}^{l_2} g (|\lambda p|^2)_x dx - \beta \int_{l_1}^{l_2} g (|p_x|^2)_x dx + 2\gamma\beta\Re \left( \int_{l_1}^{l_2} g v_{xx} \bar{p}_x dx \right) = -2\Re \left( \int_{l_1}^{l_2} g F^3 \bar{p}_x dx \right).$$

By several integration by parts and the definition of  $F^3$  given in (2.37), the following holds

$$\begin{aligned}
(2.56) \quad & \int_{l_1}^{l_2} g' (\mu|\lambda p|^2 + \beta|p_x|^2) dx - g(l_2) [\mu|\lambda p(l_2)|^2 + \beta|p_x(l_2)|^2] + g(l_1) [\mu|\lambda p(l_1)|^2 + \beta|p_x(l_1)|^2] \\
& + 2\gamma\beta\Re \left( \int_{l_1}^{l_2} g v_{xx} \bar{p}_x dx \right) = 2\Re \left( \int_{l_1}^{l_2} g f^6 \bar{p}_x dx \right) - 2\Re \left( i\lambda \mu \int_{l_1}^{l_2} (g f^5)_x \bar{p} dx \right) \\
& + 2\Re (i\lambda \mu g(l_2) f^5(l_2) \bar{p}(l_2)) - 2\Re (i\lambda \mu g(l_1) f^5(l_1) \bar{p}(l_1)).
\end{aligned}$$

Since  $p_x$  and  $\lambda p$  are uniformly bounded in  $L^2(l_1, l_2)$  and  $\|\mathcal{F}\|_{\mathcal{H}} = o(1)$ , Cauchy-Schwarz inequality is implemented to obtain

$$\left| \Re \left( \int_{l_1}^{l_2} g f^6 \bar{p}_x dx \right) \right| = o(1) \quad \text{and} \quad \left| \Re \left( i\lambda \mu \int_{l_1}^{l_2} (g f^5)_x \bar{p} dx \right) \right| = o(1).$$

Substituting the estimates above into (2.56) results in

$$\begin{aligned}
(2.57) \quad & \int_{l_1}^{l_2} g' (\mu|\lambda p|^2 + \beta|p_x|^2) dx - g(l_2) [\mu|\lambda p(l_2)|^2 + \beta|p_x(l_2)|^2] + g(l_1) [\mu|\lambda p(l_1)|^2 + \beta|p_x(l_1)|^2] \\
& + 2\gamma\beta\Re \left( \int_{l_1}^{l_2} g v_{xx} \bar{p}_x dx \right) = 2\Re (i\lambda \mu g(l_2) f^5(l_2) \bar{p}(l_2)) - 2\Re (i\lambda \mu g(l_1) f^5(l_1) \bar{p}(l_1)) + o(1).
\end{aligned}$$

Finally, adding (2.55) and (2.57), the desired result (2.52) is obtained.  $\square$

**Lemma 2.12.** *The solution  $(u, u^1, v, z, p, q, y, y^1)$  of the system (2.34)-(2.35) satisfies the following estimates*

$$(2.58) \quad \int_{l_1}^{l_2} |\lambda v|^2 dx = o(1), \quad \int_{l_1}^{l_2} |v_x|^2 dx = o(1), \quad \int_{l_1}^{l_2} |\lambda p|^2 dx = o(1) \quad \text{and} \quad \int_{l_1}^{l_2} |p_x|^2 dx = o(1).$$

**Proof.** First, define two cut-off functions  $\theta_3, \theta_4 \in C^2([l_1, l_2])$  such that  $0 \leq \theta_3, \theta_4 \leq 1$  for all  $x \in [l_1, l_2]$  and

$$(2.59) \quad \theta_3(x) = \begin{cases} 1 & \text{if } x \in [l_1, a_1 + 2\varepsilon], \\ 0 & \text{if } x \in [a_2 - 2\varepsilon, l_2] \end{cases}, \quad \text{and} \quad \theta_4(x) = \begin{cases} 0 & \text{if } x \in [l_1, a_1 + 2\varepsilon], \\ 1 & \text{if } x \in [a_2 - 2\varepsilon, l_2]. \end{cases}$$

By taking  $g(x) = (x - l_1)\theta_3(x) + (x - l_2)\theta_4(x)$ , it is easy to see that

$$g'(x) = \theta_3(x) + \theta_4(x) + (x - l_1)\theta_3'(x) + (x - l_2)\theta_4'(x) \quad \text{and} \quad g(l_1) = g(l_2) = 0.$$

Setting  $\tilde{g} = (x - l_1)\theta_3' + (x - l_2)\theta_4'$  and considering  $g$  defined above in (2.52) lead to

$$(2.60) \quad \int_{l_1}^{l_2} (\theta_3 + \theta_4) (\rho|\lambda v|^2 + \alpha|v_x|^2 + \mu|\lambda p|^2 + \beta|p_x|^2) dx - 2\gamma\beta\Re \left( \int_{l_1}^{l_2} (\theta_3 + \theta_4)p_x \overline{v_x} dx \right) \\ - \underbrace{\int_{l_1}^{l_2} \tilde{g} (\rho|\lambda v|^2 + \alpha|v_x|^2 + \mu|\lambda p|^2 + \beta|p_x|^2) dx}_{:=\mathcal{I}_1} + \underbrace{2\gamma\beta\Re \left( \int_{l_1}^{l_2} \tilde{g} p_x \overline{v_x} dx \right)}_{:=\mathcal{I}_2} + o(1).$$

Now, adopting (2.40), (2.42), (2.48), (2.50), the definitions of  $\theta_3$  and  $\theta_4$  and Cauchy-Schwarz inequality result in

$$(2.61) \quad |\mathcal{I}_1| = o(1) \quad \text{and} \quad |\mathcal{I}_2| = o(1).$$

On the other hand, since  $\alpha = \alpha_1 + \gamma^2\beta$ , it is easy to see that

$$(2.62) \quad \int_{l_1}^{l_4} (\theta_3 + \theta_4) (\alpha|v_x|^2 + \beta|p_x|^2) dx - 2\gamma\beta\Re \left( \int_{l_1}^{l_2} (\theta_3 + \theta_4)p_x \overline{v_x} dx \right) = \\ \alpha_1 \int_{l_1}^{l_4} (\theta_3 + \theta_4)|v_x|^2 dx + \beta \int_{l_1}^{l_4} (\theta_3 + \theta_4)|\gamma v_x - p_x|^2 dx.$$

Therefore, substitution of (2.61) and (2.62) into (2.60) result in

$$(2.63) \quad \int_{l_1}^{l_2} (\theta_3 + \theta_4) (\rho|\lambda v|^2 + \alpha_1|v_x|^2 + \mu|\lambda p|^2 + \beta|\gamma v_x - p_x|^2) dx = o(1).$$

Finally, by (2.40), (2.42), (2.48), (2.50), the definitions of  $\theta_3$  and  $\theta_4$  in (2.63), the desired result (2.58) is obtained.  $\square$

**Lemma 2.13.** *The solution  $(u, u^1, v, z, p, q, y, y^1)$  of the system (2.34)-(2.35) satisfies the following estimates*

$$(2.64) \quad |v(l_1)|^2 = o(\lambda^{-2}), \quad |v(l_2)|^2 = o(\lambda^{-2}), \quad |v_x(l_1)|^2 = o(1) \quad \text{and} \quad |v_x(l_2)|^2 = o(1).$$

**Proof.** Define  $g \in C^1([l_1, l_2])$  such that

$$(2.65) \quad g(l_2) = -1, \quad g(l_1) = 1, \quad \max_{x \in [l_1, l_2]} |g(x)| = m_g \quad \max_{x \in [l_1, l_2]} |g'(x)| = m'_g.$$

By using  $g$  in (2.52),

$$(2.66) \quad \mathcal{J}_1(l_1) - \mathcal{J}_1(l_2) = - \int_{l_1}^{l_2} g' (\rho|\lambda v|^2 + \alpha|v_x|^2 + \mu|\lambda p|^2 + \beta|p_x|^2) dx + 2\gamma\beta\Re \left( \int_{l_1}^{l_2} g' p_x \overline{v_x} dx \right) + \\ \mathcal{J}_2(l_2) - \mathcal{J}_2(l_1) + o(1),$$

which, together with (2.65) and (2.58), implies that

$$\int_{l_1}^{l_2} g' (\rho|\lambda v|^2 + \alpha|v_x|^2 + \mu|\lambda p|^2 + \beta|p_x|^2) dx = o(1) \quad \text{and} \quad \left| \Re \left( \int_{l_1}^{l_2} g' p_x \overline{v_x} dx \right) \right| = o(1).$$

Substituting these estimates into (2.66) and by (2.53) leads to

$$\mathcal{M}(l_1) + \mathcal{M}(l_2) = 2\gamma\beta\Re (p_x(l_1)\overline{v_x}(l_1)) - 2\gamma\beta\Re (p_x(l_2)\overline{v_x}(l_2)) + \mathcal{J}_2(l_2) - \mathcal{J}_2(l_1) + o(1)$$

where

$$(2.67) \quad \mathcal{M}(\zeta) = \rho|\lambda v(\zeta)|^2 + \alpha|v_x(\zeta)|^2 + \mu|\lambda p(\zeta)|^2 + \beta|p_x(\zeta)|^2,$$

and therefore,

$$(2.68) \quad \mathcal{M}(l_1) + \mathcal{M}(l_2) \leq 2\gamma\beta|p_x(l_1)||v_x(l_1)| + 2\gamma\beta|p_x(l_2)||v_x(l_2)| + |\mathcal{J}_2(l_2)| + |\mathcal{J}_2(l_1)| + o(1).$$

Now, use the fact that  $f^3 \in H^1(l_1, l_2) \subset C([l_1, l_2])$ ,  $f^5 \in H_*^1(l_1, l_2) \subset C([l_1, l_2])$ , and  $\|\mathcal{F}\|_{\mathcal{H}} = o(1)$  (cf. (2.33)), to obtain

$$(2.69) \quad |f^3(\zeta)| = o(1) \quad \text{and} \quad |f^5(\zeta)| = o(1), \quad \text{where} \quad \zeta \in \{l_1, l_2\}.$$

Finally, by (2.69), the definition of  $\mathcal{J}_2$  given in (2.53), and Young's inequality,

$$(2.70) \quad \begin{cases} 2\gamma\beta|p_x(\zeta)||v_x(\zeta)| \leq \gamma^2\beta|v_x(\zeta)|^2 + \beta|p_x(\zeta)|^2, \\ |\mathcal{J}_2(\zeta)| \leq \frac{1}{2}\rho|\lambda v(\zeta)|^2 + 2\rho|f^3(\zeta)|^2 + \frac{1}{2}\mu|\lambda p(\zeta)|^2 + 2\mu|f^5(\zeta)|^2 \leq \frac{1}{2}\rho|\lambda v(\zeta)|^2 + \frac{1}{2}\mu|\lambda p(\zeta)|^2 + o(1), \end{cases}$$

where  $\zeta \in \{l_1, l_2\}$ . To get the desired result (2.64), substitute (2.70) into (2.68) and use  $\alpha = \alpha_1 + \gamma^2\beta$ ,

$$(2.71) \quad \sum_{j=1}^2 \left( \frac{\rho}{2} |\lambda v(l_j)|^2 + \frac{\mu}{2} |\lambda p(l_j)|^2 + \alpha_1 |v_x(l_j)|^2 \right) \leq o(1).$$

□

**Lemma 2.14.** *The solution  $(u, u^1, v, z, p, q, y, y^1)$  of the system (2.34)-(2.35) satisfies the following estimates*

$$(2.72) \quad \int_0^{l_1} (|\lambda u|^2 + c_1 |u_x|^2) = o(1) \quad \text{and} \quad \int_{l_2}^L (|\lambda y|^2 + c_2 |y_x|^2) dx = o(1).$$

**Proof.** First, multiply (2.36)<sub>1</sub> and (2.36)<sub>4</sub> by  $-2x\bar{u}_x$  and  $-2(x-L)\bar{y}_x$ , and integrate over  $(0, l_1)$  and  $(l_2, L)$ , respectively. Since  $\|\mathcal{F}\|_{\mathcal{H}} = o(1)$ , and  $u_x$  and  $y_x$  are uniformly bounded in  $L^2(0, l_1)$  and  $L^2(l_2, L)$ , respectively,

$$-\int_0^{l_1} x (|\lambda u|^2)_x dx - c_1 \int_0^{l_1} x (|u_x|^2)_x = \underbrace{2\Re \left( \int_0^{l_1} x f^2 \bar{u}_x dx \right)}_{=o(1)} + 2\Re \left( i\lambda \int_0^{l_1} x f^1 \bar{u}_x dx \right)$$

and

$$-\int_{l_2}^L (x-L) (|\lambda y|^2)_x dx - c_2 \int_{l_2}^L (x-L) (|y_x|^2)_x = \underbrace{2\Re \left( \int_{l_2}^L (x-L) f^8 \bar{y}_x dx \right)}_{=o(1)} + 2\Re \left( i\lambda \int_{l_2}^L (x-L) f^7 \bar{y}_x dx \right).$$

