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# Understanding fraction operations: The case of fraction division

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*This qualitative and interpretative study aims to identify grade 5 students' understanding of fraction division as a result of a teaching experiment based on the use of models and their knowledge at the end of the experiment. Data was gathered using four students' work on classroom tasks, an individual test and individual interviews. During the teaching experiment, students demonstrated not knowing a division operation as a measure. In the final test, they demonstrated to conceptually understand this meaning of division, associating the common denominator algorithm with the modelling process, and demonstrating operation sense.*

*Keywords. Rational numbers, fractions, fraction operations, division.*

## Introduction

The concept of rational number is very important in the mathematical curriculum of the early years, as a fundamental basis for the later understanding of algebra, but this concept is very complex for children, who have prior knowledge of whole numbers which have some rather different properties (Hansen et al., 2017). Then, often, students apply their whole number knowledge to rational number tasks. Students need to gradually expand the sets of numbers that they know, and they need to understand the difference between whole numbers and rational numbers (Van Hoof et al., 2021). Fraction operations are a fundamental aspect for understanding rational numbers but, despite the studies carried out, students continue to demonstrate difficulty in understanding it (González-Forte et al., 2022). In addition, the teaching methodologies for these numbers are often based on procedural aspects that students do not conceptually understand (Jarrah et al., 2022). In working with rational numbers, the student should explore and understand rich mathematical situations, in which operations arise, in order to construct and understand the algorithms, but they do not usually do that (Zembar, 2017).

Behr and Post (1992) emphasize the role of models, seen as representations that reflect mathematical aspects and that can take different forms (like geometric and area models), to help children better understand the conceptual basis of each operation, that is, the relationships between the quantities and between models and algorithms and understand what each step of an algorithm represents. The present study aims to identify grade 5 students' understandings of fraction division during a teaching experiment based on the use of models and their knowledge at the end of the experiment. The research question is: How did the teaching experiment allow the development of students' operation sense regarding division?

## Operation sense and the operation of division

Rational number operations represent a challenge for students, who demonstrate difficulty in conceptually understand them. When solving tasks, many students demonstrate weaknesses in their operation sense, which, according to Huinker (2002), includes several understandings, namely: (i) meanings (e.g., division does not always represent a sharing situation) and models for operations (like

rectangular, circular or area models, used to represent the operations); (ii) compose and decompose numbers using properties of operations; (iii) the relationship between operations (e.g., understand division as successive subtractions); (iv) the relationship between context and the required operation; (v) translations between representations; (vi) multiple solution strategies and select the most effective one; (vii) attributing meaning to symbols and formal mathematical language; (viii) knowledge of the effects of an operation on numbers; and (ix) reasonableness of results.

According to Lee (2017), students often understand division only as a sharing situation, possibly because it is usually the first meaning that they learn when approaching this operation and it is relatively easy to understand. Consequently, they think that this operation always gives a smaller result, and that the dividend must be greater than the divisor, thus, students do not fully understand the concept of division (Greer, 1992). It is also possible to know the total quantity and the quantity in each group and find the number of groups (measure meaning). So, verifying how many times the divisor goes into the dividend has a conceptual structure similar to dividing integers (Hansen et al., 2017). In the field of rational numbers, this meaning of division allows students to understand that division can originate a greater quantity than the initial quantity, which is a quite challenging idea for students. However, regardless the meaning used (sharing or measure), teachers usually teach how to solve this operation using the “invert and multiply” procedure, which does not represent the concept of dividing fractions (Ratnasari, 2018). In contrast, the common denominator algorithm (CDA) involves an understanding of division closer to understanding whole number division (Zembar, 2017). As an integral part of operation sense, the use of models allows students to represent the concept of dividing fractions (especially to understand the measure meaning of division, i. e, this operation does not always yield a result less than the dividend, depending on the numbers involved), to move from the concrete to the abstract (Jarrah et al., 2022; Ratnasari, 2018), and to understand the remainder obtained if the division is not exact (Zembar, 2017).

