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An Explicit Scheme for Pathwise XVA Computations*

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Abstract

Motivated by the equations of cross valuation adjustments (XVAs) in the realistic case where capital is deemed fungible as a source of funding for variation margin, we introduce a simulation/regression scheme for a class of anticipated BSDEs, where the coefficient entails a conditional expected shortfall of the martingale part of the solution. The scheme is explicit in time and uses neural network least-squares and quantile regressions for the embedded conditional expectations and expected shortfall computations. An a posteriori Monte Carlo validation procedure allows assessing the regression error of the scheme at each time step. The superiority of this scheme with respect to Picard iterations is illustrated in a high-dimensional and hybrid market/default risks XVA use-case.

Keywords: anticipated BSDE, neural network regression and quantile regression, cross-valuation adjustments (XVA).

Mathematics Subject Classification: 62M45, 65C30, 91G20, 91G40, 91G60, 91G70.

JEL Classification: C63, G13.

1 Introduction

Anticipated BSDEs (ABSDEs) are backward stochastic differential equations which coefficient at time t depends on the time- t conditional law of the solution beyond time t . In the seminal ABSDE paper by Peng and Yang (2009), this dependence occurs via a conditional expectation of the value process Y at some later time. Motivated by the equations

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*GPU and python implementation available on <https://github.com/BouazzaSE/NeuralXVA>.

of cross valuation adjustments (XVAs) in finance, Crépey, Sabbagh, and Song (2020) establish the well-posedness of an ABSDE where it occurs via a conditional expected shortfall of a future increment of the martingale part of the solution. In the present paper we address the numerical solution of such ABSDEs and their XVA application.

The literature on these topics is relatively scarce. In a purely Brownian setup (versus also jumps in our case), Agarwal, Marco, Gobet, López-Salas, Noubiagain, and Zhou (2019) consider an ABSDE involving a conditional expected shortfall as anticipated term (by contrast with a conditional expectation in the previous ABSDE literature). Exploiting the short horizon of the anticipation in the equation (one week in their case versus one year in our XVA application), they devise approximations by standard BSDEs, which allows them to avoid the difficulty posed by the regression of the anticipated terms¹. The XVA ABSDEs received a first numerical treatment in Albanese, Caenazzo, and Crépey (2017) by nested Monte Carlo², using Picard iterations to decouple the solution from the embedded conditional risk measures and ignoring the conditionings in the latter³ to avoid multiply nested Monte Carlo. The other natural approach to address such problems numerically is regression-based Monte Carlo, i.e. iterated regressions that are used for cutting the recursively nested levels of Monte Carlo to which a naive implementation of the equations conducts. A first take in this direction, still using Picard iterations for decoupling purposes, was implemented in Albanese, Crépey, Hoskinson, and Saadeddine (2021), leveraging on the elicibility of the embedded risk measures for learning not only the XVAs, but also conditional value-at-risk for dynamic initial margin calculations and conditional expected shortfall⁴ for dynamic economic capital calculations. Proceeding in this way, Albanese, Crépey, Hoskinson, and Saadeddine (2021) were able to learn the embedded conditional risk measures, instead of treating them numerically as constants in Albanese, Caenazzo, and Crépey (2017).

In this paper we introduce an explicit time-discretization ABSDE scheme which, in conjunction with a refined neural net regression approach for the embedded conditional expectations and risk measures, leads to a direct algorithm for computing pathwise XVA metrics, without Picard iterations. The a posteriori error control of Theorem 4.1 allows the user to overcome the lack of a priori error control inherent to stochastic gradient descent training of neural networks. A numerical benchmark in a 36 dimensional XVA usecase emphasizes the scalability of the explicit scheme and its superiority with respect to the Picard one.

The paper is outlined as follows. Section 2 recasts the generic ABSDE of Crépey, Sabbagh, and Song (2020) in a Markovian setup amenable to numerical simulations. Section 3 introduces the explicit simulation/regression scheme. Section 4 deals with its a posteriori error control. Section 5 details the XVA specification of our setup. Section 6 provides an XVA numerical benchmark. Section 7 discusses the outputs of the paper in relation with the literature and introduces future research perspectives.

¹cf. the beginning of Section 3.2 in Agarwal, Marco, Gobet, López-Salas, Noubiagain, and Zhou (2019).

²cf. also Abbas-Turki, Crépey, and Saadeddine (2023).

³i.e. computing unconditional risk measures instead of conditional ones.

⁴that is elicitable jointly with value-at-risk.