Since  $\|\mathcal{F}\|_{\mathcal{H}} = o(1)$ , and  $(\lambda u)$  and  $(\lambda y)$  are uniformly bounded in  $L^2(0, l_1)$  and  $L^2(l_2, L)$ , respectively,

$$(2.73) \quad \begin{aligned} & \int_0^{l_1} (|\lambda u|^2 + c_1 |u_x|^2) dx - l_1 (|\lambda u(l_1)|^2 + c_1 |u_x(l_1)|^2) = \\ & \underbrace{-2\Re \left( i\lambda \int_0^{l_1} (x f^1)_x \bar{u} dx \right)}_{=o(1)} + 2\Re (i\lambda l_1 f^1(l_1) \bar{u}(l_1)) + o(1) \end{aligned}$$

and

$$(2.74) \quad \begin{aligned} & \int_{l_2}^L (|\lambda y|^2 + c_2 |y_x|^2) dx + (l_2 - L) (|\lambda y(l_2)|^2 + c_2 |y_x(l_2)|^2) = \\ & \underbrace{-2\Re \left( i\lambda \int_{l_2}^L ((x-L) f^7)_x \bar{y} dx \right)}_{=o(1)} - 2\Re (i\lambda (l_2 - L) f^7(l_2) \bar{y}(l_2)) + o(1). \end{aligned}$$

Recalling Remark 2.3,  $u(l_1) = v(l_1)$ ,  $v(l_2) = y(l_2)$ , (2.64) and the facts that

$$|f^1(l_1)| \leq \int_0^{l_1} |f_x^1| dx = o(1) \quad \text{and} \quad |f^7(l_2)| \leq \int_{l_2}^L |f_x^7| dx = o(1),$$

the following is obtained

$$\begin{cases} |\lambda u(l_1)|^2 + c_1 |u_x(l_1)|^2 = o(1), & |\lambda y(l_2)|^2 + c_2 |y_x(l_2)|^2 = o(1), \\ |\Re (i\lambda l_1 f^1(l_1) \bar{u}(l_1))| = o(1), & |\Re (i\lambda (l_2 - L) f^7(l_2) \bar{y}(l_2))| = o(1). \end{cases}$$

Finally, substitution of the estimates above into (2.73) and (2.74), the desired result (2.72) is obtained.  $\square$

Now, we are ready to finally prove Theorem 2.6.

**Proof of Theorem 2.6.** By (2.58) and (2.72), we obtain that  $\|U\|_{\mathcal{H}} = o(1)$ . This contradicts that  $\|U\|_{\mathcal{H}} = 1$ . Hence, (M2) holds true, and this makes the proof complete.

**Remark 2.15.** Note that the electrostatic/quasi-static approaches in modeling piezoelectric beams discard the dynamic electromagnetic effects, i.e.  $\mu \equiv 0$  in (E/P/E). Therefore, the reduced model becomes identical to the one obtained in [20], where two fully-distributed viscous damping terms for the outer wave equations are considered to achieve exponential stability. However, through the analysis of the section, it can be shown that only a local damping acting in the middle layer can also lead to an exponential stability result with the assumption (LD – P). The proof is exactly the same (even simpler) as the one presented above yet it left to the reader. This is a major improvement of the earlier result.

### 3. STABILITY RESULTS FOR THE SYSTEM (P/E)

Note that the assumption (LD – E) applies to all results in this section. For simplicity, the repetition of the assumption in the results below is avoided unless it is necessary to state.

**3.1. Well-Posedness.** In this section, the well-posedness of the system (P/E) is established by a semigroup approach. The natural energy of the system (P/E) is defined by

$$E_{PE}(t) = \frac{1}{2} \int_0^{l_1} (\rho|v_t|^2 + \alpha_1|v_x|^2 + \mu|p_t|^2 + \beta|\gamma v_x - p_x|^2) dx + \frac{1}{2} \int_{l_1}^L (|y_t|^2 + c_2|y_x|^2) dx.$$

It is straightforward to show that the energy  $E_{PE}(t)$  is dissipative along the smooth enough solutions of (P/E), i.e.

$$(3.1) \quad \frac{dE_{PE}(t)}{dt} = - \int_{l_1}^L d_1|y_t|^2 dx.$$

Define the energy space  $\mathcal{H}_{PE}$

$$\mathcal{H}_{PE} := \left\{ (v, z, p, q, y, y^1) \in (H_L^1(0, l_1) \times L^2(0, l_1))^2 \times H_R^1(l_1, L) \times L^2(l_1, L), v(l_1) = y(l_1) \right\}$$

equipped by the norm

$$(3.2) \quad \|U\|_{\mathcal{H}_{PE}}^2 = \int_0^{l_1} (\alpha_1|v_x|^2 + \rho|z|^2 + \beta|\gamma v_x - p_x|^2 + \mu|q|^2) dx + \int_{l_1}^L (c_2|y_x|^2 + |y^1|^2) dx, \quad \forall U = (v, z, p, q, y, y^1) \in \mathcal{H}_{PE}.$$

This norm is equivalent to the standard norm of  $\mathcal{H}_{PE}$  (the arguments in the proof of Lemma 2.2 can be followed mutatis mutandis). Define the unbounded linear operator  $\mathcal{A}_{PE} : D(\mathcal{A}_{PE}) \subset \mathcal{H}_{PE} \rightarrow \mathcal{H}_{PE}$  by

$$\mathcal{A}_{PE} \begin{pmatrix} v \\ z \\ p \\ q \\ y \\ y^1 \end{pmatrix} = \begin{pmatrix} z \\ \frac{1}{\rho} (\alpha v_{xx} - \gamma \beta p_{xx}) \\ q \\ \frac{1}{\mu} (\beta p_{xx} - \gamma \beta v_{xx}) \\ y^1 \\ c_2 y_{xx} - d_1 y^1 \end{pmatrix}, \quad \forall U = (v, z, p, q, y, y^1) \in D(\mathcal{A}_{PE})$$

with the domain

$$D(\mathcal{A}_{PE}) = \left\{ U := (v, z, p, q, y, y^1) \in \mathcal{H}_{PE}; z, q \in H_L^1(0, l_1), y^1 \in H_R^1(0, l_1), v, p \in H^2(0, l_1) \cap H_L^1(0, l_1), \right. \\ \left. y \in H^2(l_1, L) \cap H_R^1(l_1, L), \alpha v_x(l_1) - \gamma \beta p_x(l_1) = c_2 y_x(l_1), \beta p_x(l_1) = \gamma \beta v_x(l_1), y^1(l_1) = z(l_1) \right\}.$$

**Remark 3.1.** Obviously as in the previous section, the transmission conditions

$$\alpha v_x(l_1) - \gamma \beta p_x(l_1) = c_2 y_x(l_1), \quad \text{and} \quad \beta p_x(l_1) = \gamma \beta v_x(l_1),$$

are equivalent to the transmission conditions

$$\alpha_1 v_x(l_1) = c_2 y_x(l_1), \quad \text{and} \quad \alpha_1 p_x(l_1) = c_2 \gamma y_x(l_1).$$



If  $(v, p, y)^\top$  is a sufficiently regular solution of the system (P/E), the system can be transformed into the first order evolution equation on the Hilbert space  $\mathcal{H}_{PE}$

$$(3.3) \quad U_t = \mathcal{A}_{PE}U, \quad U(0) = U_0,$$

with  $U = (v, v_t, p, p_t, y, y_t)$  and  $U_0 = (v_0, v_1, p_0, p_1, y_0, y_1)$ . By the analogous arguments in Subsection 2.1, the solution of the Cauchy problem (3.3) admits the following representation

$$U(t) = e^{t\mathcal{A}_{PE}}U_0, \quad t \geq 0,$$

which leads to the well-posedness for (3.3).

**Theorem 3.2.** *Let  $U_0 \in \mathcal{H}_{PE}$ , the system (3.3) admits a unique weak solution  $U$  satisfying*

$$U \in C^0(\mathbb{R}^+, \mathcal{H}_{PE}).$$

*Moreover, if  $U_0 \in D(\mathcal{A}_{PE})$ , the system (3.3) admits a unique strong solution  $U$  satisfying*

$$U \in C^1(\mathbb{R}^+, \mathcal{H}_{PE}) \cap C^0(\mathbb{R}^+, D(\mathcal{A}_{PE})).$$

**3.2. Strong Stability.** In this section, the strong stability of the system (P/E) is investigated. Here is the main result.

**Theorem 3.3.** *The  $C_0$ -semigroup of contraction  $(e^{t\mathcal{A}_{PE}})$  is strongly stable in  $\mathcal{H}_{PE}$ , i.e., for all  $U_0 \in \mathcal{H}_{PE}$ , the solution of (3.3) satisfies  $\lim_{t \rightarrow \infty} \|e^{t\mathcal{A}_{PE}}U_0\|_{\mathcal{H}_{PE}} = 0$ , if and only if*

$$(SC) \quad \frac{\sigma_+}{\sigma_-} \neq \frac{2n_+ - 1}{2n_- - 1}, \quad \forall n_+, n_- \in \mathbb{N}$$

where the two positive real numbers  $\sigma_+$  and  $\sigma_-$  are defined by

$$(3.4) \quad \sigma_+ := \sqrt{\frac{(\rho\beta + \mu\alpha) + \sqrt{(\rho\beta - \mu\alpha)^2 + 4\gamma^2\beta^2\mu\rho}}{2\beta\alpha_1}} \quad \text{and} \quad \sigma_- := \sqrt{\frac{(\rho\beta + \mu\alpha) - \sqrt{(\rho\beta - \mu\alpha)^2 + 4\gamma^2\beta^2\mu\rho}}{2\beta\alpha_1}}.$$

**Proof.** It follows from the Arendt-Batty theorem (see page 837 in [6]), since the resolvent of  $\mathcal{A}_{PE}$  is compact in  $\mathcal{H}_{PE}$ , the system (P/E) is strongly stable if and only if  $\mathcal{A}_{PE}$  does not have pure imaginary eigenvalues, i.e.  $\sigma(\mathcal{A}_{PE}) \cap i\mathbb{R} = \emptyset$ . By Section 3.1,  $0 \in \rho(\mathcal{A}_{PE})$  is immediate. However,  $\sigma(\mathcal{A}_{PE}) \cap i\mathbb{R}^* = \emptyset$  must be proved. For this purpose, for a real number  $\lambda \neq 0$  and  $U = (v, z, p, q, y, y^1) \in D(\mathcal{A}_{PE})$ , consider

$$(3.5) \quad \mathcal{A}_{PE}U = i\lambda U,$$

which is equivalent to the following system

$$(3.6) \quad z = i\lambda v \text{ in } (0, l_1), \quad q = i\lambda p \text{ in } (0, l_1), \quad \text{and} \quad y^1 = i\lambda y \text{ in } (l_1, L),$$

and

$$(3.7) \quad \begin{cases} \rho\lambda^2 v + \alpha v_{xx} - \gamma\beta p_{xx} = 0, \\ \mu\lambda^2 p + \beta p_{xx} - \gamma\beta v_{xx} = 0, & x \in (0, l_1), \\ \lambda^2 y + c_2 y_{xx} - d_1 y^1 = 0, & x \in (l_1, L), \end{cases}$$

From the identity

$$\Re(\mathcal{A}_{PE}U, U)_{\mathcal{H}_{PE}} = - \int_{l_1}^L d_1 |y^1|^2 dx,$$

and (3.5),

$$(3.8) \quad 0 = \Re(i\lambda U, U)_{\mathcal{H}_{PE}} = \Re(\mathcal{A}_{PE}U, U)_{\mathcal{H}_{PE}} = - \int_{l_1}^L d_1 |y^1|^2 dx.$$

Thus,

$$(3.9) \quad d_1 y^1 = 0 \text{ in } (l_1, L), \text{ and consequently, } y^1 = y = 0 \text{ in } (a_1, b_1)$$

by (3.6), (LD-E) and (3.8). Considering (3.7)<sub>3</sub>, (3.9) and the unique continuation theorem,  $y = 0$  in  $(l_1, L)$ . Moreover, since  $y \in H^2(l_1, L) \subset C^1([l_1, L])$ ,  $y(l_1) = y_x(l_1) = 0$ . It follows from the continuity condition and

Remark 3.1 that  $v(l_1) = v_x(l_1) = p_x(l_1) = 0$ . Using the fact that  $\alpha = \alpha_1 + \gamma^2\beta$ , (3.7)<sub>1</sub> and (3.7)<sub>2</sub>, the system (3.6)-(3.7) reduced to

$$(3.10) \quad \begin{cases} v_{xx} = -\lambda^2\alpha_1^{-1}(\rho v + \gamma\mu p), \\ p_{xx} = -\lambda^2\alpha_1^{-1}(\gamma\rho v + \mu\alpha\beta^{-1}p), & x \in (0, l_1), \\ v(0) = p(0) = v(l_1) = v_x(l_1) = p_x(l_1) = 0. \end{cases}$$

