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Prospective teacher's representations of fractions

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Mathematics learning is more effective if students explore ideas in different ways by making connections between different representations. This paper aims to share an exploratory study with 34 prospective elementary school teachers that seeks to understand what kind of representations they have about some fractions. In the beginning, symbolic representations were strongly used, mainly equivalent fractions. Although this mode of representation remained, some diversity of combinations of symbolic, visual, and verbal representations emerged throughout the study, highlighting students' interconnected knowledge of fractions and their capacity to represent fractions in different ways.

Keywords: Rational numbers, fractions, multiple representations, prospective teachers.

Introduction

Effective mathematics instruction needs to engage students in making connections between mathematical representations to contribute to a deeper understanding of mathematical concepts and procedures (NCTM, 2017). In line with this idea, Portuguese mathematics teachers are currently implementing a new curriculum where the development of multiple representations is one of the six skills that students must develop along with other skills such as communication, reasoning, problem-solving, connections, and computational thinking.

The complexity of teaching and learning rational numbers is widely accepted and the use of multiple representations can be a powerful tool to increase students' understanding and interconnected knowledge (Moseley, 2005; Tripathi, 2008). In this sense, prospective teachers' instruction needs to engage them in sharing and discussing multiple representations when learning mathematics, so they could do it with their students while they are teaching. Opportunities to produce external representations and the internalization of mathematical ideas through a social activity must be given to future teachers (Pape & Tchoshanov, 2001). In addition, Dreher et al. (2016) suggest that the use of multiple representations and their interconnections for conceptual understanding of mathematics should be seen as a key element in the development of teachers' pedagogical content knowledge.

To provide such an opportunity, an exploratory study was conducted during the course of Numbers and Operations in the second semester of the first year of the bachelor's degree that prepares future teachers. This study seeks to understand what kind of representations prospective elementary school teachers have about fractions.

Theoretical background

Mathematical representation can be described as “a mental or physical construct that describes aspects of the inherent structure of a concept and the interrelationships between the concept and other ideas” (Tripathi, 2008, 438). It is a way to externalize an internal, mental abstraction representation and must be thought as a tool for thinking, explaining, and justifying mathematics (Pape &

Tchoshanov, 2001). When someone produces an external representation (e.g., a number line, a numerical expression, or a circle) a physical product is provided to communicate, and directly to teach mathematical ideas (Mainali, 2021).

In 1966, Bruner presented a learning model based on three levels of engagement with representations: enactive (e.g., manipulating concrete materials), iconic (e.g., pictures and graphs), and symbolic (e.g., numerals). In 1987, Lesh, Post, and Behr based on Bruner's ideas suggest that mathematical concepts can be explored using multiple ways such as manipulatives, pictures, and written symbols, as Bruner stressed, and added two more levels - verbal and real-world.

According to Cramer et al. (2002) the use of iconic and enactive modes of representation is essential to make connections to oral language and symbols. A lack of understanding of the relationship between symbolic representations of a fraction and its enactive or iconic representation does not allow the student to comprehend the relationship between the numerator and the denominator or how this relationship affects the size of the number. For example, a student develops a deep understanding of the meaning of $\frac{3}{4}$ (symbolic form) when they can understand it as a quantity formed by 3 parts with the size of one quarter represented in a rectangle or in a number line (iconic form), or as a measure of a string that has lengths of three-quarters of a meter (enactive form). Cramer et al. (2002) also emphasise that the fundamental concept of equivalent fractions should reflect more than just procedural knowledge for generating equal fractions, it should be rich in connections between symbols, models, images, and context.

In Behr's et al. (1983) perspective, there are five meanings for rational numbers that students need to understand: (i) part-whole relationship; (ii) measure; (iii) quotient; (iv) operator; and (v) ratio. If representing mathematical concepts in different ways allows the construction of connections between representations, highlights conceptual knowledge and various aspects of the structure of the mathematical concept, "using different representations is like examining the concept through a variety of lenses, with each lens providing a different perspective that makes the picture (concept) richer and deeper" (Tripathi, 2008, 439). In this sense, the use of multiple representations certainly contributes to students' deeper understanding of rational numbers and their different meanings.