1.1 Standing Notation

We denote by:

- $|\cdot|$, an Euclidean norm in the dimension of its arguments;
- $T \in (0, \infty)$, a constant time horizon;
- $(\Omega, \mathcal{A}, \mathfrak{F}, \mathbb{P})$, a filtered probability space, for a probability measure \mathbb{P} on the measurable space (Ω, \mathcal{A}) and a complete and right-continuous filtration $\mathfrak{F} = (\mathfrak{F}_t)_{0 \leq t \leq T}$ of sub- σ fields of \mathcal{A} ;
- \mathbb{E} , the \mathbb{P} expectation, and \mathbb{P}_t , \mathbb{E}_t , and \mathbb{ES}_t , the $(\mathfrak{F}_t, \mathbb{P})$ conditional probability, expectation, and expected shortfall at some given quantile level $\alpha \in (\frac{1}{2}, (1))$.

By the latter we mean, for each \mathfrak{F}_T measurable, \mathbb{P} integrable, random variable ℓ ,

$$\mathbb{ES}_t(\ell) = \mathbb{E}_t(\ell | \ell \geq q_t^\alpha(\ell)), \quad (1)$$

in which $q_t^\alpha(\ell)$ denotes the $(\mathfrak{F}_t, \mathbb{P})$ conditional left-quantile⁵ of level α of ℓ . We recall that⁶, for any \mathfrak{F}_T measurable, \mathbb{P} integrable random variables ℓ and ℓ' ,

$$|\mathbb{ES}_t(\ell) - \mathbb{ES}_t(\ell')| \leq (1 - \alpha)^{-1} \mathbb{E}_t[|\ell - \ell'|]. \quad (2)$$

2 Limiting Equations

In this section we specify the stochastic differential equations addressed from a numerical viewpoint in later sections.

2.1 Spaces and Martingale Representation

Given nonnegative integers d and q , we denote by W , an $(\mathfrak{F}, \mathbb{P})$ standard d variate Brownian motion, and $\nu = (\nu^\kappa)$, an integer valued random measure on $[0, T] \times \{0, 1\}^q$ ⁷ with \mathbb{P} compensatrix

$$d\mu_t^\kappa = d\nu_t^\kappa - \gamma_t^\kappa dt, \quad \kappa \in \{0, 1\}^q,$$

for some nonnegative real valued predictable processes γ^κ . Given any positive integer l , we introduce:

- \mathcal{S}_2^l , the space of \mathbb{R}^l valued \mathfrak{F} adapted càdlàg processes Y such that

$$\|Y\|_{\mathcal{S}_2^l}^2 = \mathbb{E} \left[\sup_{0 \leq t \leq T} |Y_t|^2 \right] < +\infty;$$

⁵value-at-risk.

⁶additionally assuming ℓ and ℓ' atomless given \mathfrak{F}_t , without harm for the XVA applications targeted in this work; cf. e.g. Barrera, Crépey, Diallo, Fort, Gobet, and Stazhynski (2019, Lemma A.6, Eqn. (A.16)).

⁷i.e. ν is an $\mathbb{N} \cup \{+\infty\}$ valued random measure on $[0, T] \times \{0, 1\}^q$ such that $\sum_{\kappa \in \{0, 1\}^q} \nu^\kappa(\{t\}, \omega) \leq 1$ holds $d\mathbb{P} \times dt$ everywhere, cf. Jacod and Shiryaev (2003, Definitions II.1.3, II.1.6.a and II.1.13 on pages 65, 66, and 68).

- \mathcal{H}_2^l , the space of $\mathbb{R}^{l \otimes d}$ valued \mathfrak{F} progressive processes Z such that

$$\|Z\|_{\mathcal{H}_2^l}^2 = \mathbb{E} \left[\int_0^T |Z_t|^2 dt \right] < +\infty;$$

- $\tilde{\mathcal{H}}_2^l$, the space of $\mathbb{R}^{l \otimes 2^q}$ valued \mathfrak{F} predictable processes U such that

$$\|U\|_{\tilde{\mathcal{H}}_2^l}^2 = \mathbb{E} \left[\int_0^T |U|_t^2 dt \right] < +\infty, \text{ where } |U|_t^2 = \sum_{\kappa \in \{0,1\}^q} \gamma_t^\kappa |U_t^\kappa|^2.$$

We use $\int_0^t U_s d\mu_s$ as shorthand for $\sum_{\kappa \in \{0,1\}^q} \int_0^t U_s^\kappa d\mu_s^\kappa$.

Assumption 2.1 *Every $(\mathfrak{F}, \mathbb{P})$ martingale in \mathcal{S}_2^l starting from 0 has a representation of the form*

$$\int_0^\cdot Z_t dW_t + \int_0^\cdot U_t d\mu_t, \quad (3)$$

for some $Z \in \mathcal{H}_2^l$ and $U \in \tilde{\mathcal{H}}_2^l$.