By differentiating (3.10)<sub>1</sub> twice, using (3.10)<sub>2</sub> and (3.10)<sub>3</sub>, the following system is obtained

$$(3.11) \quad \begin{cases} \alpha_1\beta v_{xxxx} + \lambda^2(\rho\beta + \mu\alpha)v_{xx} + \mu\rho\lambda^4 v = 0, \\ v(0) = v_{xx}(0) = v(l_1) = v_x(l_1) = v_{xx}(l_1) = 0. \end{cases}$$

The characteristic polynomial corresponding to (3.11) is

$$(3.12) \quad q(\varkappa) = \alpha_1\beta\varkappa^4 + \lambda^2(\rho\beta + \mu\alpha)\varkappa^2 + \mu\rho\lambda^4.$$

and therefore, define

$$q_0(m) := \alpha_1\beta m^2 + \lambda^2(\rho\beta + \mu\alpha)m + \mu\rho\lambda^4.$$

Since  $(\rho\beta + \mu\alpha)^2 - 4\beta\alpha_1\mu\rho = (\rho\beta - \mu\alpha)^2 + 4\gamma^2\beta^2\mu\rho > 0$ , the polynomial  $q_0$  has two distinct real roots  $m_-$  and  $m_+$  :

$$m_+ = -\sigma_+^2\lambda^2 \quad \text{and} \quad m_- = -\sigma_-^2\lambda^2$$

where  $\sigma_+$  and  $\sigma_-$  are defined by (3.4). Observe that  $m_+ < 0$ , and by  $\alpha > \gamma^2\beta$ ,  $m_- < 0$  is immediate. Setting  $\varkappa_+ := \sqrt{-m_+}$  and  $\varkappa_- := \sqrt{-m_-}$ ,  $q$  has in total of four roots  $i\varkappa_+, -i\varkappa_+, i\varkappa_-, -i\varkappa_-$ . Hence, the general solution of (3.11) is

$$v(x) = c_1 \sin(\varkappa_+ x) + c_2 \cos(\varkappa_+ x) + c_3 \sin(\varkappa_- x) + c_4 \cos(\varkappa_- x)$$

where  $c_j \in \mathbb{C}$ ,  $j = 1, \dots, 4$ . By the boundary conditions in (3.11) at  $x = 0$  and  $\varkappa_+^2 - \varkappa_-^2 \neq 0$ , it is deduced that  $c_2 = c_4 = 0$ . Moreover, by boundary conditions in (3.11) at  $x = l_1$ ,

$$(3.13) \quad \begin{cases} c_1 \sin(\varkappa_+ l_1) + c_3 \sin(\varkappa_- l_1) = 0, \\ c_1 \varkappa_+ \cos(\varkappa_+ l_1) + c_3 \varkappa_- \cos(\varkappa_- l_1) = 0, \\ c_1 \varkappa_+^3 \cos(\varkappa_+ l_1) + c_3 \varkappa_-^3 \cos(\varkappa_- l_1) = 0. \end{cases}$$

Now, by (3.13)<sub>2</sub> and (3.13)<sub>3</sub>,

$$M(c_1, c_3)^\top = (0, 0)^\top, \quad \text{where} \quad M = \begin{pmatrix} \varkappa_+ \cos(\varkappa_+ l_1) & \varkappa_- \cos(\varkappa_- l_1) \\ \varkappa_+^3 \cos(\varkappa_+ l_1) & \varkappa_-^3 \cos(\varkappa_- l_1) \end{pmatrix}$$

It is easy to see that  $\det(M) = \varkappa_- \varkappa_+ (\varkappa_-^2 - \varkappa_+^2) \cos(\varkappa_- l_1) \cos(\varkappa_+ l_1)$ . Utilizing  $\varkappa_+^2 - \varkappa_-^2 \neq 0$ , it is observed that  $\det(M)$  vanishes if and only if  $\cos(\varkappa_+ l_1) = 0$  or  $\cos(\varkappa_- l_1) = 0$ . We split this into three cases:

**Case 1:** Consider  $\cos(\varkappa_- l_1) = 0$  and  $\cos(\varkappa_+ l_1) \neq 0$ . It follows from (3.13)<sub>1</sub> and (3.13)<sub>2</sub> that  $c_1 = c_3 = 0$ . Consequently,  $U = 0$ .

**Case 2:** Consider  $\cos(\varkappa_- l_1) \neq 0$  and  $\cos(\varkappa_+ l_1) = 0$ . It follows from (3.13)<sub>1</sub> and (3.13)<sub>2</sub> that  $c_1 = c_3 = 0$ . Consequently,  $U = 0$ .

**Case 3** Consider  $\cos(\varkappa_+ l_1) = 0$  and  $\cos(\varkappa_- l_1) = 0$ . Then, there exists  $n_+, n_- \in \mathbb{N}$  such that

$$(3.14) \quad \varkappa_+ = \frac{2n_+ + 1}{2l_1} \pi \quad \text{and} \quad \varkappa_- = \frac{2n_- + 1}{2l_1} \pi.$$

By  $c_2 = c_4 = 0$ , and (3.13)<sub>1</sub>, (3.14), the general solution of (3.11) is given by

$$(3.15) \quad v(x) = c \left( \sin \left( \frac{2n_+ + 1}{2l_1} \pi x \right) \pm \sin \left( \frac{2n_- + 1}{2l_1} \pi x \right) \right)$$

where  $c \in \mathbb{C}$ . On the other hand, by (3.14) again,

$$(3.16) \quad \frac{\varkappa_+}{\varkappa_-} = \frac{\sigma_+}{\sigma_-} = \frac{2n_+ + 1}{2n_- + 1}, \quad n_+, n_- \in \mathbb{N}, \quad \text{and} \quad \lambda = \frac{2n_+ + 1}{2l_1 \sigma_+}$$

where  $\sigma_+$  and  $\sigma_-$  are defined by (3.4). Hence,  $\sigma(\mathcal{A}_{PE}) \cap i\mathbb{R} = \emptyset$  if and only if (SC) holds.  $\square$

**3.3. Exponential and Polynomial Stability Results.** The aim of this subsection is to prove the exponential and the polynomial stabilities of the system (P/E) if (LD – E) holds and under an appropriate assumption on the ratio  $\frac{\sigma_+}{\sigma_-}$ , which depends on its arithmetic nature. Let us consider the following hypotheses

(**H<sub>Exp</sub>**) Assume that  $\frac{\sigma_+}{\sigma_-} \in \mathbb{Q}$  is such that  $\frac{\sigma_+}{\sigma_-} = \frac{\xi_+}{\xi_-}$  where  $\gcd(\xi_+, \xi_-) = 1$ , and  $\xi_+, \xi_-$  are even and odd integers, respectively, or the other way around.

(**H<sub>Pol</sub>**) Assume that  $\frac{\sigma_+}{\sigma_-}$  is an irrational number. Then, suppose that there exists  $\varpi \left( \frac{\sigma_+}{\sigma_-} \right) \geq 2$ , depending on  $\frac{\sigma_+}{\sigma_-}$ , such that for all sequences  $\Lambda = (\xi_{1,n}, \xi_{2,n})_{n \in \mathbb{N}} \in (\mathbb{N} \times \mathbb{N}^*)^{\mathbb{N}}$  with  $\xi_{1,n} \sim \xi_{2,n}$  for sufficiently large  $n$ , there exist a positive constant  $c \left( \frac{\sigma_+}{\sigma_-}, \Lambda \right)$  and a positive integer  $N \left( \frac{\sigma_+}{\sigma_-}, \Lambda \right)$ , depending on  $\frac{\sigma_+}{\sigma_-}$  and the sequence  $\Lambda$ , such that

$$\left| \frac{\sigma_+}{\sigma_-} - \frac{\xi_{1,n}}{\xi_{2,n}} \right| > \frac{c \left( \frac{\sigma_+}{\sigma_-}, \Lambda \right)}{\xi_{2,n}^{\varpi \left( \frac{\sigma_+}{\sigma_-} \right)}}, \forall n \geq N \left( \frac{\sigma_+}{\sigma_-}, \Lambda \right).$$

**Remark 3.4.** (i) Note that it will be shown in section 4 that the number  $\varpi \left( \frac{\sigma_+}{\sigma_-} \right)$  is indeed an irrationality measure of the quotient  $\frac{\sigma_+}{\sigma_-}$ . More explanations on this notion will be under way as well as examples and some references. (ii) Note also that (**H<sub>Exp</sub>**) or (**H<sub>Pol</sub>**) implies that (SC) holds.

The main results of this section are the following theorems.

**Theorem 3.5.** Assume that (LD – E) and (**H<sub>Exp</sub>**) hold. Then, the  $C_0$ -semigroup of contractions  $(e^{t\mathcal{A}_{PE}})_{t \geq 0}$  is exponentially stable, i.e. there exist  $M \geq 1$  and  $\omega > 0$  such that

$$(3.17) \quad \|e^{t\mathcal{A}_{PE}} U_0\|_{\mathcal{H}_{PE}} \leq M e^{-\omega t} \|U_0\|_{\mathcal{H}_{PE}}, \quad \forall U_0 \in \mathcal{H}_{PE}.$$

**Theorem 3.6.** Assume that (LD – E) and (**H<sub>Pol</sub>**) hold. Then, there exists a constant  $C > 0$  such that the energy of the system (P/E) satisfies the following estimate for all  $t > 0$

$$(3.18) \quad \|e^{t\mathcal{A}_{PE}} U_0\|^2 \leq \frac{C}{t^{4\varpi \left( \frac{\sigma_+}{\sigma_-} \right) - 4}} \|U_0\|_{D(\mathcal{A}_{PE})}^2, \quad \forall U_0 \in D(\mathcal{A}_{PE}),$$

By ([27], [40] for Theorem 3.5), or ([9], [31] for Theorem 3.6), the  $C_0$ -semigroup of contractions  $(e^{t\mathcal{A}_{PE}})_{t \geq 0}$  on  $\mathcal{H}_{PE}$  satisfies (3.17) or (3.18) if the following two conditions hold

$$(N1) \quad i\mathbb{R} \subset \rho(\mathcal{A}_{PE}),$$

$$(N2) \quad \sup_{\lambda \in \mathbb{R}} \frac{1}{\lambda^\ell} \|(i\lambda I - \mathcal{A}_{PE})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty \quad \text{with} \quad \begin{cases} \ell = 0, & \text{for Theorem 3.5,} \\ \ell = 4\varpi \left( \frac{\sigma_+}{\sigma_-} \right) - 4, & \text{for Theorem 3.6,} \end{cases}$$

Since it is already proved that  $i\mathbb{R} \subset \rho(\mathcal{A}_{PE})$  (see Section 3.3), it remains to prove the condition (N2), for which a contradiction argument is applicable. Suppose that (N2) is false. Then, there exists a sequence  $\{(\lambda^n, U^n)\}_{n \geq 1} \subset \mathbb{R}^* \times D(\mathcal{A}_{PE})$  with

$$(3.19) \quad |\lambda^n| \rightarrow \infty \quad \text{and} \quad \|U^n\|_{\mathcal{H}} = \|(v^n, z^n, p^n, q^n, y^n, y^{1,n})^\top\|_{\mathcal{H}_{PE}} = 1$$

such that

$$(3.20) \quad \lambda_n^\ell (i\lambda^n I - \mathcal{A}_{PE}) U^n = \mathcal{G}^n := (g^{1,n}, g^{2,n}, g^{3,n}, g^{4,n}, g^{5,n}, g^{6,n})^\top \rightarrow 0 \quad \text{in} \quad \mathcal{H}_{PE}.$$

For simplicity, index  $n$  is dropped for the rest of the proof. Now, (3.20) is equivalent to

$$(3.21) \quad \begin{cases} i\lambda v - z = \lambda^{-\ell} g^1 \rightarrow 0 & \text{in} \quad H_L^1(0, l_1), \\ i\lambda p - q = \lambda^{-\ell} g^3 \rightarrow 0 & \text{in} \quad H_L^1(0, l_1), \\ i\lambda y - y^1 = \lambda^{-\ell} g^5 \rightarrow 0 & \text{in} \quad H_R^1(l_1, L) \end{cases}$$

and

$$(3.22) \quad \begin{cases} i\lambda \rho z - \alpha v_{xx} + \gamma \beta p_{xx} = \rho \lambda^{-\ell} g^2 \rightarrow 0 & \text{in} \quad L^2(0, l_1), \\ i\lambda \mu q - \beta p_{xx} + \gamma \beta v_{xx} = \mu \lambda^{-\ell} g^4 \rightarrow 0 & \text{in} \quad L^2(0, l_1), \\ i\lambda y^1 - c_2 y_{xx} + d_1 y^1 = \lambda^{-\ell} g^6 \rightarrow 0 & \text{in} \quad L^2(l_1, L). \end{cases}$$

Combining (3.21)<sub>3</sub> and (3.22)<sub>3</sub> leads to

$$(3.23) \quad \lambda^2 y + c_2 y_{xx} - i\lambda d_1 y = \lambda^{-\ell} (-i\lambda g^5 - d_1 g^5 - g^6).$$

**Lemma 3.7.** *The solution  $(v, z, p, q, y, y^1)$  of the system (3.21)-(3.22) satisfies the following estimates*

$$(3.24) \quad \int_{l_1}^L d_1 |y^1|^2 dx = o(\lambda^{-\ell}), \quad \int_{l_1}^L d_1 |\lambda y|^2 dx = o(\lambda^{-\ell}), \quad \int_{a_1}^{b_1} |\lambda y|^2 dx = o(\lambda^{-\ell}).$$

and

$$(3.25) \quad \int_{D_\epsilon} |y_x|^2 dx = o(\lambda^{-\ell})$$

where  $D_\epsilon := (a_1 + \epsilon, b_1 - \epsilon)$  with a small enough  $\epsilon > 0$  such that  $\epsilon < \frac{b_1 - a_1}{2}$ .

