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Commutative B_∞ -algebras are shuffle algebras

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Abstract

We here construct an explicit isomorphism between any commutative Hopf algebra which underlying coalgebra is the tensor coalgebra of a space V and the shuffle algebra based on the same space. This isomorphism uses the commutative B_∞ structure that governs the product and the eulerian idempotent, as well as the canonical projection on the space V . This generalizes Hoffman’s isomorphism between commutative quasi-shuffle and shuffle algebras, which correspond to the case when the B_∞ structure is given by an associative and commutative product.

1 Introduction

The purpose of the present article is to show a structure theorem for commutative B_∞ -algebras, that is commutative Hopf algebras that are cofree as graded, connected coalgebras. Namely, we show that any such Hopf algebra is canonically isomorphic to a shuffle Hopf algebra $(T(V), \sqcup, \Delta)$, where V is a vector space, $T(V)$ the associated graded vector space of tensors over V , \sqcup the shuffle product and Δ the deconcatenation coproduct. The canonical isomorphism can be described explicitly (Thm 4.3). Such a structure (or rigidity) theorem is relatively surprising as B_∞ -algebras and other related “up to homotopy” algebraic notions were first introduced in homotopy algebra as deformations of classical structures (recall that the shuffle Hopf algebra $(T(V), \sqcup, \Delta)$ is the fundamental object in the modern approach to free Lie algebras [18]), and there was at our knowledge no reason why such deformations should be trivial — that is, induced by an isomorphism. Notice that our results also generalize a classical theorem in Lie theory and the theory of combinatorial Hopf algebras: Hoffman’s isomorphism, relating shuffle and quasi-shuffle algebras¹.

Let us survey briefly more precisely the context of these results. A classical and fundamental theorem by C. Malvenuto in the theory of free Lie algebras, symmetric group representations and symmetric functions asserts that the dual of the Hopf algebra of descents in symmetric groups, also known as the dual of the Hopf algebra of noncommutative symmetric functions, is isomorphic to the Hopf algebra of quasi-symmetric functions [13, 14]. This isomorphism generalizes to an isomorphism (of Hopf algebras) between the shuffle algebra and the quasi-shuffle algebra of words over a graded commutative semi-group. This last isomorphism is known as Hoffman’s isomorphism [10].

¹Shuffle and quasi-shuffle products can be commutative (as usually in Lie theory) or not (as usually in algebraic topology). In the present article they will be always commutative and we write therefore systematically “shuffle product” and “quasi-shuffle product” for “commutative shuffle product” and “commutative quasi-shuffle product”.

Quasi-shuffle algebras can be used to encode certain relations between multi-zeta values and generalizations of them, see [12, 11] for a review on this topic. They play a prominent role in the study of Rota-Baxter algebras [8, 9, 4]. For completeness sake, recall also that the Hoffman isomorphism is used not only in combinatorics and algebra as it has for example applications also in stochastics, where it allows to better understand the equivalence between Itô and Stratonovich integrals, resp. solutions of stochastic differential equations (Itô calculus being encoded by quasi-shuffle algebras and Stratonovich's by shuffle algebras) [2, 3]. In a joint article with J.-Y. Thibon, we investigated and classified more generally natural deformations of the shuffle Hopf algebra structure $\text{Sh}(A)$ which can be defined on the space of tensors over a commutative algebra A (where natural means functorial) [6].

In another direction, together with C. Malvenuto, we investigated commutative B_∞ structures in the context of finite topologies (or, equivalently, preorders) and their links with shuffle Hopf algebras, introducing and featuring in particular the notion of Schur-Weyl categories of bialgebras [5]. This notion of commutative B_∞ -algebra is a far reaching generalization of shuffle and quasi-shuffle Hopf algebras of words over a commutative semi-group. The fact that such structures appear very naturally (but in a non straightforward way) in the context of finite topologies and ordered structures provides, together with the homotopical algebra origin of these notions in [7], further evidence for their naturalness.

The present article pushes the existing analyses one step forward by showing that an arbitrary commutative B_∞ -algebra is always isomorphic as a Hopf algebra to a shuffle Hopf algebra of words. As the notion of commutative B_∞ -algebra generalizes the two notions of shuffle Hopf algebra and quasi-shuffle Hopf algebra of words over a commutative semi-group, our results extend in particular the Malvenuto and Hoffman isomorphisms and also provide a new understanding of them.

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2 Cofree coalgebras

Recall first some definitions and elementary properties of coalgebras. Details and proofs can be found in [1], especially [1, Sect. 2.13 Graded and Conilpotent Coalgebras]. All vector spaces are defined over a ground field \mathbb{K} of characteristic 0.