In the case where $l = 1$ we drop the index l , e.g. we write \mathcal{S}_2 instead of \mathcal{S}_2^1 .

2.2 The Markovian Anticipated BSDE

In this section, we introduce a Markovian specification of the semimartingale ABSDE of Crépey, Sabbagh, and Song (2020)⁸

Given a positive integer p , let X in \mathcal{S}_2^p satisfy

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, \quad (4)$$

for coefficients $b(t, x)$ and $\sigma(t, x)$ Lipschitz in x uniformly in $t \in [0, T]$ and with linear growth in x . Hence the SDE (4) is classically well-posed in \mathcal{S}_2^p , for any constant initial condition $x \in \mathbb{R}^p$. We write $\mathcal{X} = (X, J)$, where a $\{0, 1\}^q$ valued “Markov chain like” model component J ⁹ satisfies

$$dJ_t = \sum_{\kappa \in \{0,1\}^q} (\kappa - J_{t-}) d\nu_t^\kappa \quad (5)$$

and the compensator $\gamma_t^\kappa dt$ of each $d\nu_t^\kappa$ satisfies $\gamma_t^\kappa = \gamma^\kappa(t, \mathcal{X}_{t-})$, for some continuous functions $\gamma^\kappa(t, x, k)$ of (t, x, k) such that $\gamma^\kappa(t, x, k) = \mathbb{1}_{\{k \neq \kappa\}} \gamma^\kappa(t, x, k)$. Hence ν_t^κ counts the number of transitions of J to the state κ on $(0, t]$.

We write $f(t, \mathcal{X}_t, \dots)$ as shorthand for $f_{J_t}(t, X_t, \dots)$, for any function $f = f_k(t, x, \dots)$ (e.g. $\gamma^\kappa(t, x, k) \equiv \gamma_k^\kappa(t, x)$). Given a positive integer l , let the terminal condition

⁸see Section 1.

⁹but with transition probabilities modulated by X .

$\phi = \phi_k(x)$ be for each k an \mathbb{R}^l valued continuous function on \mathbb{R}^p , the running cost function $f = f_k(t, x, y, \varrho)$ be for each k an \mathbb{R}^l valued continuous function on $[0, T] \times \mathbb{R}^p \times \mathbb{R}^l \times \mathbb{R}$, and the conditional expected shortfall $M \mapsto \mathbb{E}\mathbb{S}(\Phi_{\cdot}(M))$ be a map from \mathcal{S}_2^l into the space of \mathfrak{F} predictable¹⁰ processes, where¹¹

$$\Phi_{\bar{t}}(M) := \Phi(t; \mathcal{X}_{[t, \bar{t}]}, M_{[t, \bar{t}]} - M_t), \quad (6)$$

for some deterministic maps $t \mapsto \bar{t} \in [t, T]$ ¹² and $\Phi = \Phi(t, \mathbf{x}, \mathbf{m})$ of time t and càdlàg paths \mathbf{x} and \mathbf{m} on $[t, \bar{t}]$ such that $\mathbf{m}_t = 0$. Assuming $\Phi_{\bar{t}}(M)$ integrable for each time t and $M \in \mathcal{S}_2^l$, we consider the following backward stochastic differential equation (BSDE) for Y in \mathcal{S}_2^l :

$$Y_t = \mathbb{E}_t \left[\phi(\mathcal{X}_T) + \int_t^T f(s, \mathcal{X}_s, Y_s, \mathbb{E}\mathbb{S}_s(\Phi_{\bar{s}}(M))) ds \right], \quad t \leq T, \quad (7)$$

where M , also required to belong to \mathcal{S}_2^l , is the canonical Doob-Meyer martingale component of the special semimartingale Y .

Assumption 2.2 (i) The function $f = f_k(t, x, y, \varrho)$ is Λ_f Lipschitz in (y, ϱ) , i.e.

$$|f_k(t, x, y, \varrho) - f_k(t, x, y', \varrho')| \leq \Lambda_f (|y - y'| + |\varrho - \varrho'|); \quad (8)$$

(ii) The processes $\mathbb{E}\mathbb{S}(\Phi_{\cdot}(0))$ and $|f(\cdot, \mathcal{X}, 0, 0)|$ are in \mathcal{H}_2 ;

(iii) Φ is Λ_Φ Lipschitz with respect to its last argument in the sense that for every $t \in [0, T]$,

$$|\Phi(t; \mathbf{x}, \mathbf{m}) - \Phi(t; \mathbf{x}, \mathbf{m}')| \leq \Lambda_\Phi |\mathbf{m}_{\bar{t}} - \mathbf{m}'_{\bar{t}}| \quad (9)$$

holds for all càdlàg paths $\mathbf{x}, \mathbf{m}, \mathbf{m}'$ on $[t, \bar{t}]$ such that $\mathbf{m}_t = \mathbf{m}'_t = 0$.