**Proof.** The proof is split into two steps.

**Step 1.** For obtaining first estimate in (3.24), take the inner product of (3.20) and  $U$  in  $\mathcal{H}_{PE}$ , and use the fact that  $\|U\|_{\mathcal{H}_{PE}} = 1$  and  $\|\mathcal{G}\|_{\mathcal{H}_{PE}} = o(1)$ ,

$$(3.26) \quad \int_{l_1}^L d_1 |y^1|^2 dx = -\Re(\langle \mathcal{A}_{PE} U, U \rangle_{\mathcal{H}_{PE}}) = \Re(\langle (i\lambda I - \mathcal{A}_{PE})U, U \rangle_{\mathcal{H}_{PE}}) = \frac{1}{\lambda^\ell} \Re(\langle \mathcal{G}, U \rangle_{\mathcal{H}_{PE}}) = o(\lambda^{-\ell}).$$

Next, multiply (3.21)<sub>3</sub> by  $\sqrt{d_1}$ , and use the first estimate in (3.24) and  $\|\mathcal{G}\|_{\mathcal{H}} = o(1)$ . This leads to the second estimate in (3.24). Finally, the second estimate in (3.24) with (LD – E) yields the last estimate in (3.24).

**Step 2.** For proving (3.25), let  $0 < \epsilon < \frac{b_1 - a_1}{2}$  and fix the cut-off function  $\theta_5 \in C^2([l_1, L])$  such that  $0 \leq \theta_5(x) \leq 1$  for all  $x \in [l_1, L]$ , and

$$\theta_5(x) = \begin{cases} 1, & \text{if } x \in [a_1 + \epsilon, b_1 - \epsilon], \\ 0, & \text{if } x \in [l_1, a_1] \cup [b_1, L]. \end{cases}$$

Now, multiply (3.23) by  $\theta_5 \bar{y}$ , integrate by parts over  $(l_1, L)$ , and use the definition of  $\theta_5$  to obtain

$$(3.27) \quad \int_{l_1}^L \theta_5 |\lambda y|^2 dx - c_2 \int_{l_1}^L \theta_5 |y_x|^2 dx - c_2 \Re \left( \int_{l_1}^L \theta_5' y_x \bar{y} dx \right) = -\Re \left( \lambda^{-\ell} \int_{l_1}^L (i\lambda g^5 + d_1 g^5 + g^6) \theta_5 \bar{y} dx \right).$$

It is known that  $\|\mathcal{G}\|_{\mathcal{H}_{PE}} = o(1)$  implies that  $(\lambda y)$  is uniformly bounded in  $L^2(l_1, L)$  by (3.21)<sub>3</sub> and  $y_x$  is uniformly bounded in  $L^2(l_1, L)$ . Therefore, by Cauchy-Schwarz inequality, (3.24), and the definition of  $\theta_5$ ,  $\|U\|_{\mathcal{H}_{PE}} = 1$ ,

$$\left| \Re \left( \int_{l_1}^L \theta_5' y_x \bar{y} dx \right) \right| = \left| \frac{1}{2} \int_{l_1}^L \theta_5' (|y|^2)_x dx \right| = \left| \frac{1}{2} \int_{l_1}^L \theta_5'' |y|^2 dx \right| = o(\lambda^{-\ell-2}),$$

and

$$\left| \lambda^{-\ell} \int_{l_1}^L (i\lambda g^5 + d_1 g^5 + g^6) \theta_5 \bar{y} dx \right| = o(\lambda^{-\ell}).$$

Substituting the above estimates into (3.27) and using (3.24) lead to

$$\int_{l_1}^L \theta_5 |y_x|^2 dx = o(\lambda^{-\ell}).$$

Finally, by the definition of  $\theta_5$ , the desired result (3.25) is obtained.  $\square$

**Lemma 3.8.** *The solution  $(v, z, p, q, y, y^1)$  of the system (3.21)-(3.22) satisfies the following estimates*

$$(3.28) \quad \int_{l_1}^L (|\lambda y|^2 + c_2 |y_x|^2) dx = o(\lambda^{-\frac{\ell}{2}}),$$

$$(3.29) \quad |\lambda v(l_1)|^2 = o(\lambda^{-\frac{\ell}{2}}), \quad |v_x(l_1)|^2 = o(\lambda^{-\frac{\ell}{2}}) \quad \text{and} \quad |p_x(l_1)|^2 = o(\lambda^{-\frac{\ell}{2}}).$$

**Proof.** The proof is split into three steps.

**Step 1.** Letting  $h_1 \in C^1([l_1, L])$ , the following estimate is targeted to prove

$$(3.30) \quad - \int_{l_1}^L h_1' (|\lambda y|^2 + c_2 |y_x|^2) dx + h_1(L) (|\lambda y(L)|^2 + c_2 |y_x(L)|^2) - h_1(l_1) (|\lambda y(l_1)|^2 + c_2 |y_x(l_1)|^2) = 2\Re(i\lambda^{1-\ell} h_1(l_1) g^5(l_1) \bar{y}(l_1)) + o(\lambda^{-\frac{\ell}{2}}).$$

First, multiply (3.23) by  $2h_1\bar{y}_x$  and integrate over  $(l_1, L)$  to get

$$\int_{l_1}^L h_1 ( (|\lambda y|^2)_x + c_2(|y_x|^2)_x ) dx - 2\Re \left( i\lambda \int_{l_1}^L d_1 h_1 y \bar{y}_x dx \right) = -2\Re \left( \lambda^{-\ell} \int_{l_1}^L (i\lambda g^5 + d_1 g^5 + g^6) h_1 \bar{y}_x dx \right).$$

As the Integration by parts is implemented,

$$(3.31) \quad \begin{aligned} & - \int_{l_1}^L h_1' (|\lambda y|^2 + c_2|y_x|^2) dx + h_1(L) (|\lambda y(L)|^2 + c_2|y_x(L)|^2) - h_1(l_1) (|\lambda y(l_1)|^2 + c_2|y_x(l_1)|^2) \\ & - 2\Re \left( i\lambda \int_{l_1}^L d_1 h_1 y \bar{y}_x dx \right) = -2\Re \left( \lambda^{-\ell} \int_{l_1}^L (d_1 g^5 + g^6) h \bar{y}_x dx \right) + 2\Re \left( i\lambda^{1-\ell} \int_{l_1}^L (h_1 g^5)_x \bar{y} dx \right) \\ & + 2\Re \left( i\lambda^{1-\ell} h_1(l_1) g^5(l_1) \bar{y}(l_1) \right). \end{aligned}$$

Since  $y_x$  is uniformly bounded in  $L^2(l_1, L)$  and  $\|\mathcal{G}\|_{\mathcal{H}_{PE}} = o(1)$ , by Cauchy-Schwarz inequality and (3.24), the following estimates are immediate

$$(3.32) \quad \begin{aligned} & \left| -2\Re \left( i\lambda \int_{l_1}^L d_1 h_1 y \bar{y}_x dx \right) \right| = o(\lambda^{-\frac{\ell}{2}}), \quad \left| \Re \left( \lambda^{-\ell} \int_{l_1}^L (d_1 g^5 + g^6) h \bar{y}_x dx \right) \right| = o(\lambda^{-\ell}), \\ & \left| \Re \left( i\lambda \int_{l_1}^L (h g^5)_x \bar{y} dx \right) \right| = o(\lambda^{-\ell}). \end{aligned}$$

Finally, substituting these in in (3.31) lead to the desired equation (3.30).

**Step 2.** For this step, (3.28) is aimed to be proved. First, define the following cut-off functions  $\theta_6, \theta_7 \in C^2([l_1, L])$  by

$$\theta_6(x) = \begin{cases} 1 & \text{if } x \in [l_1, a_1 + \epsilon], \\ 0 & \text{if } x \in [a_2 - \epsilon, L] \end{cases}, \quad \text{and} \quad \theta_7(x) = \begin{cases} 0 & \text{if } x \in [l_1, a_1 + \epsilon], \\ 1 & \text{if } x \in [a_2 - \epsilon, L]. \end{cases}$$

so that  $0 \leq \theta_6, \theta_7 \leq 1$  for all  $x \in [l_1, L]$ .

Following the the same arguments as in Lemma 2.12, take  $h_1(x) = (x - l_1)\theta_6(x) + (x - L)\theta_7(x)$  in (3.30). This, combined with Lemma 3.7, results in (3.28).

**Step 3.** Finally, to prove (3.29), take  $h_1(x) = x - L$  in (3.30) and use (3.28) to obtain

$$|\lambda y(l_1)|^2 + c_2|y_x(l_1)|^2 = 2\Re \left( -i\lambda^{1-\ell} g^5(l_1) \bar{y}(l_1) \right) + o(\lambda^{-\frac{\ell}{2}}).$$

It follows from Young's inequality that

$$|\lambda y(l_1)|^2 + c_2|y_x(l_1)|^2 \leq 2|\lambda^{-\ell} g^5(l_1)| |\lambda y(l_1)| + o(\lambda^{-\frac{\ell}{2}}) \leq 2\lambda^{-2\ell} |g^5(l_1)|^2 + \frac{1}{2} |\lambda y(l_1)|^2 + o(\lambda^{-\frac{\ell}{2}}).$$

Now, use  $g^5 \in H_R^1(l_1, L) \subset C([l_1, L])$  and  $\|\mathcal{G}\|_{\mathcal{H}_{PE}} = o(1)$  to obtain  $|g^5(l_1)| = o(1)$ . This, together with the estimate above, provides the following estimate

$$\frac{1}{2} |\lambda y(l_1)|^2 + c_2|y_x(l_1)|^2 \leq o(\lambda^{-\frac{\ell}{2}}).$$

Hence, (3.29) is concluded from recalling Remark (3.1).  $\square$

For the next result, substitute (3.21)<sub>1</sub> and (3.21)<sub>2</sub> in (3.22)<sub>1</sub> and (3.22)<sub>2</sub>, respectively, so that the following system is obtained,

$$(3.33) \quad \begin{cases} \rho \lambda^2 v + \alpha v_{xx} - \gamma \beta p_{xx} = -\lambda^{-\ell} (\rho g^2 + i\lambda \rho g^1) \\ \mu \lambda^2 p + \beta p_{xx} - \gamma \beta v_{xx} = -\lambda^{-\ell} (\mu g^4 + i\lambda \mu g^3). \end{cases}$$

**Lemma 3.9.** *The solution  $(v, z, p, q, y, y^1)$  of the system (3.21)-(3.22) satisfies the following estimate*

$$(3.34) \quad \rho \int_0^{l_1} |\lambda v|^2 dx + \mu \int_0^{l_1} |\lambda p|^2 + \alpha_1 \int_0^{l_1} |v_x|^2 dx + \beta \int_0^{l_1} |\gamma v_x - p_x|^2 dx - 2\mu l_1 |\lambda p(l_1)|^2 \leq o(\lambda^{-\frac{\ell}{2}}).$$

**Proof.** First, use the multipliers  $-2x\bar{v}_x$  and  $-2x\bar{p}_x$  for (3.33)<sub>1</sub> and (3.33)<sub>2</sub>, respectively, and integrate over  $(0, l_1)$  to get

$$-\rho \int_0^{l_1} x (|\lambda v|^2)_x dx - \alpha \int_0^{l_1} x (|v_x|^2)_x dx + 2\gamma \beta \Re \left( \int_0^{l_1} x p_{xx} \bar{v}_x dx \right) = 2\Re \left( \lambda^{-\ell} \int_0^{l_1} x (\rho g^2 + i\lambda \rho g^1) \bar{v}_x dx \right),$$

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$$-\mu \int_0^{l_1} x(|\lambda p|^2)_x dx - \beta \int_0^{l_1} x(|p_x|^2)_x dx + 2\gamma\beta\Re \left( \int_0^{l_1} x v_{xx} \bar{p}_x dx \right) = 2\Re \left( \lambda^{-\ell} \int_0^{l_1} x(\mu g^4 + i\lambda\mu g^3) \bar{p}_x dx \right).$$