Thus, mathematics learning is more effective if students explore ideas in different ways by making connections between different representations so that they can be aware that symbols are just only one way to represent mathematical ideas (Cramer et al., 2002; Kilpatrick et al., 2001).

Methods

This research has an exploratory design framed by a qualitative methodology (Denzin & Lincoln, 2005). The participants were 34 prospective elementary school teachers (from now on named only students), even though the numbers of students that attended the lesson varied during the study between 21 to 34, from the course of Number and Operations (second semester, first year of the bachelor's degree). The data was collected weekly for five weeks, through research teacher and students' notes.

The task proposed to the students was the routine "Fraction of the day", while they were working with rational numbers in the course. The course had two lessons a week. In one lesson, the students

were challenged to write down as many representations of a given fraction as possible on a sheet of paper. The teacher collects students' notes, and for the second lesson of the week, she selects a diversity of representations that can promote a whole class discussion about different modes of representations and their relationships. These discussions intend to increase students' representations of fractions, emphasise the importance of using multiple representations of a fraction, share different representations, and relate students' knowledge to what was being covered in the course about rational numbers so that it could be deepened.

The routine "Fraction of the day" started with important benchmarks $\frac{1}{2}$ and $\frac{3}{4}$. The third fraction was $\frac{2}{3}$ which represents a repeating decimal. The fourth fraction was $\frac{4}{5}$, one very close to the unit and the last fraction used was $\frac{5}{10}$, equivalent to the first one.

Inspired by Lesh et al. (1987) the data was analysed using three main categories and several subcategories that emerged from the data, as shown in Table 1. Manipulatives were not used in the routine, and for that reason, it was not considered. All the written forms used by students that were not symbols (numbers), were included in the "verbal" category.

Table 1: Categorization of fractions representations

	Categories		
	Symbols	Visual	Verbal
Subcategories	Decimal Repeating decimal Equivalent fractions Percent Operations Others	Discrete unit Continuous unit Number line	Fraction/decimal reading Word problems Others

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"Fraction of the day" was a number routine developed with students from the course of Numbers and Operations in 2021-22. During the five weeks of the study, students used multiple representations for each fraction of the day that reflected their conceptual and procedural knowledge. Table 2 gives an overview of the representation students used for all five fractions. Students started with strong use of symbolic representations (16 out of 25 students used only symbolic representation) in the first fraction, but along the study, they integrated other representations including word problems.

According to the data, they moved from only using symbolic representations to combining them with visual and verbal representations (18 out of 27 students) by the end of the study. The last fraction of the day was $\frac{5}{10}$ to see if students' representations changed when compared with $\frac{1}{2}$, an equivalent fraction used in the first routine. Table 2 shows that no one used only symbolic representations to represent $\frac{5}{10}$ as they did before with $\frac{1}{2}$.

Table 2: Numbers of students who used a certain mode of representation

Mode of representation	Fraction of the day				
	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{4}{5}$	$\frac{5}{10}$
Symbolic	16	4	2	---	---
Visual/verbal	---	---	---	1	---
Symbolic/verbal	---	---	---	1	---
Symbolic/visual	8	24	11	21	9
Symbolic/visual/verbal	1	4	8	11	18
Total number of students	25	32	21	34	27

Table 3 presents the relation between the number of students who used a certain mode of representation in each category and the number of students who participated in the routine. Concerning symbolic representation, equivalent fractions were the most used in the study, but a considerable number of students used frequently decimals and percentages. Only in the fraction, $\frac{3}{4}$, the use of decimals had a higher frequency when compared with other symbolic representations.