Remark 2.1 Assumption 2.2(iii) strongly points out to the case where $\Phi_{\bar{t}}(M)$ only depends on M through $M_{\bar{t}} - M_t$, which indeed corresponds to our XVA use case of Sections 5-6. However, the algorithm of Section 3 is not restricted to this case and Assumption 2.2(iii) only yields a sufficient condition for the conclusion of Proposition 2.1 below to hold.

Lemma 2.1 There exists a positive constant Λ_ρ such that

$$|\mathbb{E}\mathbb{S}_t(\Phi_{\bar{t}}(M)) - \mathbb{E}\mathbb{S}_t(\Phi_{\bar{t}}(M'))|^2 \leq \Lambda_\rho^2 \mathbb{E}_t \left[\int_t^{\bar{t}} (|Z_s - Z'_s|^2 + |U - U'|_s^2) ds \right] \quad (10)$$

holds for any $M, M' \in \mathcal{S}_2^l$, where (Z, U) and (Z', U') in $\mathcal{H}_2^l \times \tilde{\mathcal{H}}_2^l$ are the integrands in the martingale representations (3) of $M - M_0$ and $M' - M'_0$.

¹⁰assuming the raw process $\Phi_{\bar{t}}(M)$ càdlàg in t , see Crépey, Sabbagh, and Song (2020, Lemma 2.1).

¹¹see e.g. the last line in (31).

¹²e.g. $\bar{t} = (t + 1) \wedge T$ in our XVA use case of Sections 5-6.

Proof. By (2), we have

$$\begin{aligned} (1 - \alpha)^2 |\mathbb{E}\mathbb{S}_t(\Phi_{\bar{t}}(M)) - \mathbb{E}\mathbb{S}_t(\Phi_{\bar{t}}(M'))|^2 &\leq \\ \left(\mathbb{E}_t \left[\Phi(t; \mathcal{X}_{[t, \bar{t}]}, M_{[t, \bar{t}]} - M_t) - \Phi(t; \mathcal{X}_{[t, \bar{t}]}, M'_{[t, \bar{t}]} - M'_t) \right] \right)^2 &\leq \\ \mathbb{E}_t \left[\left(\Phi(t; \mathcal{X}_{[t, \bar{t}]}, M_{[t, \bar{t}]} - M_t) - \Phi(t; \mathcal{X}_{[t, \bar{t}]}, M'_{[t, \bar{t}]} - M'_t) \right)^2 \right], & \end{aligned}$$

by the (conditional) Jensen inequality. Moreover, denoting $\delta M = M - M'$, the Lipschitz condition (9) yields

$$\left(\Phi(t; \mathcal{X}_{[t, \bar{t}]}, M_{[t, \bar{t}]} - M_t) - \Phi(t; \mathcal{X}_{[t, \bar{t}]}, M'_{[t, \bar{t}]} - M'_t) \right)^2 \leq \Lambda_{\Phi}^2 |\delta M_{\bar{t}} - \delta M_t|^2,$$

where, with $\delta Z = Z - Z'$ and $\delta U = U - U'$,

$$\delta M_{\bar{t}} - \delta M_t = \int_t^{\bar{t}} \delta Z_s dW_s + \int_t^{\bar{t}} \delta U_s d\mu_s,$$

hence

$$|\delta M_{\bar{t}} - \delta M_t|^2 = \sum_{\iota=1}^l \left(\int_t^{\bar{t}} \delta Z_s^{\iota} dW_s + \int_t^{\bar{t}} \delta U_s^{\iota} d\mu_s \right)^2.$$

Therefore

$$(1 - \alpha)^2 |\mathbb{E}\mathbb{S}_t(\Phi_{\bar{t}}(M)) - \mathbb{E}\mathbb{S}_t(\Phi_{\bar{t}}(M'))|^2 \leq \Lambda_{\Phi}^2 \mathbb{E}_t \sum_{\iota=1}^l \left(\int_t^{\bar{t}} \delta Z_s^{\iota} dW_s + \int_t^{\bar{t}} \delta U_s^{\iota} d\mu_s \right)^2. \quad (11)$$