Next, integrate by parts the identities above, and use (3.29) and  $\alpha = \alpha_1 + \gamma^2\beta$ ,

$$(3.35) \quad \begin{aligned} & \rho \int_0^{l_1} |\lambda v|^2 dx + \alpha_1 \int_0^{l_1} |v_x|^2 dx + \gamma^2\beta \int_0^{l_1} |v_x|^2 dx - 2\gamma\beta\Re \left( \int_0^{l_1} p_x \bar{v}_x dx \right) \\ & - 2\gamma\beta\Re \left( \int_0^{l_1} x p_x \bar{v}_{xx} dx \right) = 2\rho\Re \left( \lambda^{-\ell} \int_0^{l_1} x g^2 \bar{v}_x dx \right) - 2\Re \left( i\lambda^{1-\ell} \rho \int_0^{l_1} (xg^1)_x \bar{v} dx \right) + o(\lambda^{-\frac{\ell}{2}}), \end{aligned}$$

$$(3.36) \quad \begin{aligned} & \mu \int_0^{l_1} |\lambda p|^2 dx - \mu |\lambda p(l_1)|^2 + \beta \int_0^{l_1} |p_x|^2 dx + 2\gamma\beta\Re \left( \int_0^{l_1} x v_{xx} \bar{p}_x dx \right) = \\ & 2\mu\Re \left( \lambda^{-\ell} \int_0^{l_1} x g^4 \bar{p}_x dx \right) - 2\Re \left( i\lambda^{1-\ell} \mu \int_0^{l_1} (xg^3)_x \bar{p} dx \right) + 2\Re \left( i\lambda^{1-\ell} l_1 \mu g^3(l_1) \bar{p}(l_1) \right) + o(\lambda^{-\frac{\ell}{2}}). \end{aligned}$$

Since  $\|\mathcal{G}\|_{\mathcal{H}_{PE}} = o(1)$ , and  $v_x, p_x, \lambda v, \lambda p$  are uniformly bounded in  $L^2(0, l_1)$ ,

$$\begin{aligned} & \left| \Re \left( \lambda^{-\ell} \int_0^{l_1} x g^2 \bar{v}_x dx \right) \right| = o(\lambda^{-\ell}), & \left| 2\Re \left( i\lambda^{1-\ell} \rho \int_0^{l_1} (xg^1)_x \bar{v} dx \right) \right| = o(\lambda^{-\ell}), \\ & \left| \Re \left( \lambda^{-\ell} \int_0^{l_1} x g^4 \bar{p}_x dx \right) \right| = o(\lambda^{-\ell}), & \left| \Re \left( i\lambda^{1-\ell} \mu \int_0^{l_1} (xg^3)_x \bar{p} dx \right) \right| = o(\lambda^{-\ell}). \end{aligned}$$

Thus, (3.35) and (3.36), together with the last two estimates, reduce to

$$(3.37) \quad \begin{aligned} & \rho \int_0^{l_1} |\lambda v|^2 dx + \mu \int_0^{l_1} |\lambda p|^2 + \alpha_1 \int_0^{l_1} |v_x|^2 dx + \beta \int_0^{l_1} |\gamma v_x - p_x|^2 dx \\ & - \mu l_1 |\lambda p(l_1)|^2 = 2\Re \left( i\lambda^{1-\ell} l_1 \mu g^3(l_1) \bar{p}(l_1) \right) + o(\lambda^{-\frac{\ell}{2}}). \end{aligned}$$

On the other hand, since  $g^3 \in H_R^1(0, l_1) \subset C([0, l_1])$  and  $\|\mathcal{G}\|_{\mathcal{H}_{PE}} = o(1)$ , by Young's inequality

$$|2\Re \left( i\lambda^{1-\ell} l_1 \mu g^3(l_1) \bar{p}(l_1) \right)| \leq l_1 \mu |\lambda p(l_1)|^2 + l_1 \mu \lambda^{-2\ell} |g^3(l_1)|^2 \leq l_1 \mu |\lambda p(l_1)|^2 + o(\lambda^{-2\ell}).$$

Finally, substituting the estimate above in (3.37) lead to (3.34).  $\square$

For the next result, another form of (3.33) is needed by considering  $\alpha = \alpha_1 + \gamma^2\beta$ :

$$(3.38) \quad \begin{cases} \lambda^2 \rho v + \alpha_1 v_{xx} + \gamma \mu \lambda^2 p = \lambda^{-\ell} G^1 + i\lambda^{1-\ell} G^2, \\ \lambda^2 \mu \alpha p + \alpha_1 \beta p_{xx} + \rho \gamma \beta \lambda^2 v = \lambda^{-\ell} G^3 + i\lambda^{1-\ell} G^4, \end{cases}$$

where

$$(3.39) \quad \begin{aligned} G^1 &= -(\rho g^2 + \gamma \mu g^4), & G^2 &= -(\rho g^1 + \gamma \mu g^3), \\ G^3 &= -(\alpha \mu g^4 + \rho \gamma \beta g^2), & G^4 &= -(\alpha \mu g^3 + \rho \gamma \beta g^1). \end{aligned}$$

**Lemma 3.10.** *The solution  $(v, z, p, q, y, y^1)$  of the system (3.21)-(3.22) satisfies the following asymptotic estimate*

$$(3.40) \quad \begin{aligned} & e_1(l_1)p(l_1) = -e_2(l_1)o(\lambda^{-(\frac{\ell}{4}+1)}) - e_3(l_1)o(\lambda^{-\frac{\ell}{4}}) - e_4(l_1)o(\lambda^{-\frac{\ell}{4}}) \\ & - \lambda^{-\ell} \int_0^{l_1} (e_3(s)G^1(s) + e_4(s)G^3(s)) ds - i\lambda^{1-\ell} \int_0^{l_1} (e_3(s)G^2(s) + e_4(s)G^4(s)) ds, \end{aligned}$$

$$(3.41) \quad \begin{aligned} & e_5(l_1)p(l_1) = -e_6(l_1)o(\lambda^{-(\frac{\ell}{4}+1)}) - e_7(l_1)o(\lambda^{-\frac{\ell}{4}}) - e_8(l_1)o(\lambda^{-\frac{\ell}{4}}) \\ & - \lambda^{-\ell} \int_0^{l_1} (e_7(s)G^1(s) + e_8(s)G^3(s)) ds - i\lambda^{1-\ell} \int_0^{l_1} (e_7(s)G^2(s) + e_8(s)G^4(s)) ds, \end{aligned}$$

with

$$(3.42) \quad \begin{cases} e_1(s) = \frac{\cos(s\kappa_+) - \cos(s\kappa_-)}{b_+ - b_-}, & e_2(s) = \frac{b_+ \cos(s\kappa_-) - b_- \cos(s\kappa_+)}{b_+ - b_-}, \\ e_3(s) = -\frac{b_+ \kappa_+ \sin(s\kappa_-) - b_- \kappa_- \sin(s\kappa_+)}{\kappa_- \kappa_+ (b_+ - b_-)}, & e_4(s) = -\frac{\kappa_- \sin(s\kappa_+) - \kappa_+ \sin(s\kappa_-)}{\kappa_- \kappa_+ (b_+ - b_-)}, \\ e_5(s) = \frac{b_+ \cos(s\kappa_+) - b_- \cos(s\kappa_-)}{b_+ - b_-}, & e_6(s) = -b_+ b_- \frac{\cos(s\kappa_+) - \cos(s\kappa_-)}{b_+ - b_-}, \\ e_7(s) = -\frac{b_+ b_- (\kappa_+ \sin(s\kappa_-) - \kappa_- \sin(s\kappa_+))}{\kappa_- \kappa_+ (b_+ - b_-)}, & e_8(s) = -\frac{b_+ \kappa_- \sin(s\kappa_+) - b_- \kappa_+ \sin(s\kappa_-)}{\kappa_- \kappa_+ (b_+ - b_-)}, \end{cases}$$

and

$$(3.43) \quad \kappa_- = \lambda \sigma_- \quad \kappa_+ = \lambda \sigma_+, \quad b_+ = \frac{\alpha_1 \kappa_+^2 - \lambda^2 \rho}{\gamma \mu \lambda^2} \quad \text{and} \quad b_- = \frac{\alpha_1 \kappa_-^2 - \lambda^2 \rho}{\gamma \mu \lambda^2},$$

where  $\sigma_+$  and  $\sigma_-$  defined in (3.4).

**Proof.** Firstly, note that (3.4) and (3.43) directly imply that

$$(3.44) \quad \begin{cases} b_+ = \frac{\alpha \mu - \rho \beta + \sqrt{(\alpha \mu - \rho \beta)^2 + 4\gamma^2 \beta^2 \mu \rho}}{2\beta \gamma \mu} \neq 0, & b_- = \frac{\alpha \mu - \rho \beta - \sqrt{(\alpha \mu - \rho \beta)^2 + 4\gamma^2 \beta^2 \mu \rho}}{2\beta \gamma \mu} \neq 0, \\ b_+ - b_- = \frac{\sqrt{(\alpha \mu - \rho \beta)^2 + 4\gamma^2 \beta^2 \mu \rho}}{\beta \gamma \mu} \neq 0, & b_+ b_- = -\frac{\rho}{\mu} \neq 0, \\ \kappa_+ \kappa_- = \lambda^2 \sigma_+ \sigma_- \neq 0, & \text{and} \quad \frac{\kappa_+}{\kappa_-} = \frac{\sigma_+}{\sigma_-} \neq 0. \end{cases}$$

which, indeed, better explains the expressions of  $e_i, i = 1, \dots, 8$  in (3.42).

Let  $U^{PE} = (v, v_x, p, p_x)^\top$ . By (3.29),

$$(3.45) \quad U^{PE}(0) = (0, v_x(0), 0, p_x(0)), \quad U^{PE}(l_1) = \left( o(\lambda^{-(\frac{\ell}{4}+1)}), o(\lambda^{-\frac{\ell}{4}}), p(l_1), o(\lambda^{-\frac{\ell}{4}}) \right)^\top,$$

and therefore, the system (3.38) can be written as

$$(3.46) \quad U_x^{PE} = N^{PE} U^{PE} + G$$

where

$$(3.47) \quad N^{PE} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{\lambda^2 \rho}{\alpha_1} & 0 & -\frac{\lambda^2 \gamma \mu}{\alpha_1} & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{\lambda^2 \rho \gamma}{\alpha_1} & 0 & -\frac{\lambda^2 \mu \alpha}{\alpha_1 \beta} & 0 \end{pmatrix} \quad \text{and} \quad G = \begin{pmatrix} 0 \\ \lambda^{-\ell} G^1 + i \lambda^{1-\ell} G^2 \\ 0 \\ \lambda^{-\ell} G^3 + i \lambda^{1-\ell} G^4 \end{pmatrix}.$$

Notice that the eigenvalues  $\kappa$  of the matrix  $N^{PE}$  are the roots of the following characteristic equation

$$\varsigma(\kappa) = \frac{q(\kappa)}{\alpha_1 \beta}$$

where  $q(\kappa)$  is defined in (3.12). This characteristic equation has four distinct pure imaginary roots  $i\kappa_-, -i\kappa_-, i\kappa_+, -i\kappa_+$  where  $\kappa_+$  and  $\kappa_-$  are defined in (3.43). Since the eigenvalues of  $N^{PE}$  are simple,  $N^{PE}$  is a diagonalizable matrix, i.e.,  $N^{PE}$  can be written as  $N^{PE} = P N_1^{PE} P^{-1}$  such that

$$P = \begin{pmatrix} 1 & 1 & 1 & 1 \\ i\kappa_+ & -i\kappa_+ & i\kappa_- & -i\kappa_- \\ b_+ & b_+ & b_- & b_- \\ i\kappa_+ b_+ & -i\kappa_+ b_+ & i\kappa_- b_- & -i\kappa_- b_- \end{pmatrix}, \quad N_1^{PE} = \begin{pmatrix} i\kappa_+ & 0 & 0 & 0 \\ 0 & -i\kappa_+ & 0 & 0 \\ 0 & 0 & i\kappa_- & 0 \\ 0 & 0 & 0 & i\kappa_- \end{pmatrix},$$

and

$$P^{-1} = \frac{1}{2(b_+ - b_-)} \begin{pmatrix} -b_- & \frac{ib_-}{\kappa_+} & 1 & -\frac{i}{\kappa_+} \\ -b_- & -\frac{ib_-}{\kappa_+} & 1 & \frac{i}{\kappa_+} \\ b_+ & -\frac{ib_+}{\kappa_-} & -1 & \frac{i}{\kappa_-} \\ b_+ & \frac{ib_+}{\kappa_-} & -1 & -\frac{i}{\kappa_-} \end{pmatrix}$$

where  $b_+$  and  $b_-$  are defined in (3.43). Therefore, for all  $s \in \mathbb{R}$ ,  $E(s) := e^{N^{PE}s} = Pe^{N_1^{PE}s}P^{-1}$ , or equivalently,

$$(3.48) \quad E(s) := e^{N^{PE}s} = (E_{ij}(s))_{1 \leq i \leq 4, 1 \leq j \leq 4},$$

with

$$(3.49) \quad \left\{ \begin{array}{ll} E_{11}(s) = E_{22}(s) = \frac{b_+ \cos(s\kappa_-) - b_- \cos(s\kappa_+)}{b_+ - b_-}, & E_{12}(s) = \frac{b_+ \kappa_+ \sin(s\kappa_-) - b_- \kappa_- \sin(s\kappa_+)}{\kappa_- \kappa_+ (b_+ - b_-)}, \\ E_{13}(s) = \frac{\cos(s\kappa_+) - \cos(s\kappa_-)}{b_+ - b_-}, & E_{14}(s) = \frac{\kappa_- \sin(s\kappa_+) - \kappa_+ \sin(s\kappa_-)}{\kappa_- \kappa_+ (b_+ - b_-)}, \\ E_{21}(s) = \frac{\kappa_+ b_- \sin(s\kappa_+) - \kappa_- b_+ \sin(s\kappa_-)}{b_+ - b_-}, & E_{23}(s) = \frac{\kappa_- \sin(s\kappa_-) - \kappa_+ \sin(s\kappa_+)}{b_+ - b_-}, \\ E_{32}(s) = \frac{b_+ b_- (\kappa_+ \sin(s\kappa_-) - \kappa_- \sin(s\kappa_+))}{\kappa_- \kappa_+ (b_+ - b_-)}, & E_{33}(s) = E_{44}(s) = \frac{b_+ \cos(s\kappa_+) - b_- \cos(s\kappa_-)}{b_+ - b_-}, \\ E_{34}(s) = \frac{b_+ \kappa_- \sin(s\kappa_+) - b_- \kappa_+ \sin(s\kappa_-)}{\kappa_- \kappa_+ (b_+ - b_-)}, & E_{43}(s) = \frac{b_- \kappa_- \sin(s\kappa_-) - b_+ \kappa_+ \sin(s\kappa_+)}{b_+ - b_-}, \\ E_{24}(s) = E_{13}(s), \quad E_{31}(s) = -b_+ b_- E_{13}(s), \\ E_{41}(s) = -b_+ b_- E_{23}(s), \quad E_{42}(s) = -b_+ b_- E_{24}(s). \end{array} \right.$$