Table 3: Relation between the number of students who used a certain mode of representation in each category and the number of students who participated in the routine

Mode of Representation		Fraction of the day				
		$\frac{1}{2}$	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{4}{5}$	$\frac{5}{10}$
Symbolic	Decimal	23/25	23/32	---	12/34	23/27
	Repeating decimal	---	---	6/21	---	---
	Percent	15/25	21/32	3/21	8/34	20/27
	Equivalent fractions	24/25	21/32	19/21	24/34	25/27
	Operations	5/25	14/32	10/21	19/34	11/27
	Others	2/25	---	---	4/34	1/27
Visual	Discrete unit	5/25	9/32	6/21	13/34	12/27
	Continuous unit	7/25	24/32	17/21	28/34	24/27
	Number line	---	4/32	7/21	15/34	7/27
Verbal	Fraction reading	2/25	4/32	5/21	9/34	12/27
	Word problems	---	---	2/21	5/34	8/27
	Others	---	---	---	1/34	1/27

Figure 1a) shows some students' representations of $\frac{1}{2}$ used for discussion by the teacher. Many equivalent fractions can be observed, although there are others such as visual representations using discrete and continuous units or fraction as an operator ($\frac{1}{2}$ of 10 = 5). Despite emerging in the discussion, no one used number line or word problems in the first fraction.

One student tried to generalize fractions that represent $\frac{1}{2}$, through the expression $x \div 2$. This expression was included in the subcategory of "Others" since it is an algebraic expression. To discuss this expression and its veracity, some divisions between two numbers (that emphasise fractions as a quotient) were suggested by students. They started with $1 \div 2$ and then $4 \div 2$. This second quotient leads them to understand that $x \div 2$ was not a generalization of fractions that represents $\frac{1}{2}$. It needs to be $x \div 2x$. This conclusion that arose from the whole class discussion highlights the relation between the numerator and denominator in this kind of fraction.

In Figure 1b), to the fraction $\frac{3}{4}$ it is possible to observe that students added some more modes of representation when compared with those used for $\frac{1}{2}$. Number line appears as well as a lot of operations that emphasised important relationships such as the decomposition of $\frac{3}{4}$ into $\frac{1}{4} + \frac{1}{2}$ or $3 \times \frac{1}{4}$ using multiplication, repeated addition with fractions or with decimals, as well as $\frac{3}{4}$ as half of $\frac{3}{2}$, for example. Verbal representations appeared only through fraction/decimal reading (e.g., three-quarters or seventy-five hundredths).

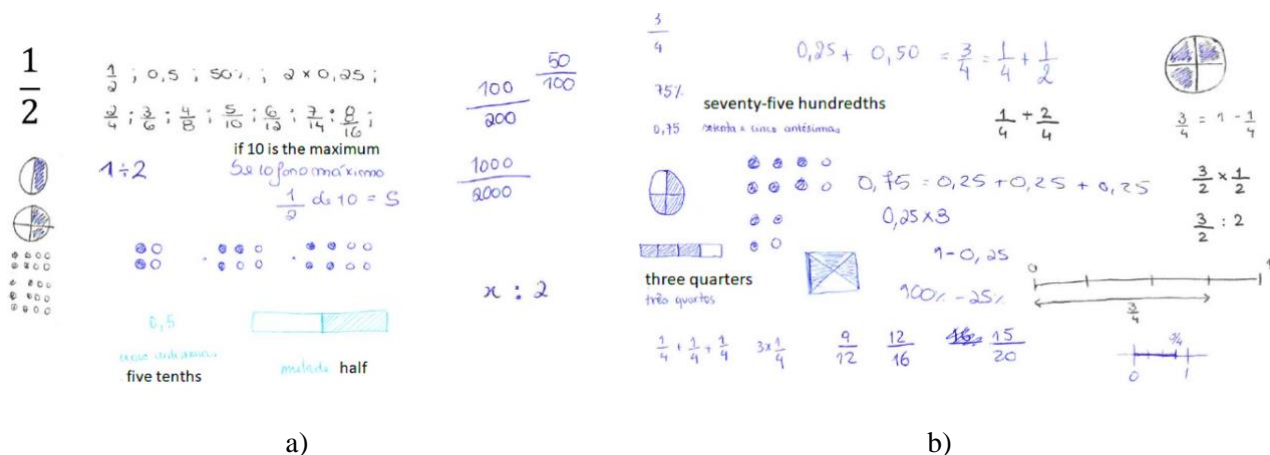


Figure 1: Students' mode of representation a) of $\frac{1}{2}$ and b) of $\frac{3}{4}$

During the discussion, the teacher challenged students to think in a context that could give meaning to $\frac{3}{4}$. This was an important input for students to start using word problems afterward (Figure 2).

