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Partitive division and quotative division in children's thinking: New individual case studies and considerations for teaching

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Studies of conceptual understanding of division at the secondary school level show widespread problems in associating meanings (actions of either sharing or measuring and related everyday situations and word problems) with division terms (symbolic level). Those students who manage to make such connections at all, tend to interpret division in terms of sharing, while interpretations in the sense of measuring are relatively rare. However, when first introduced to division in primary school, children seem to find the measurement interpretation at least as accessible. The paper reports on a small longitudinal study of the development of 11 children's mental models of division from second to fourth grade. The study suggests that many children had problems linking the interpretations of division they showed when using material, which very often corresponded to measuring, to viable prototypes of everyday situations. Implications for teaching are discussed.

Keywords: Partitive division, quotative division, mental models, conceptual understanding.

Two ways to divide: a short didactic analysis

In the mathematics education literature, two different kinds of division problems and thereto related concepts are considered to be of fundamental importance at both primary and secondary levels, namely quotative (measurement) and partitive (sharing) division (Carvalho & Ponte, 2019).

A didactic analysis of the subject matter suggests that at the *enactive* level of representation when material is used to solve a given division problem, the solution can be found more directly for *quotative* problems: it is sufficient to repeatedly take away groups of the size given by the divisor from the known dividend until this is no longer possible. In contrast, in partitive problems, the divisor gives the number of subsets among which the dividend is to be distributed equally. If you try to solve such a problem at the enactive level, you do not know at the start but must work out how many elements to assign to each subset. A possible, but with higher divisors and dividends rather tedious way to do this is to distribute one item at a time to each recipient until nothing is left. It is also possible to assign elements in pairs or even larger groups to the number of recipients indicated by the divisor based on an a priori estimate of how many elements each individual would presumably receive. Any misestimates must then be subsequently compensated for by moving individual elements.

At the *symbolic* level, the quotative interpretation of a division term asks for the interpretation of both given numbers, dividend, and divisor, as sets/quantities. This aligns with an interpretation that a child may have acquired by this time in learning addition and subtraction, which usually precedes learning division. In addition, and subtraction, the two given numbers can also be interpreted as sets. However, such an interpretation does not work for either multiplication or partitive division, where one of the

numbers in the term acts as a factor or “number of sets” of the same size. In the case of multiplication, this is known to cause difficulty for some children in learning (Gaidoschik, 2014).

Empirical framework and theoretical background

How do children cope with the above-described duality of division? Taken as a whole, research on what might be called “individual models” (Prediger, 2008, p. 7) of division and their development in the course of mathematic lessons over the years is relatively scarce. Empirical studies, however, show that solving division problems is associated with considerable difficulties for a significant number of students, in primary school (e.g., Mulligan, 1992) and beyond (e.g., Ehlert et al., 2013).

Evidence suggests that students at the *secondary* level and young adults are generally more successful at solving division word problems with a partitive structure (Bell et al., 1984; De Corte & Verschaffel, 1996; Fischbein et al., 1985; Mulligan, 1993). Similarly, when asked to invent stories to fit a given division term, secondary school students predominantly formulated stories that could be assigned to partitive division (Bell et al., 1984). Fischbein et al. (1985) asked students in grades 5, 7, and 9 to identify the operation needed to solve, inter alia, division word problems. Typical errors in this task seem to stem from the “tacit model” that “division makes smaller”, and, connected to this, the larger number in a division must be the dividend (Fischbein et al., 1985, p. 12). In the same study, only in grade 9 did more than 50% of students give the correct term for quotative division word problems with dividends less than 1. Thus, it was only at this age that the performance was consistent with the authors’ assumption that “in quotative problems, the negative effect of the divisor being a decimal is diminished” (Fischbein et al., 1985, p. 13). Therefore, the authors (p.14) conclude, “there is only one intuitive primitive model for division problems – the partitive model”. In contrast, the quotative model would be acquired only later, with instruction, in the secondary grades.

Subsequent studies of division at the secondary level have focused on the dependence of solution frequencies on the number type (integer or decimal) and size (smaller or bigger than 1) of the numbers that occur, the “number type effects” (De Corte & Verschaffel, 1996, p. 224). Whether the availability of models for quotative and partitive division can play a role in solving division tasks is not part of the analysis in De Corte and Verschaffel (1996).