By the classical arguments from the theory of ordinary differential equations, the solution of (3.46) is given by

$$U^{PE}(x) = e^{N^{PE}(x-l_1)} U^{PE}(l_1) - \int_x^{l_1} e^{N^{PE}(x-s)} G(s) ds.$$

with

$$U^{PE}(0) = e^{-N^{PE}l_1} U^{PE}(l_1) - \int_0^{l_1} e^{-N^{PE}s} G(s) ds.$$

Next, substitute (3.45) and (3.47) and (3.49) into the above equation to obtain

$$\begin{pmatrix} 0 \\ v_x(0) \\ 0 \\ p_x(0) \end{pmatrix} = E(-l_1) \begin{pmatrix} o(\lambda^{-(\frac{\ell}{4}+1)}) \\ o(\lambda^{-\frac{\ell}{4}}) \\ p(l_1) \\ o(\lambda^{-\frac{\ell}{4}}) \end{pmatrix} + \int_0^{l_1} E(-s) \begin{pmatrix} 0 \\ \lambda^{-\ell} G^1(s) + i\lambda^{1-\ell} G^2(s) \\ 0 \\ \lambda^{-\ell} G^3(s) + i\lambda^{1-\ell} G^4(s) \end{pmatrix} ds.$$

This together with (3.49) yields

$$\begin{aligned} E_{13}(-l_1)p(l_1) &= -E_{11}(-l_1)o(\lambda^{-(\frac{\ell}{4}+1)}) - E_{12}(-l_1)o(\lambda^{-\frac{\ell}{4}}) - E_{14}(-l_1)o(\lambda^{-\frac{\ell}{4}}) \\ &- \lambda^{-\ell} \int_0^{l_1} (E_{12}(-s)G^1(s) + E_{14}(-s)G^3(s)) ds - i\lambda^{1-\ell} \int_0^{l_1} (E_{12}(-s)G^2(s) + E_{14}(-s)G^4(s)) ds, \end{aligned}$$

and

$$\begin{aligned} E_{33}(-l_1)p(l_1) &= -E_{31}(-l_1)o(\lambda^{-(\frac{\ell}{4}+1)}) - E_{32}(-l_1)o(\lambda^{-\frac{\ell}{4}}) - E_{34}(-l_1)o(\lambda^{-\frac{\ell}{4}}) \\ &- \lambda^{-\ell} \int_0^{l_1} (E_{32}(-s)G^1(s) + E_{34}(-s)G^3(s)) ds - i\lambda^{1-\ell} \int_0^{l_1} (E_{32}(-s)G^2(s) + E_{34}(-s)G^4(s)) ds, \end{aligned}$$

which, thus, lead to (3.40) and (3.41).  $\square$

**Lemma 3.11.** *The solution  $(v, z, p, q, y, y^1)$  of the system (3.21)-(3.22) satisfies the following asymptotic estimates*

$$(3.50) \quad (\cos(l_1 \kappa_+) - \cos(l_1 \kappa_-)) \lambda p(l_1) = o(\lambda^{-\frac{\ell}{4}}), \quad (b_+ \cos(l_1 \kappa_+) - b_- \cos(l_1 \kappa_-)) \lambda p(l_1) = o(\lambda^{-\frac{\ell}{4}}).$$

**Proof.** First, the following estimates are immediate by (3.42), (3.44), (3.39), and  $\|\mathcal{G}\|_{\mathcal{H}_{PE}} = o(1)$ ,

$$(3.51) \quad \begin{cases} |e_2(l_1)o(\lambda^{-(\frac{\ell}{4}+1)})| = o(\lambda^{-(\frac{\ell}{4}+1)}), & |e_3(l_1)o(\lambda^{-\frac{\ell}{4}})| = o(\lambda^{-(\frac{\ell}{4}+1)}), & |e_4(l_1)o(\lambda^{-\frac{\ell}{4}})| = o(\lambda^{-(\frac{\ell}{4}+1)}), \\ |e_6(l_1)o(\lambda^{-(\frac{\ell}{4}+1)})| = o(\lambda^{-(\frac{\ell}{4}+1)}), & |e_7(l_1)o(\lambda^{-\frac{\ell}{4}})| = o(\lambda^{-(\frac{\ell}{4}+1)}), & |e_8(l_1)o(\lambda^{-\frac{\ell}{4}})| = o(\lambda^{-(\frac{\ell}{4}+1)}), \\ \left| \lambda^{-\ell} \int_0^{l_1} (e_3(s)G^1(s) + e_4(s)G^3(s)) ds \right| = o(\lambda^{-(1+\ell)}), \\ \left| \lambda^{-\ell} \int_0^{l_1} (e_7(s)G^1(s) + e_8(s)G^3(s)) ds \right| = o(\lambda^{-(1+\ell)}). \end{cases}$$



Now, integrate by parts to obtain

$$\begin{aligned}
i\lambda^{1-\ell} \int_0^{l_1} e_3(s)G^2(s)ds &= \frac{i\lambda^{1-\ell}G^2(l_1)}{\varkappa_- \varkappa_+ (b_+ - b_-)} \left[ b_+ \frac{\varkappa_+}{\varkappa_-} \cos(l_1 \varkappa_-) - b_- \frac{\varkappa_-}{\varkappa_+} \cos(l_1 \varkappa_+) \right] \\
&\quad - \frac{i\lambda^{1-\ell}}{\varkappa_- \varkappa_+ (b_+ - b_-)} \int_0^{l_1} \left[ b_+ \frac{\varkappa_+}{\varkappa_-} \cos(s \varkappa_-) - b_- \frac{\varkappa_-}{\varkappa_+} \cos(s \varkappa_+) \right] G_s^2(s) ds, \\
i\lambda^{1-\ell} \int_0^{l_1} e_4(s)G^4(s)ds &= \frac{i\lambda^{1-\ell}G^4(l_1)}{\varkappa_- \varkappa_+ (b_+ - b_-)} \left[ \frac{\varkappa_-}{\varkappa_+} \cos(l_1 \varkappa_+) - \frac{\varkappa_+}{\varkappa_-} \cos(l_1 \varkappa_-) \right] \\
&\quad - \frac{i\lambda^{1-\ell}}{\varkappa_- \varkappa_+ (b_+ - b_-)} \int_0^{l_1} \left[ \frac{\varkappa_-}{\varkappa_+} \cos(l_1 \varkappa_+) - \frac{\varkappa_+}{\varkappa_-} \cos(l_1 \varkappa_-) \right] G_s^4(s) ds, \\
i\lambda^{1-\ell} \int_0^{l_1} e_7(s)G^2(s)ds &= \frac{i\lambda^{1-\ell}b_+b_-G^2(l_1)}{\varkappa_- \varkappa_+ (b_+ - b_-)} \left[ \frac{\varkappa_+}{\varkappa_-} \cos(l_1 \varkappa_-) - \frac{\varkappa_-}{\varkappa_+} \cos(l_1 \varkappa_+) \right] \\
&\quad - \frac{i\lambda^{1-\ell}b_+b_-}{\varkappa_- \varkappa_+ (b_+ - b_-)} \int_0^{l_1} \left[ \frac{\varkappa_+}{\varkappa_-} \cos(s \varkappa_-) - \frac{\varkappa_-}{\varkappa_+} \cos(s \varkappa_+) \right] G_s^2(s) ds, \\
i\lambda^{1-\ell} \int_0^{l_1} e_8(s)G^4(s)ds &= \frac{i\lambda^{1-\ell}G^4(l_1)}{\varkappa_- \varkappa_+ (b_+ - b_-)} \left[ b_+ \frac{\varkappa_-}{\varkappa_+} \cos(l_1 \varkappa_+) - b_- \frac{\varkappa_+}{\varkappa_-} \cos(l_1 \varkappa_-) \right] \\
&\quad - \frac{i\lambda^{1-\ell}}{\varkappa_- \varkappa_+ (b_+ - b_-)} \int_0^{l_1} \left[ b_+ \frac{\varkappa_-}{\varkappa_+} \cos(s \varkappa_+) - b_- \frac{\varkappa_+}{\varkappa_-} \cos(s \varkappa_-) \right] G_s^4(s) ds.
\end{aligned}$$

Next, by (3.51) and  $G^2, G^4 \in H_L^1(0, l_1) \subset C([0, l_1])$ ,

$$\begin{aligned}
(3.52) \quad & \left| i\lambda^{1-\ell} \int_0^{l_1} e_3(s)G^2(s)ds \right| \leq \left| \frac{i\lambda^{1-\ell}G^2(l_1)}{\varkappa_- \varkappa_+ (b_+ - b_-)} \left[ b_+ \frac{\varkappa_+}{\varkappa_-} \cos(l_1 \varkappa_-) - b_- \frac{\varkappa_-}{\varkappa_+} \cos(l_1 \varkappa_+) \right] \right| \\
& + \left| \frac{i\lambda^{1-\ell}}{\varkappa_- \varkappa_+ (b_+ - b_-)} \int_0^{l_1} \left[ b_+ \frac{\varkappa_+}{\varkappa_-} \cos(s \varkappa_-) - b_- \frac{\varkappa_-}{\varkappa_+} \cos(s \varkappa_+) \right] G_s^2(s) ds \right| \leq o(\lambda^{-(1+\ell)}), \\
& \left| i\lambda^{1-\ell} \int_0^{l_1} e_4(s)G^4(s)ds \right| \leq \left| \frac{i\lambda^{1-\ell}G^4(l_1)}{\varkappa_- \varkappa_+ (b_+ - b_-)} \left[ \frac{\varkappa_-}{\varkappa_+} \cos(l_1 \varkappa_+) - \frac{\varkappa_+}{\varkappa_-} \cos(l_1 \varkappa_-) \right] \right| \\
& + \left| \frac{i\lambda^{1-\ell}}{\varkappa_- \varkappa_+ (b_+ - b_-)} \int_0^{l_1} \left[ \frac{\varkappa_-}{\varkappa_+} \cos(l_1 \varkappa_+) - \frac{\varkappa_+}{\varkappa_-} \cos(l_1 \varkappa_-) \right] G_s^4(s) ds \right| \leq o(\lambda^{-(1+\ell)}),
\end{aligned}$$

$$\begin{aligned}
(3.53) \quad & \left| i\lambda^{1-\ell} \int_0^{l_1} e_7(s)G^2(s)ds \right| \leq \left| \frac{i\lambda^{1-\ell}b_+b_-G^2(l_1)}{\varkappa_- \varkappa_+ (b_+ - b_-)} \left[ \frac{\varkappa_+}{\varkappa_-} \cos(l_1 \varkappa_-) - \frac{\varkappa_-}{\varkappa_+} \cos(l_1 \varkappa_+) \right] \right| \\
& + \left| \frac{i\lambda^{1-\ell}b_+b_-}{\varkappa_- \varkappa_+ (b_+ - b_-)} \int_0^{l_1} \left[ \frac{\varkappa_+}{\varkappa_-} \cos(s \varkappa_-) - \frac{\varkappa_-}{\varkappa_+} \cos(s \varkappa_+) \right] G_s^2(s) ds \right| \leq o(\lambda^{-(1+\ell)}), \\
& \left| i\lambda^{1-\ell} \int_0^{l_1} e_8(s)G^4(s)ds \right| \leq \left| \frac{i\lambda^{1-\ell}G^4(l_1)}{\varkappa_- \varkappa_+ (b_+ - b_-)} \left[ b_+ \frac{\varkappa_-}{\varkappa_+} \cos(l_1 \varkappa_+) - b_- \frac{\varkappa_+}{\varkappa_-} \cos(l_1 \varkappa_-) \right] \right| \\
& + \left| \frac{i\lambda^{1-\ell}}{\varkappa_- \varkappa_+ (b_+ - b_-)} \int_0^{l_1} \left[ b_+ \frac{\varkappa_-}{\varkappa_+} \cos(s \varkappa_+) - b_- \frac{\varkappa_+}{\varkappa_-} \cos(s \varkappa_-) \right] G_s^4(s) ds \right| \leq o(\lambda^{-(1+\ell)}).
\end{aligned}$$