A *graded vector space* is a vector space decomposing as a direct sum $V = \bigoplus_{n \in \mathbb{N}} V_n$. It is *reduced* if $V_0 = 0$. A graded vector space is locally finite dimensional if all the V_n are finite-dimensional vector spaces. The graded dual of a graded vector space is the graded vector space $V^* = \bigoplus_{n \in \mathbb{N}} V_n^*$, where V_n^* stands for the dual of V_n . It is locally finite dimensional when V is such. The tensor product of two graded vector spaces is defined by $V \otimes W = \bigoplus_{n \in \mathbb{N}} (V \otimes W)_n$, where $(V \otimes W)_n := \bigoplus_{p+q=n} V_p \otimes W_q$. The ground field \mathbb{K} identifies with the graded vector space, still written \mathbb{K} , with only one non zero component, $\mathbb{K}_0 = \mathbb{K}$. It is the unit of the tensor product ($V \otimes \mathbb{K} = V = V \otimes \mathbb{K}$).

A *filtered* vector space is a vector space V equipped with an increasing filtration by subspaces $V_0 \subset V_1 \subset \dots \subset V_n \subset \dots$, such that $V = \bigcup V_n$. The tensor product of two filtered vector spaces is defined by requiring $(V \otimes W)_n := \sum_{i+j=n} V_i \otimes W_j$. Each graded vector space V is canonically filtered: writing F for the functor from graded to filtered vector spaces, $FV_n := \bigoplus_{p \leq n} V_p$. We will be mostly interested in the present article in Hopf algebras that are filtered as algebras (in the sense that the product map is a map of filtered vector spaces) and graded as coalgebras.

A graded coalgebra is a coalgebra in the tensor category of graded vector spaces. That

is, graded coalgebras and other graded structures are defined as usual except for the fact that structure morphisms, for example the coproduct Δ of a graded coalgebra C , has to be a morphism of graded vector spaces, so that

$$\Delta : C_n \rightarrow \bigoplus_{p+q=n} C_p \otimes C_q.$$

The counit map from C to \mathbb{K} , that we will write ε (or ε_C if we want to emphasize what is the underlying coalgebra) has to be null map except in degree 0. A graded coalgebra C is connected if $C_0 = \mathbb{K}$; for such a coalgebra, $\Gamma(C) = \{1\}$, where $\Gamma(C)$ stands for the set of group-like elements in C , that is

$$\Gamma(C) := \{c \in C, c \neq 0 \text{ and } \Delta(c) = c \otimes c\}.$$

Definition 2.1. *The category **FgCoalg** of filtered-graded coalgebras is the category whose objects are graded coalgebras and the set of morphisms between two graded coalgebras C and D the set of filtered vector spaces morphisms of coalgebras from C to D .*

That is, a morphism of filtered-graded coalgebras from C to D is a morphism ϕ of coalgebras such that $\phi(C_n) \subseteq \bigoplus_{p \leq n} D_p$ for any $n \in \mathbb{N}$.

Let now (C, Δ, ε) be a coaugmented coalgebra with coaugmentation $\eta : \mathbb{K} \rightarrow C$ (a coaugmentation is a map of coalgebras from the ground field to C , where the ground field is equipped with the identity coproduct: $\mathbb{K} = \mathbb{K} \otimes \mathbb{K}$). The coaugmentation is a section of the projection ε to the ground field and one has the decomposition $C = \bar{C} \oplus \mathbb{K}$, where $\bar{C} := \text{Ker}(\varepsilon)$. Notice that a graded connected coalgebra is canonically coaugmented: its coaugmentation is the isomorphism between the ground field and the degree 0 component of the coalgebra. In general, for a coaugmented coalgebra one can define the reduced coproduct from \bar{C} to $\bar{C} \otimes \bar{C}$ by

$$\bar{\Delta}(c) := \Delta(c) - c \otimes 1 - 1 \otimes c.$$

The iterated reduced coproduct from \bar{C} to $\bar{C}^{\otimes n+2}$ is inductively defined by

$$\bar{\Delta}_{n+2} := (\bar{\Delta} \otimes \text{Id}_{\bar{C}}^{\otimes n}) \circ \bar{\Delta}_{n+1},$$

with $\bar{\Delta}_2 := \bar{\Delta}$. Let $F_n := \text{Ker}(\bar{\Delta}_{n+1}) \subset \bar{C}$. Notice that $\bar{\Delta}_{n+2} = (\bar{\Delta} \otimes \text{Id}_{\bar{C}}^{\otimes n}) \circ \bar{\Delta}_{n+1}$ implies $F_n \subset F_{n+1}$.