## **Methodology**

This teaching experiment was carried out in 2018/19 and the participants are 20 grade 5 students from a public school in Portugal studying the concept of division of fractions for the first time. All students attend support mathematics classes. Data were collected through students’ individual work on classroom tasks, from an individual final test applied after the teaching experiment, and from individual interviews to identify students’ reasoning in each question. The test had problems involving the four operations and other rational number concepts. The interviews took place in the same day the students solved the test and were all audiotaped, transcribed and analysed in depth. This allowed us to better understand the students' solving strategies in the test. Four students were interviewed: Laura, Maria, Tiago and Pedro (pseudonyms). Previous test results indicate that Laura and Maria had difficulty in mathematics and Tiago and Pedro showed more ease with the contents in relation to other students in the class, but all of them were fluent in oral communication. This method allowed us to analyse students’ operation sense according to the Huinker’s (2002) framework.

### **The teaching experiment**

The teaching experiment lasted eight weeks, consisting of fifteen tasks, and intended to address concepts related to rational numbers, including their symbolic representations, the meanings of fractions and operations with fractions, with special focus on division. The tasks involving addition, subtraction,

multiplication and division (that lasted a month, with nearly two weeks to division) in problem solving contexts were addressed separately, in this order. The tasks involved the use of models for their understanding and solution. The teaching experiment followed an exploratory approach (Ponte & Quaresma, 2016), that is, tasks were presented to children, and they worked individually, followed by whole-class discussion and synthesis.

The first author of this article taught the classes of the teaching experiment and conducted the whole-class discussions and the interviews with the students. Division of fractions focused on the “measure” meaning and the CDA. Most students were only familiar with division as sharing, but they had already approached the division as measure. It was necessary to recall the different meanings of division. To solve the tasks, the children were guided to use (circular or rectangular) models. First, they identified the initial quantity (dividend) and the divisor, and they wrote down the symbolic expression. In the next step, they represented the initial quantity on models. Then, they tried to verify how many times the divisor fit into that quantity. To do so, they had to subdivide the initial models according to the denominator of the dividend (when they are different). When carrying out these subdivisions, the amount of the dividend could be represented by another (equivalent) fraction. The students then had to substitute the fraction in the expression. All actions performed in the models were accompanied by the corresponding numerical sentence, so that the children could understand the relationship between them and discover the (common denominator) algorithm for themselves. When children finally had the same unit of measure (the fractions have the same denominator), they could count how many times (the numerator of) the divisor goes into the (numerator of) dividend. At a later stage, they could choose the solution strategy they preferred. The goal was to progress from the use of models to symbolic manipulation with understanding. In case the result of the division was not a whole number, the models also made it easier to interpret the “remainder”. However, in support classes for these students, the regular teacher presented the “invert and multiply” algorithm for dividing fractions. During the teaching experiment, the students initially insisted on using this algorithm because they considered it easier. I always insisted that they should not use this process.

## **Results**

### **During the teaching experiment**

During the teaching experiment, this was the first problem presented to the students: “Rita bought 2 boxes of ice cream and she wants to give each friend  $\frac{2}{3}$  of a box. How many portions can Rita give?” After recalling the different meanings of division (sharing, measuring), the starting point was “how many times the divisor fit in the dividend?” I drew two equal rectangular models on the board (the initial quantity), divided them into three equal parts and showed that the quantity  $\frac{2}{3}$  fit three times into the two units. The students understood that each portion corresponded to two subdivisions, but it was necessary to clarify that  $\frac{2}{3}$  referred to a box. Then, they wrote the numerical expression  $2 : \frac{2}{3} = 3$ .