Finally, (3.50) follows from (3.44) and substituting (3.51), (3.52) and (3.53) into (3.40) and (3.41).  $\square$

**Lemma 3.12.** *Assume (LD – E) . Let  $\ell = 0$  (if  $(\mathbf{H}_{\text{Exp}})$  holds) or  $\ell = 4\varpi \left( \frac{\sigma_+}{\sigma_-} \right) - 4$  (if  $(\mathbf{H}_{\text{Pol}})$  holds). Then, the solution  $(v, z, p, q, y, y^1)$  of the system (3.21)-(3.22) satisfies the following estimate*

$$(3.54) \quad |\lambda p(l_1)| = o(1).$$

**Proof.** Since  $\ell \geq 0$ , it is easy to see that  $|\lambda p(l_1)| = O(1)$  from (3.37). Now assume that (3.54) does not hold. Then, there exists a positive constant  $cst$  and a subsequence such that  $|\lambda p(l_1)| \geq cst$ . By (3.50),

$$\cos(l_1 \varkappa_+) - \cos(l_1 \varkappa_-) = o(\lambda^{-\frac{\ell}{4}}) \quad \text{and} \quad b_+ \cos(l_1 \varkappa_+) - b_- \cos(l_1 \varkappa_-) = o(\lambda^{-\frac{\ell}{4}}),$$

from which, the following is obtained

$$\mathcal{M}(b_+, b_-) \begin{pmatrix} \cos(l_1 \varkappa_+) \\ \cos(l_1 \varkappa_-) \end{pmatrix} = \begin{pmatrix} o(\lambda^{-\frac{\ell}{4}}) \\ o(\lambda^{-\frac{\ell}{4}}) \end{pmatrix}, \quad \text{and } \mathcal{M}(b_+, b_-) = \begin{pmatrix} 1 & -1 \\ b_+ & -b_- \end{pmatrix}.$$

From (3.44),  $\det(\mathcal{M}(b_+, b_-)) = b_+ - b_- \neq 0$ , and thus,

$$\cos(l_1 \varkappa_+) = o(\lambda^{-\frac{\ell}{4}}) \quad \text{and} \quad \cos(l_1 \varkappa_-) = o(\lambda^{-\frac{\ell}{4}}).$$

This together with (3.43) imply that there exists  $n_+^1, n_-^1 \in \mathbb{Z}$  such that

$$\lambda = \frac{(2n_+^1 + 1)\pi}{2\sigma_+ l_1} + o(\lambda^{-\frac{\ell}{4}}) \quad \text{and} \quad \lambda = \frac{(2n_-^1 + 1)\pi}{2\sigma_- l_1} + o(\lambda^{-\frac{\ell}{4}}).$$

Since  $\lambda$  is large enough, i.e.,  $\lambda \sim \frac{(2n_+^1 + 1)\pi}{2\sigma_+ l_1} \sim \frac{(2n_-^1 + 1)\pi}{2\sigma_- l_1}$ ,

$$\frac{(2n_+^1 + 1)\pi}{2\sigma_+ l_1} + o(\lambda^{-\frac{\ell}{4}}) = \frac{(2n_-^1 + 1)\pi}{2\sigma_- l_1} + o(\lambda^{-\frac{\ell}{4}}).$$

and therefore,

$$(3.55) \quad (2n_+^1 + 1) - (2n_-^1 + 1) \frac{\sigma_+}{\sigma_-} = o(\lambda^{-\frac{\ell}{4}}).$$

- Assume that  $(\mathbf{H}_{\text{Exp}})$  holds and take  $\ell = 0$ . Then, by (3.55)

$$\frac{(2n_+ + 1)\xi_- - (2n_- + 1)\xi_+}{\xi_-} = o(1).$$

It is known that  $|(2n_+^1 + 1)\xi_- - (2n_-^1 + 1)\xi_+| \geq 1$  since  $(2n_-^1 + 1)\xi_+$  is an even number and  $(2n_+^1 + 1)\xi_-$  is an odd number, or  $(2n_-^1 + 1)\xi_+$  is an odd number and  $(2n_+^1 + 1)\xi_-$  is an even number. However, this contradicts with  $\frac{1}{\xi_-} \leq o(1)$ . Consequently, (3.54) is obtained as  $(\mathbf{H}_{\text{Exp}})$  holds.

- Assume  $(\mathbf{H}_{\text{Pol}})$  and choose  $\ell = 4\varpi \left(\frac{\sigma_+}{\sigma_-}\right) - 4$ . Since  $\lambda \sim \frac{(2n_+^1 + 1)\pi}{2\sigma_+ l_1} \sim \frac{(2n_-^1 + 1)\pi}{2\sigma_- l_1}$  and by (3.55),

$$\frac{\sigma_+}{\sigma_-} - \frac{2n_+^1 + 1}{2n_-^1 + 1} = o(\lambda^{-\varpi \left(\frac{\sigma_+}{\sigma_-}\right)}).$$

However, the contradiction  $c \left(\frac{\sigma_+}{\sigma_-}, \Lambda\right) \leq o(1)$  is immediate by  $(\mathbf{H}_{\text{Pol}})$  with the sequence  $\Lambda = ((2n_+^1 + 1, 2n_-^1 + 1))_{n \in \mathbb{N}}$ . Consequently, (3.54) is obtained as  $(\mathbf{H}_{\text{Pol}})$  holds.  $\square$

**Proof of Theorem 3.5.** Assume  $(\mathbf{H}_{\text{Exp}})$ . Then, (3.28), (3.34) and (3.54) results in  $\|U\|_{\mathcal{H}_{PE}} = o(1)$ , which contradicts by (3.19). Consequently, the condition (N2) holds true.

**Proof of Theorem 3.6.** Assume  $(\mathbf{H}_{\text{Pol}})$  and take  $\ell = 4\varpi \left(\frac{\sigma_+}{\sigma_-}\right) - 4$ . Then, (3.28), (3.34) and (3.54) result in  $\|U\|_{\mathcal{H}_{PE}} = o(1)$ , which contradicts by (3.19). Consequently, the condition (N2) holds true.

#### 4. ILLUSTRATION OF THE HYPOTHESIS $(\mathbf{H}_{\text{Pol}})$

In this section, some examples are provided for the hypothesis  $(\mathbf{H}_{\text{Pol}})$  to hold true. For this purpose, we start with the notion of badly approximable real numbers.

**Definition 4.1.** [10, Definition 1.3] *A real number  $\xi$  is badly approximable if there is a positive constant  $c(\xi)$  such that for every rational number  $\frac{p}{q} \neq \xi$ ,*

$$(4.1) \quad \left| \xi - \frac{p}{q} \right| > \frac{c(\xi)}{q^2}.$$

It is well known (see [10, Theorem 1.1 and Corollary 1.2]) that rational and irrational quadratic numbers are badly approximable. However the set  $\mathcal{B}$  of badly approximable numbers is larger since  $\xi \in \mathcal{B}$  if and only if the sequence  $\{x_n\}_{n \in \mathbb{N}}$  is bounded, denoting  $[x_0, x_1, \dots, x_n, \dots]$  its expansion as a continued fraction, see [10, Theorem 1.9]. Note also that the Lebesgue measure of  $\mathcal{B}$  is equal to zero.

Now, by Definition 4.1, it safe to deduce that if  $\frac{\sigma_{\pm}}{\sigma_{-}}$  is a badly approximable irrational number,  $(\mathbf{H}_{\mathbf{P}\mathbf{o}1})$  holds with  $\varpi(\frac{\sigma_{\pm}}{\sigma_{-}}) = 2$ , and consequently, Theorem 3.6 yields a polynomial energy decay in  $t^{-1/2}$ . The case  $\frac{\sigma_{\pm}}{\sigma_{-}} \notin \mathcal{B}$ , though, requires the notion of the irrationality measure (sometimes called the Liouville-Roth constant or irrationality exponent). For this, recall the following result from [11].

**Definition 4.2.** *Let  $\xi$  be an irrational real number. Then, the real number  $\mu \geq 1$  is called to be the irrationality measure of  $\xi$ , if there exists a positive constant  $C(\xi, \mu, \varepsilon)$  for every positive real number  $\varepsilon$  such that*

$$(4.2) \quad \left| \xi - \frac{p}{q} \right| > \frac{C(\xi, \mu, \varepsilon)}{q^{\mu+\varepsilon}}, \quad \forall p, q \in \mathbb{Z} \text{ with } q \geq 1.$$

The irrationality exponent  $\mu(\xi)$  of  $\xi$  is defined as the infimum of the irrationality measures of  $\xi$ .

Notice that  $\mu(\xi)$  is always  $\geq 2$ , see [10, Theorem E.2]. A direct consequence of this definition is that if  $\frac{\sigma_{\pm}}{\sigma_{-}} \notin \mathcal{B}$  is an irrational real number such that its irrationality exponent  $\mu(\xi)$  is finite, then  $(\mathbf{H}_{\mathbf{P}\mathbf{o}1})$  holds with  $\varpi(\frac{\sigma_{\pm}}{\sigma_{-}}) = \mu(\frac{\sigma_{\pm}}{\sigma_{-}}) + \varepsilon$  for any  $\varepsilon > 0$ . Let us then give some examples of irrational real numbers with finite irrationality exponent. First by the Roth's theorem, for every algebraic number of degree  $\geq 2$ ,  $\mu(\xi) = 2$ , see [10]. However, for many irrational real numbers  $\xi$ , the exact value of  $\mu(\xi)$  is not explicitly known but some upper bound for their irrationality exponent is available, see Table 1. Note that  $\nu(\xi)$  is an upper bound of  $\mu(\xi)$  if  $\mu(\xi) \leq \nu(\xi)$ , therefore, we automatically have

$$\left| \xi - \frac{p}{q} \right| > \frac{C(\xi, \mu, \varepsilon)}{q^{\nu(\xi)+\varepsilon}}, \quad \forall p, q \in \mathbb{Z} \text{ with } q \geq 1,$$

for all  $\varepsilon > 0$ . Consequently if  $\nu(\xi)$  is finite,  $(\mathbf{H}_{\mathbf{P}\mathbf{o}1})$  holds with  $\varpi(\frac{\sigma_{\pm}}{\sigma_{-}}) = \nu(\frac{\sigma_{\pm}}{\sigma_{-}}) + \varepsilon$ , for any  $\varepsilon > 0$ .

Now, in order to give other illustrations stated in the literature, an equivalent formulation of the irrationality measure, frequently used as the definition of the irrationality measure, may be needed, e.g. see [51, 55].

**Lemma 4.3.** *Fix an irrational real number  $\xi$  and a real number  $\mu \geq 1$ . Then, the following are equivalent*

- (1) *For every positive real number  $\varepsilon$ , there exists a positive constant  $C(\xi, \mu, \varepsilon)$  such that (4.2) holds.*
- (2) *For every positive real number  $\varepsilon$ , there exists a positive integer  $N(\xi, \mu, \varepsilon)$  such that*

$$(4.3) \quad \left| \xi - \frac{p}{q} \right| > \frac{1}{q^{\mu+\varepsilon}}, \quad \forall p, q \in \mathbb{Z} \text{ with } q \geq N(\xi, \mu, \varepsilon).$$

**Proof.** “(2)  $\Rightarrow$  (1):” Fix  $\varepsilon > 0$  and suppose that (1) does not hold. Then, for all  $n \in \mathbb{N}^*$ , there exists  $p_n, q_n \in \mathbb{Z}$  with  $q_n \geq 1$  such that

$$\left| \xi - \frac{p_n}{q_n} \right| \leq \frac{1}{nq_n^{\mu+\varepsilon}}, \quad \forall n \in \mathbb{N}^*.$$

This trivially implies that

$$\left| \xi - \frac{p_n}{q_n} \right| \leq \frac{1}{n}, \quad \forall n \in \mathbb{N}^*,$$

and consequently,  $\frac{p_n}{q_n}$  converges to  $\xi$  as  $n$  goes to infinity. Therefore,  $q_n$  (as well as  $p_n$ ) approaches infinity as  $n$  goes to infinity. For large enough  $n$ ,  $q_n$  will be greater than  $N(\xi, \mu, \varepsilon)$ . As (2) holds, by (4.3), the following is deduced

$$\frac{1}{q_n^{\mu+\varepsilon}} < \left| \xi - \frac{p}{q} \right| \leq \frac{1}{nq_n^{\mu+\varepsilon}}, \quad \forall n \in \mathbb{N}^* : q_n \geq N(\xi, \mu, \varepsilon),$$

which leads to a contradiction by letting  $n$  go to infinity.