For $\frac{2}{3}$, sharing and measurement context appeared in students' word problems as well as a triangle (not common representation) as a continuous unit. The visual representation of the $\frac{2}{3}$ shaded in the triangle, circle, or rectangle, or even in the discrete model, can be easily connected with the operation $1 - \frac{1}{3}$ where a piece with a size of $\frac{1}{3}$ was removed from the unit.

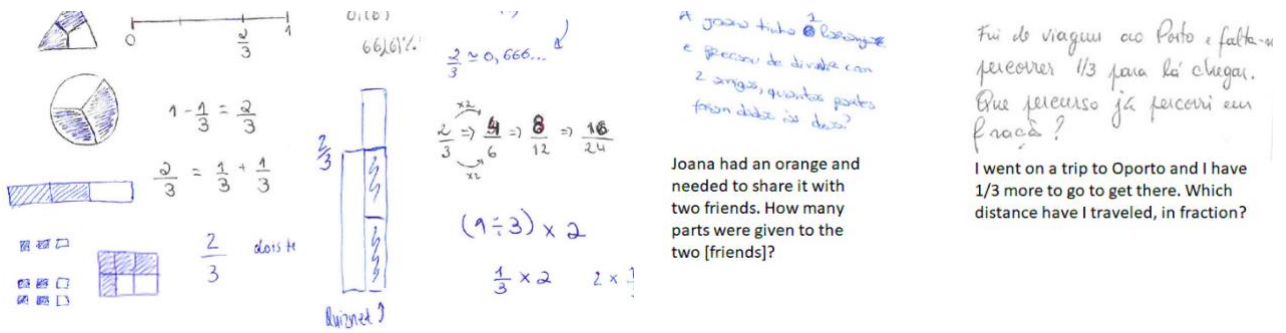


Figure 2: Students' mode of representation of $\frac{2}{3}$

New modes of representation emerged in the fraction $\frac{4}{5}$, influenced not only by the whole class discussion of the previous fractions but certainly by the rational number approach that was ongoing in the class. Several students also analysed the magnitude of the fraction comparing it with $\frac{1}{2}$ and with the unit as we can see in Figure 3. Students' word problems stressed part-whole relationships and measurement in the context of a pizza and the length of a table.