The second problem was: “Mr. Joaquim has a wooden bar that is 75% of a meter long. He intends to cut it into smaller bars,  $\frac{1}{8}$  m. How many bars can he cut?” Some students represented a rectangular model divided into four parts and shaded three to represent 75%. They divided each shaded part in half, to obtain eighths ( $\frac{1}{8}$ ), and verified that they could obtain six parts with the shaded quantity. They

also wrote the numerical expression  $\frac{3}{4} : \frac{1}{8} = 6$  but they said they did not know how to solve. The result (6) was placed at the end of the numerical expression because it was the result of modelling process. However, several students demonstrated difficulty in dividing a model previously divided, and to understand if the quantity  $\frac{1}{8}$  would be related to the shaded part or to the entire model:

- Gustavo: I think it's in the shaded area... he can only cut the bars with the wood he has...  
 Dora: I think we divide everything into eight parts, but we only count the ones that are shaded...  
 Researcher: The bars have 1/8... Isn't it 1/8 of a meter?

The students understood that the unit related to the fraction  $\frac{1}{8}$  was the entire model, and not just the shaded part. The third problem was: "Lia has  $2\frac{1}{2}$  kg of sugar to make cakes. Each cake takes  $\frac{3}{4}$  kg. How many cakes can she make?" All students indicated the numerical expression  $\frac{5}{2} : \frac{3}{4}$  but only a few students, like Laura (Figure 1), also modelled the problem, using rectangles or squares, to demonstrate that they understood the symbolic expression. These students wrote  $\frac{5}{2} : \frac{3}{4} = 3$ , ignoring the remainder.



**Figure 1: Laura's solution**

Laura represented the initial sugar quantity using squares. To determine how many times she could get  $\frac{3}{4}$ , she divided them into quarters. Then, she made sets of three parts to verify how many cakes she could make. The discussion focused on the analysis of the remaining part in two ways:  $\frac{1}{4}$  of 1 kg, since the 1 kg was divided into four parts and one of these parts was left over; or  $\frac{1}{3}$  of a cake, since the cake involved three parts and the remaining part represented one of them. Then, the students and I try to conceptual understanding the symbolic representation. At this point, I approached the CDA. Students understood the expression  $\frac{5}{2} : \frac{3}{4}$  and Laura's initial modelling. The first author explained that, when performing the subdivisions, to get quarters, the quantities would be represented in a different way. Not  $\frac{5}{2}$ , but  $\frac{10}{4}$ . Laura mentioned that all quantities were "fourths". Afterwards, analysing Laura's model again and the numerical expression  $\frac{10}{4} : \frac{3}{4}$ , I needed to emphasize that it was necessary to divide just the numerators, because we already had the same units, i.e., to verify how many times 3 fits into 10. All changes in the symbolic expression were associated with steps in the modelling process. In the following questions, the students resorted mostly to procedures ("invert and multiply" algorithm or CDA). When they realized they could not get a whole number as an answer, they used modelling to interpret the remaining part.

### **After the teaching experiment**

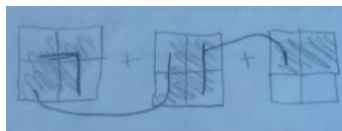
In the final test, the students had to determine the number of cakes that could be made with  $2\frac{1}{2}$  kg of flour, knowing that a recipe used  $\frac{3}{4}$  kg of this ingredient. Pedro using the CDA, obtaining  $\frac{10}{3}$ . He said that " $\frac{10}{3}$  is 3 complete units plus a little bit [...] each unit is  $\frac{3}{3}$ , so  $\frac{3}{3}$  plus  $\frac{3}{3}$  plus  $\frac{3}{3}$  equals  $\frac{9}{3}$ ". He also

analysed the remaining part: “ $\frac{1}{3}$  is left over...  $\frac{1}{3}$  for another cake [...] it is  $\frac{1}{3}$  of the part that each cake takes, because each one takes  $\frac{3}{4}$ ”. Maria (Figure 2) also used the CDA and justified: “when we have something to share by several people, but we don’t know how many people there are, we divide it by the amount each person takes, and we get the number of people!” She converted the mixed number to an improper fraction and multiplied it by 2 to get equal denominators.