Lemma 2.2. *The coproduct Δ and the coalgebra C are called conilpotent if $\bar{C} = \bigcup_n F_n$, that is if for every $c \in \bar{C}$ there exists an integer $n \geq 2$ such that $\bar{\Delta}_n(c) = 0$. When the coalgebra C is graded and connected, the coproduct is automatically conilpotent.*

Proof. Indeed, when C is graded connected, $\bar{C}_0 = 0$. As $\bar{\Delta}_k : \bar{C}_n \rightarrow \bigoplus_{i_1+\dots+i_k=n} \bar{C}_{i_1} \otimes \dots \otimes \bar{C}_{i_k}$, the iterated coproduct $\bar{\Delta}_k$, $k \geq 2$, vanishes on C_n for $1 \leq n \leq k-1$. \square

Let now V be a vector space over \mathbb{K} . We denote by $T(V) := \bigoplus_{n=0}^{\infty} V^{\otimes n}$ the tensor algebra of V , and use the word notation for tensors (that is, $v_1 \dots v_n$ will stand for $v_1 \otimes \dots \otimes v_n$). The tensor algebra is graded: the degree of a word w is its length $\ell(w)$, that is, the number of its letters. We use the terminology of [1] and call algebra a vector space that can be equipped with several algebraic structures — this allows in particular to avoid calling “tensor algebra” the vector space $T(V)$ without equipping it with the algebra structure obtained from the concatenation product of words.

We instead equip the tensor algebra with the deconcatenation coproduct, that makes it, together with the canonical projection ε to $\mathbb{K} = V^{\otimes 0}$, a graded (counital, conilpotent, connected, coassociative) coalgebra: for any $v_1, \dots, v_n \in V$, with $n \geq 0$,

$$\Delta(v_1 \dots v_n) = \sum_{i=0}^n v_1 \dots v_i \otimes \dots \otimes v_{i+1} \dots v_n.$$

We write $T_+(V) := \bigoplus_{n=1}^{\infty} V^{\otimes n}$ for $\overline{T(V)}$. It is equipped with the (coassociative but not counital) reduced deconcatenation coproduct: for any $v_1, \dots, v_n \in V$, with $n \geq 1$,

$$\overline{\Delta}(v_1 \dots v_n) = \sum_{i=1}^{n-1} v_1 \dots v_i \otimes \dots v_{i+1} \dots v_n.$$

The following Lemma justifies the introduction of the category of filtered-graded coalgebras.

Lemma 2.3. *Let C be graded connected coalgebra and ϕ be a morphism of coalgebras from C to $(T(V), \Delta)$. Then, ϕ is a morphism of filtered graded coalgebras (that is, $\phi(C_n) \subseteq FT(V)_n = \bigoplus_{k=0}^n V^{\otimes k}$).*

Proof. As ϕ is a coalgebra morphism between connected coalgebras, $(\phi \otimes \phi) \circ \overline{\Delta} = \overline{\Delta} \circ \phi$, which implies that for any $n \in \mathbb{N}$,

$$\phi^{\otimes(n+1)} \circ \overline{\Delta}_n = \overline{\Delta}_n \circ \phi.$$

Therefore, $\phi(\text{Ker}(\overline{\Delta}_{n+1})) \subseteq \text{Ker}(\overline{\Delta}_{n+1})$, and it is an easy exercise to show that in $T(V)$, $\text{Ker}(\overline{\Delta}_{n+1}) = FT(V)_n$. Moreover, in C , by connectivity, $C_n \subseteq \text{Ker}(\overline{\Delta}_{n+1})$, and finally $\phi(C_n) \subseteq FT(V)_n$. \square

Definition 2.4 (Cofree filtered-graded coalgebras). *Let V be a vector space. A cofree filtered-graded coalgebra over V is a connected graded coalgebra C such that $C_1 = V$ and such that for each connected graded coalgebra D , any vector space morphism ϕ from \overline{D} to V lifts uniquely to a morphism Φ of coalgebras from D to C in $\mathbf{FgCoalg}$.*

As usual for cofree objects, any two cofree filtered-graded coalgebras over V are isomorphic (by a unique isomorphism). This justifies to call (slightly abusively) any cofree graded coalgebra over V , “the” cofree graded coalgebra over V .

Lemma 2.5. *The tensor gebr over V , equipped with the deconcatenation coproduct Δ is a cofree filtered-graded coalgebra over V . With our previous notation, the morphism Φ from D to $T(V)$ is obtained as*

$$\varepsilon + \sum_{n \geq 1} \phi^{\otimes n} \circ \overline{\Delta}_n.$$

This follows from the fact that $T(V)$ is a cofree conilpotent coalgebra (see e.g. [1, Exercise 2.13.3, Remark 2.13.1]) and that graded connected coalgebras are conilpotent. The simplest way to understand and prove the Lemma and the formula for Φ is by dualizing the statement and using the fact that the tensor gebr equipped with the concatenation product of words is a free associative algebra. Given indeed a map ϕ^* from the dual of V , V^* , to the projective limit of vector spaces $\dots \rightarrow FD_{n+1}^* \rightarrow FD_n^* \rightarrow \dots \rightarrow FD_1^*$, where $FD_n^* = D_1^* \oplus \dots \oplus D_n^*$ and D^* is a graded connected algebra, this map uniquely extends to an algebra map from $T(V^*)$ to $\hat{D} = \mathbb{K} \oplus \prod_{n \in \mathbb{N}^*} D_n^*$, the completion of the graded dual of D with respect to its canonical filtration.