As a local martingale in \mathcal{S}_2 , each process $\int_t^{\cdot} \delta Z_s^{\iota} dW_s + \int_t^{\cdot} \delta U_s^{\iota} d\mu_s$ is a square integrable martingale over $[t, \bar{t}]$. The (conditional) Burkholder inequality¹³ applied to this process then yields

$$\mathbb{E}_t \left(\int_t^{\bar{t}} \delta Z_s^{\iota} dW_s + \int_t^{\bar{t}} \delta U_s^{\iota} d\mu_s \right)^2 \leq C \mathbb{E}_t \left[\int_t^{\bar{t}} (|\delta Z_s^{\iota}|^2 + |\delta U_s^{\iota}|^2) ds \right],$$

so that (11) entails (10). ■

Proposition 2.1 *The ABSDE (7) has a unique special semimartingale solution Y in \mathcal{S}_2^l with martingale component M in \mathcal{S}_2^l . The process Y is the limit in \mathcal{S}_2^l of the Picard iteration defined by $Y^0 = 0$ and, for $j \geq 1$,*

$$Y_t^j = \mathbb{E}_t \left[\phi(\mathcal{X}_T) + \int_t^T f(s, \mathcal{X}_s, Y_s^{j-1}, \mathbb{E}\mathbb{S}_s(\Phi_{\bar{s}}(M^{j-1}))) ds \right], \quad (12)$$

where $M^{j-1} \in \mathcal{S}_2^l$ is the martingale part of the special semimartingale $Y^{(j-1)} \in \mathcal{S}_2^l$.

¹³see e.g. He, Wang, and Yan (1992, Theorem 10.36 and Remark page 285).

Proof. Assumptions 2.2(i) and (ii) imply that the processes

$$\sup_{|y| \leq c} |f(\cdot, \mathcal{X}, y, \mathbb{E}\mathbb{S}.\Phi(\cdot)) - f(\cdot, \mathcal{X}, 0, \mathbb{E}\mathbb{S}.\Phi(\cdot))|^{\frac{1}{2}}$$

(for every $c > 0$), as well as $|f(\cdot, \mathcal{X}, 0, \mathbb{E}\mathbb{S}.\Phi(\cdot))|$, are in \mathcal{H}_2 , which is (Crépey et al., 2020, Assumption 3.2(iii)), whereas (Crépey et al., 2020, Assumption 3.2 (i), (ii), and (iv)) are implied by our Assumption 2.2(i) and the Lipschitz property of the functions ϕ_k combined with the standard a priori bound estimate $\|X\|_{\mathcal{S}_2^p}^2 \leq C(1 + |x|^2)$ on X (with constant initial condition x). Moreover, (10) corresponds to (Crépey et al., 2020, Assumption 3.1). Hence (Crépey et al., 2020, Assumptions 3.1 and 3.2) hold and the result follows by an application of (Crépey et al., 2020, Theorem 3.1). ■

3 The Explicit Simulation/Regression Scheme

3.1 Time Discretization

Let there be given a deterministic time-grid $0 = t_0 < t_1 < \dots < t_n = T$ with mesh size¹⁴ h . We write $\Delta t_{i+1} = t_{i+1} - t_i$. Let \bar{t}_i denote an approximation on the grid of t_i ¹⁵. Let there also be given, on this time grid, simulatable approximations \mathcal{X}^h (assumed Markovian with respect to its own filtration) to \mathcal{X} ¹⁶ and $\Phi_{\bar{t}_i}^h(M^h)$ to $\Phi_{\bar{t}_i}(M)$, with M^h approximating M on the time-grid and $\Phi_{\bar{t}_i}^h(M^h)$ of the form¹⁷

$$\Phi^h(t_i; \mathcal{X}_{\{t_i, \dots, \bar{t}_i\}}^h, M_{\{t_i, \dots, \bar{t}_i\}}^h - M_{t_i}^h), \quad (13)$$

for some deterministic map Φ^h of grid times t_i and discrete paths \mathbf{x}^h and \mathbf{m}^h on $\{t_i, \dots, \bar{t}_i\}$ such that $\mathbf{m}_{t_i}^h = 0$. Φ^h is assumed Λ_{Φ}^h Lipschitz in the sense mimicking (9) that

$$\left| \Phi^h(t_i; \mathbf{x}^h, \mathbf{m}^h) - \Phi^h(t_i; \mathbf{x}^h, (\mathbf{m}^h)') \right| \leq \Lambda_{\Phi}^h \left| \mathbf{m}_{\bar{t}_i}^h - (\mathbf{m}^h)_{\bar{t}_i}' \right| \quad (14)$$

holds for any paths \mathbf{m}^h and $(\mathbf{m}^h)'$ on $\{t_i, \dots, \bar{t}_i\}$ such that $\mathbf{m}_{t_i}^h = (\mathbf{m}^h)_{t_i}' = 0$.