$$2\frac{1}{2} = \frac{5}{2} \div \frac{3}{4} = \frac{10}{4} \div \frac{3}{4} = \frac{10}{3}$$

**Figure 2: Maria’s solution**

Maria also used modelling to show she understood the algorithm: “I took this amount [ $\frac{3}{4}$ ] several times and, each time, I could bake a cake! [...] since they are fourths, because there is a four at the bottom, I had to divide the rectangles into four parts, to be the same size! So, here it is  $\frac{10}{4}$  and every three pieces is one cake!” For the remaining part, she suggested that it would be “half of half a kilogram”. Then, analysing the model, Maria was able to interpret the remaining part using the amount of each cake as a unit of measure. She said it would be “ $\frac{1}{3}$ , because this little bar here, if we put 3 together, that’s what each cake takes!” Tiago obtained the improper fraction  $\frac{10}{4}$  from the mixed number  $2\frac{1}{2}$  and explained: “it has  $\frac{10}{4}$  of flour and each recipe takes  $\frac{3}{4}$ . She can make 3 cakes because it’s  $\frac{3}{4}$  plus  $\frac{3}{4}$  plus  $\frac{3}{4}$ . For the fourth cake there isn’t enough!” When he was invited to model the problem, during the interview, he made the representation in Figure 3:



**Figure 3: Tiago’s solution**

Tiago said that “every 3 pieces of these, are a cake! [...] there are 3 cakes, but not enough for another one with the flour we have”. To associate the modelling process to the addressed algorithm and establish an association between the result obtained and the quantity that was left in his modelling, I suggested using it as well. He performed the operation and understood the resulting fraction,  $\frac{10}{3}$ , as a division of two integers, which resulted in 3,333 (using calculator). The integer part of the numeral was easily associated with the number of cakes that could be made. The decimal part, Tiago did not interpret so quickly. However, analysing his modelling, he was able to verify that it was the part corresponding to one cake.

Laura also performed the division  $\frac{10}{4} : \frac{3}{4}$  using the CDA, like Maria. She explained “I used to think that... for example... we have an amount of food to divide between three people, and we would see what each person gets!” Laura justified the use of equal denominators: “I remember that, if we divide the amount of flour to see how many times there are  $\frac{3}{4}$ , the flour also must be divided into 4 parts. For this reason, I put fourths!” She divided 10 by 3, obtaining the result 3,3 and indicated: “Maria makes three cakes and there is a little left over!” After modelling the question, as suggested by me during the interview, she understood that the remaining part corresponded to  $\frac{1}{3}$  of a cake “because each cake

has three pieces of these” or  $\frac{1}{4}$  of a kilogram “because each model represents a kilogram, and it is divided into four parts!” It should be noted that, after obtaining equal denominators, some students dividing numerators and denominators. This happened when dividing the numerators yielded an integer. When it did not, the students gave an approximate whole number result.

## Discussion

During the teaching experiment, these students showed difficulty in the operation sense and misunderstandings of its components. They knew the division only in the meaning of sharing and they had some difficulties in understanding that this operation can also increase, and that the divisor can be greater than the dividend, as in Greer (1992), and in Lee (2017). According to Huinker (2002), this is a fundamental aspect associated with the operation sense. The modelling process was also a challenge for students. They demonstrated unfamiliarity with this solution strategy. In general, at the beginning, it was not easy and straightforward to subdivide models into equal parts, especially, to subdivide an already divided model. The students showed no flexibility with the reference unit and difficulty to interpret the remainder (similar difficulties were identified by Zembat, 2017). Nevertheless, this may be because the problems were not designed to elicit this kind of interpretation. The students were only questioned orally regarding this part. The preference for the procedures was visible almost from the beginning. This happened, possibly, due to the influence of their teacher, who conveyed to them the speed and efficiency of using the algorithm. The students, at an early stage, used modelling to obtain the result but later they used it just to interpret the remaining part. However, the symbolic expressions demonstrated that these students knew the meaning of each fraction in the context of the problems and some flexibility in converting between symbolic representation (task 2 and 3).