This algebra map ξ is given as usual on $T_+(V^*)$ by

$$\xi(v_1^* \dots v_n^*) := \phi^*(v_1^*) \dots \phi^*(v_n^*),$$

which is dual to the formula for Φ in the Lemma. The fact that D^* is a graded algebra insures that ξ is well-defined (as the degree p component of $\phi^*(v_1) \dots \phi^*(v_n)$ is obtained as the sum of a finite number of terms, for any p).

3 B_∞ -algebras

The notion of B_∞ -algebra was first introduced by Getzler and Jones in [7] (for cochain complexes — their definition extends to other tensor categories, we consider here B_∞ -algebras in the category of vector spaces). We survey here the fundamental definitions and properties, using systematically the properties of cofree coalgebras.

A B_∞ -algebra structure on a vector space V is equivalent to the definition of a Hopf algebra structure on the coalgebra $(T(V), \Delta)$ (see Proposition 3.5 below). As $T(V) \otimes T(V)$ is a connected graded coalgebra, by Lemma 2.3, the coproduct Δ , which is a map of coalgebras, necessarily respects the filtrations: $\Delta(T_n(V) \otimes T_m(V)) \subset \bigoplus_{k \leq n+m} T_k(V)$. In particular, structure theorems obtained in the previous section for coalgebras in **FgCoalg** apply to B_∞ -algebras. More specifically, the product map from $T(V) \otimes T(V)$ to $T(V)$ is entirely characterized by its projection to the subspace V . This observation and the application of Lemma 2.5 lead to the Lemma:

Lemma 3.1. *Let $*$: $T(V) \otimes T(V) \rightarrow T(V)$ be a coalgebra map. We define $\langle -, - \rangle : T(V) \otimes T(V) \rightarrow V$ by*

$$\langle v_1 \dots v_k, v_{k+1} \dots v_{k+l} \rangle = \pi_V(v_1 \dots v_k * v_{k+1} \dots v_{k+l}),$$

where $\pi_V : T(V) \rightarrow V$ is the canonical surjection.

Then, $1 * 1 = 1$ and for any words $w, w' \in T_+(V)$,

$$w * w' = \sum_{k=1}^{\infty} \sum_{\substack{w=w_1 \dots w_k, \\ w'=w'_1 \dots w'_k}} \langle w_1, w'_1 \rangle \dots \langle w_k, w'_k \rangle. \quad (1)$$

Note that in the sum, the words w_i or w'_j can be empty (in what case w_i and w'_j stand for 1 in the terms $\langle w_i, w'_i \rangle$ or $\langle w_j, w'_j \rangle$).

Moreover,

$$w * 1 = \sum_{k=1}^{\infty} \sum_{w=w_1 \dots w_k} \langle w_1, 1 \rangle \dots \langle w_k, 1 \rangle. \quad (2)$$

$$1 * w' = \sum_{k=1}^{\infty} \sum_{w'=w'_1 \dots w'_k} \langle 1, w'_1 \rangle \dots \langle 1, w'_k \rangle. \quad (3)$$

Proof. As $*$ is a coalgebra map, it maps $1 \otimes 1$, the unique group-like element in $T(V) \otimes T(V)$, to 1, the unique group-like element in $T(V)$. The Lemma follows then from Lemma 2.5: $*$ is entirely characterized by $\langle -, - \rangle := \pi_V \circ *$, and applying the formula expressing $*$ in terms of $\langle -, - \rangle$ yields to Eqs (1,2,3). \square

Lemma 3.2. *With the same notation, the product $*$ is unital, with unit 1 if, and only if,*

$$\langle -, 1 \rangle = \langle 1, - \rangle = \pi_V.$$

Proof. The assertion follows from the definitions of π_V , $\langle -, - \rangle$ and Eqs (2,3). \square

We assume from now on that the product $*$ is unital, with unit 1.

Lemma 3.3. *With the same notation,*

1. *the product $*$ is associative if, and only if, for any $w, w', w'' \in T_+(V)$,*

$$\langle w, w' * w'' \rangle = \langle w * w', w'' \rangle.$$

2. *It is commutative if, and only if, for any $w, w' \in T_+(V)$,*

$$\langle w, w' \rangle = \langle w', w \rangle.$$

Proof. As the product is a map of coalgebra, so are the maps $* \circ (Id \otimes *)$ and $* \circ (* \otimes Id)$ from $T(V)^{\otimes 3}$ to $T(V)$. They are therefore entirely characterized by the composition with π_V and associativity follows from

$$\pi_V \circ (* \circ (Id \otimes *)) = \pi_V \circ (* \circ (* \otimes Id)).$$

The second assertion is proved similarly, noticing that the twist map $w \otimes w' \rightarrow w' \otimes w$ is a morphism of coalgebras. \square

Definition 3.4. A B_∞ -structure on V is a map $\langle -, - \rangle : T(V) \otimes T(V) \rightarrow V$, such that:

- For any word $v_1 \dots v_n \in T(V)$,

$$\langle 1, v_1 \dots v_n \rangle = \langle v_1 \dots v_n, 1 \rangle = \begin{cases} v_1 & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

- For any words $w, w', w'' \in T_+(V)$,

$$\langle w, w' * w'' \rangle = \langle w * w', w'' \rangle,$$

where $*$ is defined by

$$w * w' = \sum_{k=1}^{\infty} \sum_{\substack{w=w_1 \dots w_k, \\ w'=w'_1 \dots w'_k}} \langle w_1, w'_1 \rangle \dots \langle w_k, w'_k \rangle. \quad (4)$$

We shall say that $\langle -, - \rangle$ is commutative if for any $w, w' \in T_+(V)$, $\langle w, w' \rangle = \langle w', w \rangle$.

A B_∞ -algebra (resp. commutative) is a vector space V equipped with a B_∞ -structure (resp. commutative).

Proposition 3.5. Let $P(V)$ be the set of products $*$ on $T(V)$ making $(T(V), *, \Delta)$ a bialgebra and by $B_\infty(V)$ the set of B_∞ structures on V . The following map is a bijection:

$$\Theta : \begin{cases} P(V) & \longrightarrow & B_\infty(V) \\ * & \longmapsto & \pi_V \circ *. \end{cases}$$

Denoting by $P^c(V)$ the set of commutative products on $T(V)$ making $(T(V), *, \Delta)$ a bialgebra and by $B_\infty^c(V)$ of commutative B_∞ structures on V , Θ induces a bijection from $P^c(V)$ to $B_\infty^c(V)$.

Proof. As 1 is the unique group-like of $T(V)$, it is necessarily the unit for the product $*$. By Lemmas 3.1, 3.2 and 3.3, Θ is well-defined. If $* \in P(V)$, then it is a coalgebra morphism from $T(V) \otimes T(V)$ to $T(V)$. By Lemma 2.5, Θ is injective.

Conversely, let $\langle -, - \rangle$ in $B_\infty(V)$. The product $*$ associated to it by (1) is a coalgebra map. It is associative by Lemma 3.3 and has 1 for a unit: $* \in P(V)$, and $\Theta(*) = \langle -, - \rangle$. Thus, Θ is a bijection.

By the last item of Lemma 3.3, $\Theta(P^c(V)) = B_\infty^c(V)$. \square

Example 3.1 (Shuffle and quasi-shuffle Hopf algebra). Let \cdot be an associative, not necessarily unitary, product on V . We extend it as a B_∞ -structure on V by putting, for any words $w, w' \in T(V)$,

$$\langle w, w' \rangle = \begin{cases} 0 & \text{if } w = w' = 1, \\ w' & \text{if } w = 1 \text{ and } \ell(w') = 1, \\ w & \text{if } \ell(w) = 1 \text{ and } w' = 1, \\ w \cdot w' & \text{if } \ell(w) = \ell(w') = 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $\ell(w)$ stands for the length of the word w . The associated product on $T(V)$ is called the quasi-shuffle product \boxplus . It is commutative, if, and only if, \cdot is commutative. In the particular case where $\cdot = 0$, we obtain the shuffle product \boxdot .

The quasi-shuffle product is classically inductively defined by the equations $v \boxplus 1 = 1 \boxplus v = v$ and

$$v_1 \dots v_k \boxplus v_{k+1} \dots v_{k+l} := v_1(v_2 \dots v_k \boxplus v_{k+1} \dots v_{k+l}) + v_{k+1}(v_1 \dots v_k \boxplus v_{k+2} \dots v_{k+l}) \\ + (v_1 \cdot v_{k+1})(v_2 \dots v_k \boxplus v_{k+2} \dots v_{k+l}),$$

which restricts to

$$v_1 \dots v_k \boxdot v_{k+1} \dots v_{k+l} := v_1(v_2 \dots v_k \boxdot v_{k+1} \dots v_{k+l}) + v_{k+1}(v_1 \dots v_k \boxdot v_{k+2} \dots v_{k+l}),$$

for the shuffle product. See e.g. [10, 6] for details on shuffle and quasi-shuffle algebras and their relationships, to be generalized below in the present article.