The explicit time discretization for $(Y, \mathbb{E}\mathbb{S}(\Phi(\cdot)(M)))$ (with M the martingale part of the solution Y to (7)) is the process (Y^h, ρ^h) defined at grid times by $Y_{t_n}^h = \phi(\mathcal{X}_T^h)$, $\rho_{t_n}^h = \Phi_T^h(0)$ and, for i decreasing from $n-1$ to 0,

$$\begin{aligned} Y_{t_i}^h &= \mathbb{E}_{t_i}^h \left[Y_{t_{i+1}}^h + f(t_i, \mathcal{X}_{t_i}^h, Y_{t_{i+1}}^h, \rho_{t_{i+1}}^h) \Delta t_{i+1} \right] \\ \rho_{t_i}^h &= \mathbb{E}_{t_i}^h \left(\underbrace{\Phi_{\bar{t}_i}^h \left(Y_{t_i}^h + \sum_{\iota < \cdot} f(t_\iota, \mathcal{X}_{t_\iota}^h, Y_{t_{\iota+1}}^h, \rho_{t_{\iota+1}}^h) \Delta t_{\iota+1} \right)}_{M^h} \right), \end{aligned} \quad (15)$$

¹⁴maximum time step.

¹⁵cf. after (6).

¹⁶e.g. the Euler scheme for X and a related approximation for J .

¹⁷cf. (6).

where the \cdot^h in $\mathbb{E}_{t_i}^h$ and $\mathbb{ES}_{t_i}^h$ mean that the corresponding conditional expectations and expected shortfalls are in reference to the natural filtration of \mathcal{X}^h .

The study of the time-consistency of the scheme, i.e. the convergence of the Y^h to Y as h goes to 0, based on suitable Feynman-Kac representation for the solution of the limiting ABSDE (7), is left for a separate work (we only provide empirical evidence of the stability of the time discretization in Section 6.1). Our main focus in this paper is on the discretization in space of (15).

3.2 Fully Discrete Algorithm

Whenever a process M^h on the time grid is such that $M_{\{t=t_i, \dots, \bar{t}_i\}}^h - M_{t_i}^h$ is a measurable functional of $(t_i, \mathcal{X}_{t_i}^h), \dots, (\bar{t}_i, \mathcal{X}_{\bar{t}_i}^h)$ with $\Phi_{\bar{t}}^h(M^h)$ square integrable, Barrera, Crépey, Gobet, Nguyen, and Saadeddine (2022, Theorem 2.3)¹⁸ yields, with $\varphi = \varphi_k(t, x)$ and $\psi = \psi_k(t, x)$: $\mathbb{ES}_{\bar{t}}^h(\Phi_{\bar{t}}^h(M^h)) = (1 - \alpha)^{-1} \psi^*(t, \mathcal{X}_t^h)$, where

$$\psi^*(t, \cdot) = \arg \min_{\psi: (t, \cdot) \in \mathcal{B}} \mathbb{E}[(\Phi_{\bar{t}}^h(M^h) \mathbb{1}_{\{\Phi_{\bar{t}}^h(M^h) \geq \varphi^*(t, \mathcal{X}_t^h)\}} - \psi(t, \mathcal{X}_t^h))^2], \quad (16)$$

in which

$$\varphi^*(t, \cdot) = \arg \min_{\varphi: (t, \cdot) \in \mathcal{B}} \mathbb{E}[(\varphi(t, \mathcal{X}_t^h) + (1 - \alpha)^{-1}(\Phi_{\bar{t}}^h(M^h) - \varphi(t, \mathcal{X}_t^h))^+)], \quad (17)$$

both minimizations bearing over the set \mathcal{B} of the Borel functions of (x, k) .

By nonparametric quantile regression estimates of $\varphi^*(t, \mathcal{X}_t^h)$ and $\psi^*(t, \mathcal{X}_t^h)$, we mean any functions $\hat{\varphi}^*(t, \cdot)$ and $\hat{\psi}^*(t, \cdot)$ obtained by solving the respective problems (17) and (16) with \mathcal{B} approximated by some hypothesis space of functions (e.g. neural networks), \mathbb{E} by the sample mean over a sufficiently large number of independent realizations of \mathcal{X}^h , minimization by numerical minimization through Adam stochastic gradient descent (Kingma and Ba, 2014), and φ^* in (16) by $\hat{\varphi}^*$.

The fully (time and space) discrete counterpart of (15) follows by estimating, at each grid time $t = t_i$ going backward, the embedded conditional expectations (resp. expected shortfalls) through nonparametric least-squares regression against \mathcal{X}_t^h (resp. quantile regression against \mathcal{X}_t^h as explained above). Using the notations $\hat{\mathbb{E}}_{t_i}^h$ and $\hat{\mathbb{ES}}_{t_i}^h$ for the embedded conditional expectation and expected shortfall estimators, we obtain the following ABSDE numerical scheme: $\hat{Y}_{t_n}^h = \phi(\mathcal{X}_T^h)$, $\hat{\rho}_{t_n}^h = \Phi_T^h(0)$ and, for i decreasing from $n - 1$ to 0,