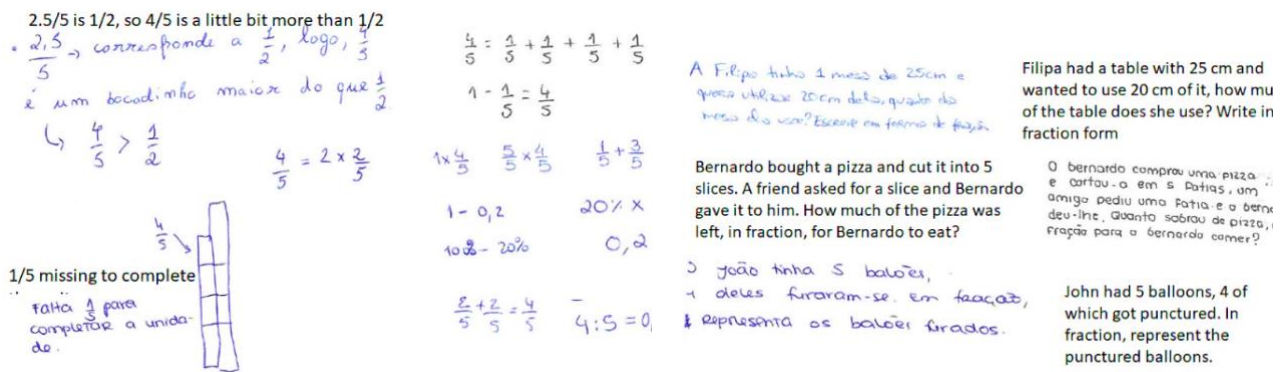


Figure 3: Students' mode of representation of $\frac{4}{5}$

In the last fraction, $\frac{5}{10}$ equivalent to the first one used, a diversity of combinations between modes of representation emerged when compared with those used by students in the case of $\frac{1}{2}$. For example, the set of representations inside the rectangle in Figure 4, belongs just to one student. This shows a big effort to share different meanings of $\frac{5}{10}$ using symbolic, visual, and verbal representations. In the case of this student, it is possible to perceive that there are less symbolic representations when compared to the visual ones. One of the students used the square root of $\frac{25}{100}$ to represent $\frac{5}{10}$. Although it does not highlight the concept of a fraction like other modes of representation, it is interesting to observe that the student knows a different equivalent symbolic representation of $\frac{5}{10}$.

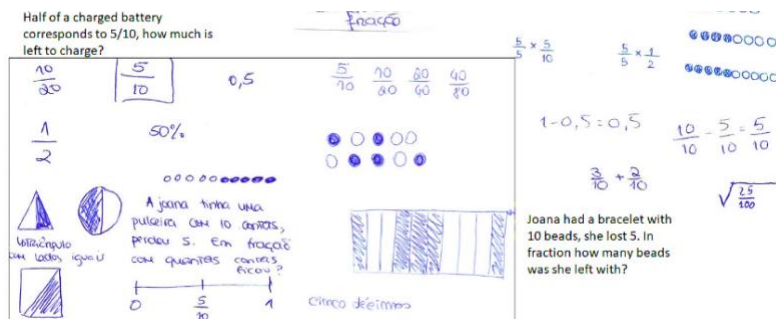


Figure 4: Students' mode of representation of $\frac{5}{10}$

Discussion and conclusion

The routine “Fraction of the day” provided students the opportunity to discuss their fractions representations through whole-class discussions. A high number of students used strongly, at the beginning of the study, symbolic representations (e.g., equivalent fractions), but over the five weeks, a diversity of combinations of modes of representations (symbolic, visual, and verbal) emerged. From the second routine on, students started to present several interconnected representations. For example, to the fraction $\frac{2}{3}$, they shaded two parts with the size of $\frac{1}{3}$ in a continuous unit (triangle, circle, and rectangle) divided into three equal parts that can be related with $\frac{1}{3} + \frac{1}{3}$, $2 \times \frac{1}{3}$ or even visually with $1 - \frac{1}{3}$ (one unit with $\frac{1}{3}$ not shaded). They also selected 4 objects from a set of 6 in a discrete unit (where the disposition of objects emphasised that 2 groups of 2 objects were shadowed) and used a point on a number line divided into 3 or 6 equal parts to represent $\frac{2}{3}$ as well as the multiplicative relations between numerators and denominators to express the equality $\frac{2}{3} = \frac{4}{6}$. All these representations highlight that students understand how to represent equivalent fractions beyond the symbolic form (Cramer et al., 2002; Kilpatrick et al., 2001). Note that for $\frac{1}{2}$, the first fraction, 24 out of 25 students used equivalent fractions in symbolic form, but during the study, word problems with context that could be related to part-hole relationships (e.g., shown in the circle representation) and with measurement (e.g., shown in the number line) also appear. Thus, the use of multiple representations seems to have increased the students' interconnected knowledge of fractions (Moseley, 2005; Tripathi, 2008) and enabled them to represent fractions in different ways. The work developed in this study proved to be essential for discussing mathematical ideas and the concept of fractions in a relational way using different lenses (Tripathi, 2008). Considering the students' evolution, this routine seems to have contributed to improving the students' representation of fractions and, therefore, deepening their knowledge of rational numbers and their meanings (Behr et al., 1983), especially the meanings of part-whole, measure, quotient, and operator.

This exploratory study highlights the importance of being aware of prospective teachers' multiple representations of mathematical concepts and its importance to develop a more consistent, interconnected, and deeper mathematical knowledge. This study can be a starting point for further research about prospective teachers' multiple representations in the field of rational numbers (not only fractions) as well as with in-service teachers.

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