Remark 3.1. Equation (4) can be rewritten in this way: for any $v_1, \dots, v_{k+l} \in V$,

$$v_1 \dots v_k * v_{k+1} \dots v_{k+l} = \sum_{n=1}^{k+l} \sum_{\substack{\sigma: [k+l] \rightarrow [n], \\ \sigma(1) \leq \dots \leq \sigma(k), \\ \sigma(k+1) \leq \dots \leq \sigma(k+l)}} \langle v_{\sigma^{-1}(1)} \rangle \dots \langle v_{\sigma^{-1}(n)} \rangle,$$

with the following notation: if $I = \{i_1, \dots, i_q\} \subseteq [n]$, with $i_1 < \dots < i_p \leq k < i_{p+1} < \dots < i_q$,

$$v_I = v_{i_1} \dots v_{i_p} \otimes v_{i_{p+1}} \dots v_{i_q}.$$

4 The structure of commutative B_∞ -algebras

Recall first that the eulerian idempotents are elements in the symmetric group algebras $\mathbb{Q}(S_n)$ given by:

$$\sum_{\sigma \in S_n} \frac{(-1)^{d(\sigma)}}{n} \binom{n-1}{d(\sigma)}^{-1} \sigma,$$

where S_n stands for the n th symmetric group and $d(\sigma)$ for the number of descents of a permutation (the number of $i < n$ such that $\sigma(i) > \sigma(i+1)$). They are also called canonical projections (in the tensor algebra), Solomon idempotents, Barr idempotents... They are studied in detail in [18].

One of the key properties of these idempotents is that they project the tensor (the free associative) algebra $T(V)$ over a vector space V to the free Lie algebra over V according to the decomposition of $T(V)$ induced by the Poincaré-Birkhoff-Witt theorem. This idea was generalized to arbitrary graded connected cocommutative or commutative Hopf algebras H in the second author PhD thesis [17, Th. I,5,6 and Th. I,6,4], see also [15, 16]. However, the constructions and proofs in these references only require the Hopf algebras to be unipotent (a weaker hypothesis than being graded connected, it is the one suited for the forthcoming study of commutative B_∞ -algebras) — this observation was developed systematically in [1, Chap. 4], to which we refer.

In the cocommutative case, these generalized canonical idempotents project on the primitive part of the Hopf algebra; in the commutative case, which is the one interesting for us here, they project onto a (canonically constructed) vector space $Q(V)$ that freely generates H as a commutative algebra, orthogonally to the square of H^+ , the augmentation ideal of H . It is important for what follows to notice that these generalized idempotents *cannot* be constructed in general as elements of the symmetric group algebras. In particular they are not directly governed by the statistics of descents in symmetric groups — the very reason for the name “eulerian idempotents” used in the classical case.

A Hopf algebra in $P^c(V)$ is not graded connected in general (since it is not graded as an algebra), but it is conilpotent as a coalgebra (by Lemma 2.2), which is enough to insure that it is unipotent as a commutative Hopf algebra, so that the structure theorems just mentioned apply (see in particular [1, Thm 4.4.1]). They imply in particular the following result, that we state as a Theorem in view of its importance for the structure of commutative B_∞ algebras:

Theorem 4.1. *Let $*$ $\in P^c(V)$. The canonical idempotent e_* is defined by $e_*(1) = 0$ and for any non-empty word w ,*

$$e_*(w) = \sum_{k=1}^{\infty} \sum_{\substack{w=w_1 \dots w_k, \\ w_1, \dots, w_k \neq \emptyset}} \frac{(-1)^{k-1}}{k} w_1 * \dots * w_k.$$

Then e_ is a projector, vanishing on $T_+(V) * T_+(V)$, and for any $v \in V$, $e_*(v) = v$. Moreover, the image of e_* freely generates $(T(V), *)$ as a commutative algebra.*

Proposition 4.2. *We set $\varpi_* := \pi_V \circ e_*$ and call ϖ the canonical commutative B_∞ idempotent. It acts as the identity map on V and sends any non-empty word w to*

$$\varpi_*(w) = \sum_{k=1}^{\infty} \sum_{\substack{w=w_1 \dots w_k, \\ w_1, \dots, w_k \neq \emptyset}} \frac{(-1)^{k-1}}{k} \langle w_1, w_2 * \dots * w_k \rangle.$$

Proof. The map ϖ_* is indeed an idempotent since π_V is an idempotent and $e_* \circ \pi_V = \pi_V$. The explicit formula follows from the formula for e_* in Thm 4.1 and the observation that, since $\langle w, w' \rangle = \pi_V(w * w')$,

$$\pi_V(w_1 * w_2 * \dots * w_k) = \pi_V(w_1 * (w_2 * \dots * w_k)) = \langle w_1, w_2 * \dots * w_k \rangle. \quad \square$$

Example 4.1. Let $v_1, v_2, v_3 \in V$.

$$\begin{aligned} \varpi_*(v_1) &= v_1, \\ \varpi_*(v_1 v_2) &= -\frac{1}{2} \langle v_1, v_2 \rangle, \\ \varpi_*(v_1 v_2 v_3) &= -\frac{1}{2} (\langle v_1 v_2, v_3 \rangle + \langle v_1, v_2 v_3 \rangle) + \frac{1}{3} \langle v_1, v_2 v_3 + v_3 v_2 + \langle v_2, v_3 \rangle \rangle. \end{aligned}$$

Theorem 4.3. *Let $\langle -, - \rangle \in B_\infty^c(V)$. The associated product $\Theta^{-1}(\langle -, - \rangle)$ on $T(V)$ given by (4) is denoted by $*$.*