$$\begin{aligned} \hat{Y}_{t_i}^h &= \hat{\mathbb{E}}_{t_i}^h \left[\hat{Y}_{t_{i+1}}^h + f(t_i, \mathcal{X}_{t_i}^h, \hat{Y}_{t_{i+1}}^h, \hat{\rho}_{t_{i+1}}^h) \Delta t_{i+1} \right], \\ \hat{\rho}_{t_i}^h &= \hat{\mathbb{ES}}_{t_i}^h \left(\underbrace{\Phi_{t_i}^h \left(\hat{Y}_{t_i}^h + \sum_{\nu < \cdot} f(t_i, \mathcal{X}_{t_i}^h, \hat{Y}_{t_{i+1}}^h, \hat{\rho}_{t_{i+1}}^h) \Delta t_{i+1} \right)}_{\hat{M}^h} \right). \end{aligned} \quad (18)$$

¹⁸additionally assuming $\Phi_{\bar{t}}^h(M^h)$ atomless given $\mathfrak{F}_{\bar{t}}$.

Note that

$$\widehat{M}_{t_i}^h = \widehat{M}_{t_{i+1}}^h + \widehat{Y}_{t_i}^h - \mathcal{Y}_{t_{i+1}}^h, \text{ where } \mathcal{Y}_{t_{i+1}}^h = \widehat{Y}_{t_{i+1}}^h + f(t_i, \mathcal{X}_{t_i}^h, \widehat{Y}_{t_{i+1}}^h, \widehat{\rho}_{t_{i+1}}^h) \Delta t_{i+1},$$

which is exploited line 19 of Algorithm 1 to compute $\widehat{M}_{t_i}^h$ iteratively (for decreasing i).

As in Abbas-Turki, Crépey, and Saadeddine (2023), given weight matrices $A^{[L+1]} \in \mathbb{R}^{1 \times u}, \dots, A^{[\ell]} \in \mathbb{R}^{u \times u}, \dots, A^{[1]} \in \mathbb{R}^{u \times (p+q)}$, biases $b^{[L+1]} \in \mathbb{R}, \dots, b^{[\ell]} \in \mathbb{R}^u, \dots, b^{[1]} \in \mathbb{R}^u$, and a scalar non-linearity ς applied element-wise, let, for every $z = (x, k) \in \mathbb{R}^p \times \mathbb{R}^q \equiv \mathbb{R}^{p+q}$,

$$\begin{aligned} \zeta^{[0]}(z; A, b) &= z \\ \zeta^{[\ell]}(z; A, b) &= \varsigma(A^{[\ell]} \zeta^{[\ell-1]}(z; A, b) + b^{[\ell]}), \quad \ell = 1, \dots, L \\ \zeta^{[L+1]}(z; A, b) &= A^{[L+1]} \zeta^{[L]}(z; A, b) + b^{[L+1]}, \end{aligned}$$

where A and b denote the respective concatenations of the $A^{[\ell]}$ and of the $b^{[\ell]}$. The function $\mathbb{R}^{p+q} \ni z \mapsto \zeta^{[L+1]}(z; A, b) \in \mathbb{R}$ then implements a neural network with L hidden layers, u neurons per hidden layer, ς as an activation function applied on each hidden unit, and no non-linearity at the output layer. Using the corresponding parameterized set of functions $\mathbb{R}^{p+q} \ni z \mapsto \zeta^{[L+1]}(z; A, b) \in \mathbb{R}$ as hypothesis space in the description following (16)-(17), we obtain Algorithm 1, using Algorithm 0 as an elementary learning block therein. At the beginning of the algorithms, i.e. before any learning is performed, the weights are initialized randomly the way explained in Goodfellow, Bengio, and Courville (2016). In our numerical experiments, we use softplus activation functions in the hidden layers, in combination with the related weight initialization scheme in He, Zhang, Ren, and Sun (2015).

Algorithm 0: Elementary learning block for least squares (ls) or quantile (qle) neural net regressions of cash flows ξ^j against features χ^j indexed by $j \in \mathcal{J}$

name : NNRegress
input : $\{(\chi^j, \xi^j), j \in \mathcal{J}\}$, a partition B of \mathcal{J} , a number of epochs $E \in \mathbb{N}^*$, a learning rate $\eta > 0$, initial values for the network parameters A and b , type of regression *regr*
output: Trained parameters A and b