Studies on the understanding of division in *primary* school (e.g., Mulligan, 1992; Murray et al., 1992; Neuman, 1999) show a quite different picture. The results are inconsistent regarding preferences for one of the two types of division. In any case, a general preference of *primary* school children for partitive division when solving division problems cannot be deduced from the available data. Nor can the assumption be confirmed that younger children find quotative division problems fundamentally more complex than partitive ones. On the contrary, Neuman (1999) shows, for example, that before the division was discussed in class, children were more successful in solving quotative (Neuman herself uses the notion “quotitive”) division problems than in solving partitive division problems. Interestingly, Neuman reports that some students resorted to quotative strategies when solving partitive problems. For example, in the task in which 28 marbles had to be divided fairly between 7 boys, “the number of items [...] represented by the divisor was repeatedly taken from the dividend, as in quotitive division, but then one marble dealt to each boy every round” (Neuman, 1999, p. 112).

Research Questions

The didactic analysis in section 1 provided some reasons to assume that, at the enactive level, *quotative* problems are easier to solve than *partitive* ones. However, looking at the empiric studies of mental models of division and their development in primary and secondary schools, some of which were reported in section 2 of this paper, it seems that older children make less use of the *quotative* interpretations that were present and perhaps even dominant when they were first confronted with division problems. Instead, if they associate formal arithmetic with ideas of action and everyday situations at all, they seem increasingly to associate division with *partitive* division. Why this?

As mentioned, research on children's mental models of division and their development is scarce, and even more so research that also relates this to what kind of instruction children receive in class. Of particular interest would be studies that examine the development of mental models of division over a more extended period of time, paying particular attention to whether and in what ways children can activate both *quotative* and *partitive* ideas. At the same time, it should be recorded whether and how the children's lessons deal with both *quotative* and *partitive* division over this longer period, as it is plausible to assume that teaching has an influence on the development of these mental models.

Against this backdrop, the first author of this paper conducted a small-scale longitudinal study with the overarching aim of gaining a broader empirical basis for an educational design research project planned in the future. In the following, some findings of this study are presented with the intention of contributing further perspectives to the issues raised in the first two sections of this paper. In particular, we report findings in relation to the following Research Questions (RQs) 1 to 3:

- 1) When asked to invent word problems matching division terms and to represent terms with material, do individual children show a preference for one of the two types of division, i.e., *partitive* and *quotative* division?
- 2) (How) does this change between the second year of school, when (in Austrian schools) division is introduced, and the end of the fourth year, when they leave (Austrian) primary school?
- 3) In what ways, if any, do the children demonstrate an awareness of or, conversely, problems with the existence of these two different types of division over the years?

Methods

Burtscher (2022) conducted a longitudinal study of repeated qualitative interviews with the same children from the middle of second to the end of fourth grade to explore these questions. The children were selected, subject to parental consent, from two urban primary schools in Austria whose second-year teachers had volunteered to participate in the project. To cover a wider range of mathematical abilities, six children were selected from each school, two of whom, according to the teachers' personal judgement, were low, high, and average achievers in mathematics. Eleven of them took part in all the interviews and are considered below. The children were interviewed four times each, in a quiet room separate from the classroom: at the end of Year 2, after division had been introduced and practised for about six weeks; at the beginning of Year 3, after division had been revisited after the summer break; and finally at the end of Year 3 and Year 4, to capture possible developments from the establishment of division understanding to the end of primary school.

During the interviews, the children were asked to (a) present a division term shown on a card with material (wooden cubes), (b) tell a story that fits a given division term, and (c) write down a term suitable to solve a given word problem. Throughout, they should give verbal explanations. The tasks were basically the same in all four interviews, with slight changes in numbers and contexts underlying the word problems. The word problems were both quotative and partitive division and, as distractors, multiplication problems. In (a) and (b), if the child represented a term with the material in the sense of quotative division or embedded it in a quotative division story, the child was first asked to elucidate this solution and to explain why and how it would fit the given term. The child was then confronted with an alternative interpretation of the term corresponding a partitive division (sharing), and vice versa if the child's first interpretation had been partitive. The child was then asked whether this other word problem or the alternative representation would also fit the given task, or maybe even fit better.

All interviews were videotaped, transcribed, and subjected to a qualitative content analysis according to Mayring (2015). The criteria for analysing were formed partly deductively based on the theoretic assumption that, especially in *translating* between symbolic and enactive representations as well as word problems, together with verbal explanations, children indicate how they interpret a given term (e.g., Bönig, 1995). Therefore, we coded whether a translation matched the term, whether it corresponded to a partitive or a quotative division, and whether the child arrived at the translation completely alone or with more or less clear support, e.g. hints on inconsistencies in the story being told, help in formulating the question, and the like. In addition, it was necessary to create inductive categories to adequately reflect the wide range of behaviours, particularly when analysing the children's performance when confronted with the alternative interpretation of a given term.