Let us then define

$$H_* : \begin{cases} T(V) & \longrightarrow & T(V) \\ 1 & \longmapsto & 1, \\ w \in T_+(V) & \longmapsto & \sum_{k=1}^{\infty} \sum_{\substack{w=w_1 \dots w_k, \\ w_1, \dots, w_k \neq \emptyset}} \varpi_*(w_1) \dots \varpi_*(w_k). \end{cases}$$

Then H_ is a bialgebra isomorphism from $(T(V), *, \Delta)$ to $(T(V), \sqcup, \Delta)$.*

Proof. Let us notice first that H_* is a coalgebra morphism: it is indeed the unique coalgebra morphism from $(T(V), \Delta)$ to $(T(V), \Delta)$ induced by the map ϖ_* from $T(V)$ to V by Lemma 2.5.

Furthermore, as ϖ_* acts as the identity map on V , for any word w of length $n \geq 2$, $H_*(w) = w + l.o.t.$, where *l.o.t.* stands for lower order terms: a linear combination of words of length $< n$. By a standard triangularity argument, H_* is invertible and a coalgebra isomorphism.

It remains to prove that H_* is an algebra morphism, that is $H_1 = H_2$, where

$$H_1 := \sqcup \circ (H_* \otimes H_*), \quad H_2 := H_* \circ *.$$

As both are coalgebra morphisms from $T(V)^{\otimes 2}$ to $T(V)$, it is actually enough to show that $\pi_V \circ H_1 = \pi_V \circ H_2$. Let $w, w' \in T_+(V)$.

- As $1 * 1 = 1 \sqcup 1 = 1$, $\pi_V \circ H_1(1 \otimes 1) = \pi_V \circ H_2(1 \otimes 1) = 0$.
- $H_1(1 \otimes w') = 1 \sqcup H_*(w') = H_*(w')$, whereas $H_2(1 \otimes w') = H_*(1 * w') = H_*(w')$. Consequently, $\pi_V \circ H_1(1 \otimes w') = \pi_V \circ H_2(1 \otimes w')$. Similarly, $\pi_V \circ H_1(w \otimes 1) = \pi_V \circ H_2(w \otimes 1)$.
- As $H_*(T_+(V)) \subseteq T_+(V)$, $H_1(w \otimes w') \in T_+(V) \sqcup T_+(V)$, so is a linear span of words of length at least 2: $\pi_V \circ H_1(w \otimes w') = 0$. On the other side, it follows from the definition of H_* that $\pi_V \circ H_*(w) = \varpi_V(w)$ and we get

$$\pi_V \circ H_2(w \otimes w') = \pi_V \circ H_*(w * w') = \varpi_V(w * w') = \pi_V \circ e_*(w * w') = 0,$$

as e_* vanishes on $T_+(V) * T_+(V)$.

This concludes the proof that $\pi_V \circ H_1 = \pi_V \circ H_2$: the map H_* is an algebra and a bialgebra isomorphism. \square

Remark 4.1. In the quasi-shuffle case, this simplifies. Indeed, given two non empty words w and w' , $\langle w, w' \rangle$ vanishes excepted when w and w' are both of length 1. Therefore, using also that the quasi-shuffle product of two words of length p and q is a linear combination of words of length at least $\max(p, q)$, we get:

$$\begin{aligned} \varpi_*(v_1 \dots v_n) &= \frac{(-1)^{n-1}}{n} \langle v_1, v_2 \boxplus \dots \boxplus v_n \rangle. \\ &= \frac{(-1)^{n-1}}{n} \langle v_1, v_2 \cdot \dots \cdot v_n \rangle \\ &= \frac{(-1)^{n-1}}{n} v_1 \cdot v_2 \cdot \dots \cdot v_n, \end{aligned}$$

and we obtain that for any $v_1, \dots, v_n \in V$, with $n \geq 1$,

$$\varpi_*(v_1 \dots v_n) = \frac{(-1)^{n-1}}{n} v_1 \cdot \dots \cdot v_n.$$

Applying Theorem 4.3, we recover the Hoffman “logarithmic” isomorphism from $(T(V), \boxplus, \Delta)$ to $(T(V), \sqcup, \Delta)$ [10].

Proposition 4.4. *Let notation be as in Theorem 4.3. Let us define inductively ζ (by induction on the length of tensors) by*

$$\zeta : \begin{cases} T_+(V) \longrightarrow V \\ v \in V \longmapsto v, \\ w \in T_n(V), n \geq 2 \longmapsto \zeta(w) = - \sum_{k=2}^n \sum_{\substack{w=w_1 \dots w_k, \\ w_1, \dots, w_k \neq \emptyset}} \varpi_*(\zeta(w_1) \dots \zeta(w_k)). \end{cases}$$

Let us then define

$$E_* : \begin{cases} T(V) \longrightarrow T(V) \\ 1 \longmapsto 1, \\ w \in T_+(V) \longmapsto \sum_{k=1}^{\infty} \sum_{\substack{w=w_1 \dots w_k, \\ w_1, \dots, w_k \neq \emptyset}} \zeta(w_1) \dots \zeta(w_k). \end{cases}$$

Then E_* is the bialgebra isomorphism from $(T(V), \sqcup, \Delta)$ to $(T(V), *, \Delta)$ inverse to H_* as defined in Proposition 4.3.