“(1)  $\Rightarrow$  (2):” For a fixed  $\varepsilon > 0$ , assume (1) with  $\frac{\varepsilon}{2}$ . Then, there exists a positive constant  $C(\xi, \mu, \frac{\varepsilon}{2})$  such that

$$\left| \xi - \frac{p}{q} \right| > \frac{C(\xi, \mu, \frac{\varepsilon}{2})}{q^{\mu+\frac{\varepsilon}{2}}}, \quad \forall p, q \in \mathbb{Z} \text{ with } q \geq 1,$$

which is equivalent to

$$(4.4) \quad \left| \xi - \frac{p}{q} \right| > \frac{C(\xi, \mu, \frac{\varepsilon}{2})q^{\frac{\varepsilon}{2}}}{q^{\mu+\varepsilon}}, \quad \forall p, q \in \mathbb{Z} \text{ with } q \geq 1.$$

Since  $C(\xi, \mu, \frac{\varepsilon}{2})q^{\frac{\varepsilon}{2}}$  approaches infinity as  $q$  goes to infinity, restrict ourselves to  $q$  such that

$$C(\xi, \mu, \frac{\varepsilon}{2})q^{\frac{\varepsilon}{2}} \geq 1,$$

$\xi$	upper bound of $\mu(\xi)$	Reference
$\pi$	7.10320...	[55]
$\pi^2$ or $\frac{\pi^2}{6}$	5.09541...	[56]
$\ln(2)$	3.57455391	[33]
$\ln(3)$	5.116201	[8]
$\zeta(3)$ Apéry's constant	5.513891	[44]
$\text{Ln}_q(2)$ q-harmonic series	2.9384	[34]
$h_q(1)$ q-harmonic series	2.4650	[57]
$T_2(b^{-1}) = \sum_{n=1}^{\infty} t_n b^{n-1}$ , $b \geq 2$ and $t_n$ is the $n$ -th term of the True-Morse sequence	4	[1]

TABLE 1. Upper bounds of some irrationality exponents

or equivalently

$$q \geq C(\xi, \mu, \frac{\varepsilon}{2})^{-\frac{2}{\varepsilon}}.$$

By choosing  $N(\xi, \varepsilon) \geq C(\xi, \mu, \frac{\varepsilon}{2})^{-\frac{2}{\varepsilon}}$ , (4.4) implies (4.3) for  $q \geq N(\xi, \varepsilon)$ .  $\square$

In relation to Remark 3.4, the following equivalent statements can be formulated

**Lemma 4.4.** *Let us fix  $\xi$  an irrational real number and a real number  $\nu \geq 1$ . Then the following results are equivalent:*

(1) *There exists a positive constant  $C(\xi, \nu)$  such that*

$$\left| \xi - \frac{p}{q} \right| > \frac{C(\xi, \nu)}{q^\nu}, \quad \forall p, q \in \mathbb{Z} \text{ with } q \geq 1.$$

(2) *For all sequences  $\Lambda = (p_n, q_n)_{n \in \mathbb{N}} \in (\mathbb{N} \times \mathbb{N}^*)^{\mathbb{N}}$  with  $p_n \sim q_n$  for sufficiently large  $n$ , there exist a positive constant  $c_1(\xi, \nu, \Lambda)$  and a positive integer  $N_1(\xi, \nu, \Lambda)$  such that*

$$\left| \xi - \frac{p_n}{q_n} \right| > \frac{c_1(\xi, \nu, \Lambda)}{q_n^\nu}, \quad \forall n \geq N_1(\xi, \nu, \Lambda).$$

(3) *For all sequences  $\Lambda = (p_n, q_n)_{n \in \mathbb{N}} \in (\mathbb{N} \times \mathbb{N}^*)^{\mathbb{N}}$  for which  $\frac{p_n}{q_n}$  approaches  $\xi$  as  $n$  goes to infinity, there exist a positive constant  $c_2(\xi, \nu, \Lambda)$  and a positive integer  $N_2(\xi, \nu, \Lambda)$  such that*

$$\left| \xi - \frac{p_n}{q_n} \right| > \frac{c_2(\xi, \nu, \Lambda)}{q_n^\nu}, \quad \forall n \geq N_2(\xi, \nu, \Lambda).$$

**Proof.** Obviously (1)  $\Rightarrow$  (2)  $\Rightarrow$  (3). Hence it suffices to show that (3)  $\Rightarrow$  (1). We prove this by a contradiction argument. Assume that (3) holds but not (1). Then, for all  $n \in \mathbb{N}^*$ , there exists  $p_n, q_n \in \mathbb{Z}$  with  $q_n \geq 1$  such that

$$\left| \xi - \frac{p_n}{q_n} \right| \leq \frac{1}{nq_n^{\mu+\varepsilon}}, \quad \forall n \in \mathbb{N}^*.$$

As in Lemma 4.3, this trivially implies that  $\frac{p_n}{q_n}$  approaches  $\xi$  as  $n$  goes to infinity. Combining this with (3) results in

$$\frac{c_2(\xi, \nu, \Lambda)}{q_n^\nu} < \left| \xi - \frac{p_n}{q_n} \right| \leq \frac{1}{nq_n^\nu}, \quad \forall n \in \mathbb{N}^* : q_n \geq N_2(\xi, \nu, \Lambda),$$

where  $\Lambda = ((p_n, q_n))_{n \in \mathbb{N}}$ . This leads to a contradiction by letting  $n$  go to infinity.  $\square$

This result shows in particular that if  $(\mathbf{H}_{\mathbf{P}01})$  holds,  $\varpi(\frac{\sigma_+}{\sigma_-})$  is an irrationality measure for  $\frac{\sigma_+}{\sigma_-}$ . Moreover, this result also shows that the condition (4.2) in the definition of an irrationality measure can be replaced by a condition on sequences (as in (2) or (3) above).

Finally, upper bounds of some irrationality exponents are presented in Table 1. To keep it short, only a few of them are provided. The interested readers can refer to, e.g.[51], for various other upper bounds.

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**Remark 4.5.** As a final remark, notice that there exist irrational numbers  $\xi$  for which  $\mu(\xi) = \infty$ . These numbers are called Liouville numbers. For example,  $L = \sum_{k=1}^{\infty} b^{-k!}$  is a Liouville number. Therefore if  $\frac{\sigma_{\pm}}{\sigma_{-}}$  is a Liouville number, the decay rate of the energy of the system (P/E) is still an open problem.

## 5. CONCLUSIONS AND OPEN PROBLEMS

In this paper, two different transmission problems are investigated: (i) a transmission problem of an Elastic-Piezoelectric-Elastic design with only one local damping acting on the longitudinal displacement of the center line of the piezoelectric layer and (ii) a transmission problem of a Piezoelectric-Elastic design with only one local damping acting on the elastic part. An exponential stability result is immediate for (i). However, for (ii), the nature of the stability (polynomial or exponential) entirely depends on the arithmetic nature of a quotient involving all the physical parameters of the system.

An interesting open problem is the stability of the following Piezoelectric-Elastic-Piezoelectric design

$$(P/E/P) \quad \left\{ \begin{array}{ll} \rho_1 v_{tt}^1 - \alpha v_{xx}^1 + \gamma_1 \beta_1 p_{xx}^1 + b_1(x) v_t^1 = 0, & (x, t) \in (0, l_1) \times (0, \infty) \\ \mu_1 p_{tt}^1 - \beta_1 p_{xx}^1 + \gamma_1 \beta_1 v_{xx}^1 = 0, & (x, t) \in (0, l_1) \times (0, \infty), \\ u_{tt} - c_1 u_{xx} + b_2(x) u_t = 0, & (x, t) \in (l_1, l_2) \times (0, \infty), \\ \rho_2 v_{tt}^2 - \tilde{\alpha} v_{xx}^2 + \gamma_2 \beta_2 p_{xx}^2 + b_3(x) v_t^2 = 0, & (x, t) \in (l_2, L) \times (0, \infty), \\ \mu_2 p_{tt}^2 - \beta_2 p_{xx}^2 + \gamma_2 \beta_2 v_{xx}^2 = 0, & (x, t) \in (l_2, L) \times (0, \infty), \\ v^1(0, t) = p^1(0, t) = v^2(L, t) = p^2(L, t) = 0, \\ v^1(l_1, t) = u(l_1, t), \\ \alpha v_x^1(l_1, t) - \gamma_1 \beta_1 p_x^1(l_1, t) = c_1 u_x(l_1, t), \\ \beta_1 p_x^1(l_1, t) = \gamma_1 \beta_1 v_x^1(l_1, t), \\ v^2(l_2, t) = u(l_2, t), \\ \tilde{\alpha} v_x^2(l_2, t) - \gamma_2 \beta_2 p_x^2(l_2, t) = c_1 u_x(l_2, t), \\ \beta_2 p_x^2(l_2, t) = \gamma_2 \beta_2 v_x^2(l_2, t), & t \in (0, \infty), \\ (v^1, p^1, u, v^2, p^2)(\cdot, 0) = (v_0^1, p_0^1, u_0, v_0^2, p_0^2)(\cdot), \\ (v_t^1, p_t^1, u_t, v_t^2, p_t^2)(\cdot, 0) = (v_1^1, p_1^1, u_1, v_1^2, p_1^2)(\cdot) \end{array} \right.$$

with the assumptions  $b_2 = 0$ ,  $b_1 \in L^\infty(0, l_1)$ ,  $b_3 \in L^\infty(l_2, L)$ , and

$$\left\{ \begin{array}{l} b_1(x) \geq b_{1,0} > 0 \quad \text{in } (r_1, r_2) \subset (0, l_1), \text{ and } b_1(x) = 0, \quad x \in (0, l_1) \setminus (r_1, r_2), \\ b_3(x) \geq b_{3,0} > 0 \quad \text{in } (r_5, r_6) \subset (l_2, L), \text{ and } b_3(x) = 0 \quad \text{in } (l_2, L) \setminus (r_5, r_6). \end{array} \right.$$

By adopting analogous arguments as in Section 2, we conjecture that one may prove that the system (P/E/P) is exponentially stable. Furthermore, by assuming that  $b_1 = b_3 = 0$  and  $b_2 \in L^\infty(l_1, l_2)$  such that

$$b_2(x) \geq b_{2,0} > 0 \quad \text{in } (r_3, r_4) \subset (l_1, l_2), \text{ and } b_2(x) = 0 \quad \text{in } (l_1, l_2) \setminus (r_3, r_4),$$

analogous stability results as in Section 3 may also be obtained.

Another open problem, which deserves to be investigated, is the stability of the Elastic-Piezoelectric-Elastic design with two dampings terms acting only on the elastic part

$$\left\{ \begin{array}{ll} u_{tt} - c_1 u_{xx} + k_1(x) u_t = 0, & (x, t) \in (0, l_1) \times (0, \infty), \\ \rho v_{tt} - \alpha v_{xx} + \gamma \beta p_{xx} = 0, & (x, t) \in (l_1, l_2) \times (0, \infty), \\ \mu p_{tt} - \beta p_{xx} + \gamma \beta v_{xx} = 0, & (x, t) \in (l_1, l_2) \times (0, \infty), \\ y_{tt} - c_2 y_{xx} + k_2(x) y_t = 0, & (x, t) \in (l_2, L) \times (0, \infty), \\ u(0, t) = y(L, t) = 0, \\ v(l_1, t) = u(l_1, t), \\ v(l_2, t) = y(l_2, t), \\ \alpha v_x(l_1, t) - \gamma \beta p_x(l_1, t) = c_1 u_x(l_1, t), \\ \alpha v_x(l_2, t) - \gamma \beta p_x(l_2, t) = c_2 y_x(l_2, t), \\ \beta p_x(l_1, t) = \gamma \beta v_x(l_1, t), \\ \beta p_x(l_2, t) = \gamma \beta v_x(l_2, t), & t \in (0, \infty), \\ (u, v, p, y, u_t, v_t, p_y, y_t)(\cdot, 0) = (u_0, v_0, p_0, y_0, u_1, v_1, p_1, y_1)(\cdot), \end{array} \right.$$

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with  $k_1 \in L^\infty(0, l_1)$  and  $k_2 \in L^\infty(l_2, L)$  such that

$$\begin{cases} k_1(x) \geq k_{1,0} > 0 & \text{in } (r_1, r_2) \subset (0, l_1), \text{ and } k_1(x) = 0 & \text{in } (0, l_1) \setminus (r_1, r_2), \\ k_2(x) \geq k_{2,0} > 0 & \text{in } (r_5, r_6) \subset (l_2, L), \text{ and } k_2(x) = 0 & \text{in } (l_2, L) \setminus (r_5, r_6). \end{cases}$$

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