In line with the assumption (see above) that instruction may influence mental models and should therefore be considered when analysing their development, at the end of each school year, the teachers were asked to report on how they had dealt with division during the year. In addition, both teachers had agreed in preliminary meetings that when they introduced division in Year 2, they would focus on quotative division (which they did for about three weeks) and only a few weeks later would they introduce partitive division. According to their reports, this second phase lasted about two weeks in both classes. In the last two weeks before the summer break, verbal problems in both aspects were dealt with within the same lessons. Both teachers stated that in Year 2 they had put a lot of emphasis on the children representing division terms on their own with material and inventing word problems to go with the terms, but without comparing and contrasting the two types of division. The latter had not been a concern of the teachers in Years 3 and 4 either, but they stated that they had paid more attention to word problems than in previous years as a result of their participation in the project.

Results

Compared with other studies (e.g., Bönig, 1995), the first thing that stands out is that almost all the attempts made during the interviews to (a) present and explain the meaning of a given division task with material were successful (Table 1). Over the three years of the study, 43 out of 44 changes of representation of this category were adequate, accompanied mainly by comprehensible verbal explanations by the children. Only during the first interview one child (originally classified as a low achiever by his teacher, see above) was found unable to solve this task. A preference for quotative

division (already noted by Bönig, 1995, and Neuman, 1999) is clearly evident in this type of change of representation: In 38 cases (88%), the given term was represented as quotative division, and only five times as partitive division. These preferences were relatively stable over the whole period, without evidence of correlations with the initial assignment of children to achievement levels by teachers apart from the one case explained above.

When (b) the children had to invent a word problem to fit a given division term, in only 13 of the 44 interviews did they manage to do so completely on their own. Four out of eleven children could not solve this task without very concrete prompts at any of the four interview time points. Among these four children were two that had originally been assigned as high achieving. The other two alleged high achievers were able to complete the task in each interview. Notably, in ten of the 13 successful cases (77 %), the children invented partitive division problems. Overall, clearly more children in Year 4 could invent a matching word problem than in Years 2 and 3 (Table 1).

	a) term → cubes	b) term → word problem	c) word problem → term
T1 – end of Year 2	90,9 %	18,2 %	63,6 %
T2 – early Year 3	100 %	18,2 %	86,4 %
T3 – end of Year 3	100 %	27,3 %	90,9 %
T4 – end of Year 4	100 %	54,5 %	100 %

Table 1: Correct changes of representation given without help in 44 interviews from Years 2 to 4

Translations of type (c), i.e. finding the term that fits a given word problem, did not cause similar difficulties and were eventually successfully made by all children in Grade 4, regardless of whether it was a division of this or that type (Table 1).

In terms of the research questions, it is important to note that in the types (a) and (b) of tasks, there was some kind of interference between the ideas of quotative and partitive division. In the case of (b), the translation *from the division term to a word problem*, these interferences were essential to the fact that precisely this change of representation often did not succeed in the end. Typically, in such cases, a child would start with a fair sharing story but then get bogged down, often because they found it difficult to formulate a question. In such a case, the child would sometimes take the material or make a drawing. However, this often led to a change to quotative division, as the child would now represent the divisor as the size of a subset and divide the dividend accordingly by measuring how many subsets of that size could be made. Thus, the representation was no longer in line with the story that the child had started with. Unravelling this knot was usually impossible. Of course, the children often did not seem to notice that their story, which they had changed several times during the interview, had not been put into a form that fits the term until the very end. Interferences of this kind occurred in 10 of 44 interviews, quite evenly distributed over T1 to T4.

On the other hand, when (a) asked to *show and explain a division using wooden cubes*, in 8 out of 44 cases, there were discrepancies in the way that the children performed an action in the sense of a

quotative division but tried to explain it in terms of a story of partitive division. This happened in all interviews, one time in the first, each two times in the second and third, and three times in the fourth.

As mentioned above, after each change of representation, the children were also confronted with the alternative model. If a child, for example, had previously represented $24:6$ as *four sets of six each* in the sense of measuring, it was shown *six sets of four each* ($24:6$ in the sense of fair sharing) and asked if this would also fit the term. In each interview, between seven and nine of the eleven children said that their original interpretation was the *better* one or that it was indeed the *only legitimate* one. Regarding the translation term to material representation, this was usually the quotative interpretation.