Proof. Let us write L_* for the inverse isomorphism H_*^{-1} to H_* . As L_* is an endomorphism of the coalgebra $(T(V), \Delta)$, it is entirely characterized by the map $\zeta := \pi_V \circ L_*$ from $T(V)$ to V and

$$L_*(w) = \sum_{k=1}^{\infty} \sum_{\substack{w=w_1 \dots w_k, \\ w_1, \dots, w_k \neq \emptyset}} \zeta(w_1) \dots \zeta(w_k).$$

Moreover, as $H_* \circ L_*$ is the identity map for the coalgebra $(T(V), \Delta)$, L_* is also characterized by the identity

$$\pi_V \circ H_* \circ L_* = \pi_V,$$

or, equivalently, (as necessarily $L_*(v) = v$ for $v \in V$), by the identities $\varpi_* \circ L_*(w) = 0$ for $w \in T_n(V)$, $n \geq 2$. We get

$$\varpi_* \left(\sum_{k=1}^n \sum_{\substack{w=w_1 \dots w_k, \\ w_1, \dots, w_k \neq \emptyset}} \zeta(w_1) \dots \zeta(w_k) \right) = 0,$$

and, as ϖ is the identity map on V and $\zeta(w) \in V$,

$$\zeta(w) = -\varpi_* \left(\sum_{k=2}^n \sum_{\substack{w=w_1 \dots w_k, \\ w_1, \dots, w_k \neq \emptyset}} \zeta(w_1) \dots \zeta(w_k) \right).$$

The Proposition follows. □

Remark 4.2. In the quasi-shuffle case, this simplifies again and a closed formula for ζ and E_* can be obtained. Indeed, using Remark 4.1, we get in that case the inductive formula

$$\zeta(w) = - \sum_{k=2}^n \sum_{\substack{w=w_1 \dots w_k, \\ w_1, \dots, w_k \neq \emptyset}} \frac{(-1)^{k-1}}{k} \zeta(w_1) \dots \zeta(w_k),$$

which, using for example the identity of coefficients resulting from the formal power series expansion of the identity $\log \circ \exp(x) = x$, is solved by

$$\zeta(w) = \frac{1}{n!} v_1 \dots v_n,$$

where $w = v_1 \dots v_n \in T_n(V)$. Applying Proposition 4.4, we recover Hoffman's "exponential" isomorphism from $(T(V), \sqcup, \Delta)$ to $(T(V), \boxplus, \Delta)$ [10].

We conclude with a remark on the naturality of the constructions presented in the article. Morphisms of commutative B_∞ -algebras are defined in the obvious way: given $(V, \langle -, - \rangle_V)$ and $(W, \langle -, - \rangle_W)$ two commutative B_∞ -algebras, a linear map f from V to W is a commutative B_∞ morphism if and only if, for any v_1, \dots, v_{p+q} in V ,

$$\langle F(v_1 \dots v_p), F(v_{p+1} \dots v_{p+q}) \rangle_W = f(\langle v_1 \otimes \dots \otimes v_p, v_{p+1} \otimes \dots \otimes v_{p+q} \rangle_V),$$

where F is defined by $F(v_1 \dots v_i) := f(v_1) \otimes \dots \otimes f(v_i)$.

We put $*_V = \Theta^{-1}(\langle -, - \rangle_V)$ and $*_W = \Theta^{-1}(\langle -, - \rangle_W)$ and let the reader check that if f is a commutative B_∞ morphism from V to W , F is a bialgebra morphism from $(T(V), *_V, \Delta)$ to $(T(W), *_W, \Delta)$.

Proposition 4.5. *Let $(V, \langle -, - \rangle_V)$ and $(W, \langle -, - \rangle_W)$ be two commutative B_∞ -algebras. Let $f : V \rightarrow W$ be a morphism of B_∞ -algebras. The following diagram is commutative:*

$$\begin{array}{ccc} (T(V), *_{V}, \Delta) & \xrightarrow{F} & (T(W), *_{W}, \Delta) \\ H_{*_{V}} \downarrow & & \downarrow H_{*_{W}} \\ (T(V), \sqcup, \Delta) & \xrightarrow{F} & (T(W), \sqcup, \Delta) \end{array}$$

In other terms, the commutative B_∞ /shuffle isomorphism is functorial.

Proof. The Proposition directly follows from the definition of the maps H_* , ϖ and e_* , from the definition of a commutative B_∞ morphism, and the fact that F is a morphism of bialgebras (so that its action commutes in particular with taking products or computing coproducts). \square

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