After the teaching experiment, it was possible to verify that the students made progress in their operation sense. They showed more deeper and complete understanding of the division operation, namely by referring “when we have something to share [...] but we don’t know how many people there are, we divide it by the amount each person takes, and we get the number of people!” (Maria). This also demonstrates to associate context/operation. These students showed knowledge of models for this operation and some flexibility in the modelling process, which was explained by Helena: “[...] to see how many times there are  $\frac{3}{4}$ , the flour also must be divided into 4 parts”. Understanding fraction size, through their decomposition, was a strategy that was also verified, especially by Pedro and Tiago. This helped to obtain the results more easily or assign meaning to the results obtained, which is a fundamental aspect in the operation sense. These solutions and statements also demonstrate that these students associated and understanding the relationship between division and repeated subtractions (“here, I took this amount  $[\frac{3}{4}]$  several times and, each time, I could bake a cake!”, Maria), which is another indicator of operation sense. While explaining their reasoning during the interviews, these students reasoned again about their solutions and were confident with the results, especially when the modelling process reinforced the results they had obtained. Regarding the remaining part, the students mostly interpreted it by observing the modelling. Only Pedro did not need a model to do so, since he associated the remaining quantity,  $\frac{1}{3}$ , with the quantity each cake took,  $\frac{3}{3}$  (“it is  $\frac{1}{3}$  of the part that each cake takes, because each one takes  $\frac{3}{3}$ !”). As verified in the studies by Behr and Post (1992), the

modelling process made it possible a better understanding of division. Modelling allowed the students to associate the division as a measure to a set of successive subtractions, to visually represent the fraction size, to establish quantitative relationships, to understand the meaning of the denominator, to interpret the remaining part, and, consequently, to assign meaning to calculation procedures.

## **Conclusion**

This teaching experiment allowed the students to conceptually understand the division operation, namely regarding the measure meaning, and overcome the misconception that “division always decreases”. This was due to building and manipulating models, which also allowed to construct the algorithm and, consequently, to conceptually understand it. It should be noted that, in an initial phase, the models were used to reason about the problems and quantities involved. At a later stage, most students used models just to show that they had understood the algorithmic procedures carried out. The use of modelling showed to yield a deeper understanding of each situation, which the algorithm does not allow (Jarrah et al., 2022). The CDA, unlike to the “invert and multiply” algorithm, allowed these students to associate fraction division with whole number division, previously known to them. A context for the operations encouraged the students to interpret the quantities more adequately, and the relationship between them. The content of the task as well as the question have an influence on its understanding and solution, due to the language used and the level of difficulty. Difficulties persisted in some students in dividing previously subdivided models, especially when the initial quantity was greater than a unit. Likewise, it is not easy for these students to interpret the remainder, according to the reference unit, without the aid of the model.

Thereby, the teaching methodology used in this study proved to be effective to develop some components of operation sense. These students gained a better understanding of contexts, meanings, and models for operations, especially regarding division, and the relationship between operations (e.g., division as successive subtractions). Translations between representations and the use of different solution strategies were also demonstrated, as well as the justification of its use. It was also verified that the students analysed the results, both by models and symbolically, according to the context. This represents a different approach to rational number operations, promoting students’ conceptual understanding because they build the algorithm themselves. However, the difficulties that still arose reflect the difficulty of this subject and the limited instructional time to focus on situations involving fraction division. These results, globally, suggest that it is advisable to shift the emphasis on teaching fraction operations from learning procedures to developing operation sense. It should be noted that the sample of this study is small, so we cannot generalize these data, but it may indicate possible ways to approach this topic. Perhaps a study with a more significant sample can confirm these results or bring new data.

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