In summary, with regard to RQ 1, the interviews show a clear preference for interpreting division in terms of quotative division *when asked to show and explain a given term using material*. Conversely, translating a term into a *word problem* was more often successful when children tried to invent a partitive story. In relation to RQ 2, this remained relatively stable over the good two years of the study. As far as RQ 3 is concerned, during the course of the interviews, and again quite consistently over the whole period, there was interference in the sense that children mixed up partitive and quotative interpretations in explaining their material representations or when trying to find a fitting word problem. Finally, in a clear majority of cases, once the children had interpreted a given term on their own according to one of the two division models, they were at least reluctant to accept an alternative interpretation according to the other model as equally fitting.

Discussion and outlook

The eleven children in the study in many cases showed problems adopting *both* perspectives, that of dividing by measuring and dividing by sharing, and giving meaning to a given division term in *both* variants. The majority of the children seemed to associate division, when asked to invent a word problem, more with *everyday situations* corresponding *fair sharing*. In contrast, when asked to explain division using the *material*, they tended to represent the divisor as a quantity and thus ended up with *dividing as measuring*. Both preferences would be understandable: In everyday life, children are probably confronted more often with fair sharing situations than with dividing as measuring. However, as explained in section 1, a division may be easier to represent with material in the sense of a quotative division. It seems that the two meanings of division coexisted in an unclear relationship in the minds of many children interviewed for this study. This did not prevent them from successfully solving one-step word problems in Year 4. However, the *individual division models* of many of the interviewed children were at least incomplete in this respect.

It could be argued that these individual models should be developed further so that children are clear about the two basic ways of interpreting a given division term and the relationship between them, because “for students, the division is a difficult operation [...] and becomes more complicated with fractions and decimals” (Carvalho & Ponte, 2019, p. 387). The broader and more profound the understanding of division children develop in primary school, the better they probably will be able to cope with the challenges they face later when division is extended to rational numbers. For example, a meaningful interpretation of $2.4:0.48$, as used in Bell et al. (1994), without the clear idea that division can also be interpreted as measurement is difficult. However, as mentioned in section 2, several studies suggest that the quotative interpretation is pushed back during the school years.

In light of the results presented here, the following tentative explanation is suggested: The quotative interpretation is unproblematic and maybe even more accessible to children than the partitive if and as long as they actively solve divisions with material or associate ideas of material actions with division. As a rule, material actions become less frequent in the upper classes, already in primary school. Therefore, when division applications are discussed in the upper grades, they are mainly word problems. Of course, in reality and in textbooks you will also find problems that require a measurement interpretation of division. However, partitive situations are probably more common. Furthermore, even prospective teachers in the teacher training courses often find it challenging to come up with *realistic* word problems that require quotative division. Some textbooks contain quotative problems that are more like a riddle than a real problem, such as asking how many grandchildren granny has if she hands out 12 sweets and gives each grandchild three pieces. Hopefully, the grandmother knows how many grandchildren she has before deciding how many sweets to give each! In the teacher interviews that formed part of the research design, it became clear that both teachers had unintentionally given such artificial examples of word problems more than once in class. This might have contributed to what became apparent in the interviews: On the one hand, the children tended to associate measurement with division when confronted with *material*. On the other hand, they had learned in class to associate the arithmetic operation of division with problems of fair sharing, a context that was quite familiar to them in everyday life. However, they struggled to also connect their *action experiences of measuring* to *significant* (in that case: quotative) division problems in everyday life, and develop viable prototypes for word problems in the sense of *quotative* division, too.

In addition, it became clear from the teachers' interviews that these children's lessons did not include any substantial work on *explicitly comparing* quotative and partitive division with the aim of creating an awareness that there are such different but related variants of the same operation. Considering standard textbooks, we assume that, at least in German-speaking countries, teaching is *usually* not explicitly aimed at children consciously distinguishing between the two types of division and understanding what they have in common. The widespread problems with division in secondary school mentioned in section 2 may, at least to some extent, be related to insufficiently clear mental models of division following *the insufficiently clear teaching* of the two division variants in primary school. In any case, we suggest that this be considered in future research on children's problems with rational number division as a possible contributing factor in addition to the "conceptual change" (Prediger, 2008) required by the extension to rational numbers. More research in this area, including in particular educational design research, seems to us to be an urgent desideratum.

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