- 1 define $\mathcal{L}(A, b, \text{batch}) =$

$$\begin{cases} \frac{1}{|\text{batch}|} \sum_{j \in \text{batch}} (\zeta^{[L+1]}(\chi^j; A, b) - \xi^j)^2 & \text{if } \text{regr} = \text{ls} \\ \frac{1}{|\text{batch}|} \sum_{j \in \text{batch}} (\xi^j - \zeta^{[L+1]}(\chi^j; A, b))^+ + (1 - \alpha) \zeta^{[L+1]}(\chi^j; A, b) & \text{if } \text{regr} = \text{qle} \end{cases}$$
- 2 **for** $\text{epoch} = 1, \dots, E$ **do** // loop over epochs
- 3 **for** $\text{batch} \in B$ **do** // loop over batches
- 4 $A, b \leftarrow A - \eta \nabla_A \mathcal{L}(A, b, \text{batch}), b - \eta \nabla_b \mathcal{L}(A, b, \text{batch})$
- 5 **end**
- 6 **end**

Algorithm 1: Explicit backward learning scheme for the ABSDE (7) based on time-discretized simulated paths $\mathcal{X}^{h,j}$ of \mathcal{X} indexed by $j \in \mathcal{J}$

input : current state $\mathcal{X}_0 = x$, $\{\{\mathcal{X}_{t_i}^{h,j}, (1) \leq i \leq n\}, \phi(\mathcal{X}_T^{h,j}), j \in \mathcal{J}\}$, a partition B of \mathcal{J} , a number of epochs $E \in \mathbb{N}^*$, a learning rate $\eta > 0$

output: \widehat{Y}_0^h and learned parameters
 $\{(A_i^{\text{VaR}}, b_i^{\text{VaR}}), (A_i^{\text{ES}}, b_i^{\text{ES}}), (A_i^\iota, b_i^\iota), \iota \in \{1, \dots, l\}, i \in \{1, \dots, n\}\}$

- 1 For all $j \in \mathcal{J}$, let $y^j \in \mathbb{R}^l$ and, for each $i = 0 \dots n$, $\widehat{M}_{t_i}^{h,j} = 0 \in \mathbb{R}^l$
- 2 Initialize parameters $\{(A_n^{\text{VaR}}, b_n^{\text{VaR}}), (A_n^{\text{ES}}, b_n^{\text{ES}})\}$ of the networks approximating the VaR and ES at terminal time-step n
- 3 Initialize the parameters $\{(A_n^\iota, b_n^\iota), \iota \in \{1, \dots, l\}\}$ of the least-squares networks, indexed by $\iota \in \{1, \dots, l\}$, at terminal time-step n
- 4 **foreach** $j \in \mathcal{J}$ **do** $y^j \leftarrow \phi(\mathcal{X}_T^{h,j}), \widehat{M}_{t_n}^{h,j} \leftarrow 0$
- 5 **for** $i = n - 1 \dots 1$ **do**
 - 6 **foreach** $j \in \mathcal{J}$ **do** $\xi^j \leftarrow \Phi_{\tilde{t}_i}^h(\widehat{M}^{h,j})$ // We first learn the VaR using a quantile regression
 - 7 $A_i^{\text{VaR}}, b_i^{\text{VaR}} \leftarrow \text{NNRegress}(\{(\mathcal{X}_{t_i}^{h,j}, \xi^j), j \in \mathcal{J}\}, B, E, \eta, A_{i+1}^{\text{VaR}}, b_{i+1}^{\text{VaR}}, \text{ql})$ // Then we deduce the ES using an ls regression
 - 8 $\xi^j \leftarrow \frac{1}{1-\alpha} \xi^j \mathbf{1}_{\{\xi^j \geq \zeta^{[L+1]}(\mathcal{X}_{t_i}^{h,j}; A_i^{\text{VaR}}, b_i^{\text{VaR}})\}}$
 - 9 $A_i^{\text{ES}}, b_i^{\text{ES}} \leftarrow \text{NNRegress}(\{(\mathcal{X}_{t_i}^{h,j}, \xi^j), j \in \mathcal{J}\}, B, E, \eta, A_{i+1}^{\text{ES}}, b_{i+1}^{\text{ES}}, \text{ls})$
// We compute the integrand ξ that needs to be projected to get the solution of the BSDE at the current time step
 - 10 $\varrho^j \leftarrow \zeta^{[L+1]}(\mathcal{X}_{t_{i+1}}^{h,j}; A_{i+1}^{\text{ES}}, b_{i+1}^{\text{ES}})$
 - 11 $\xi^j \leftarrow y^j + f(t_i, \mathcal{X}_{t_i}^{h,j}, y^j, \varrho^j) \Delta t_{i+1}$
 - 12 **for** $\iota = 1 \dots l$ **do**
 - 13 | $A_i^\iota, b_i^\iota \leftarrow \text{NNRegress}(\{(\mathcal{X}_{t_i}^{h,j}, \xi_i^\iota), j \in \mathcal{J}\}, B, E, \eta, A_{i+1}^\iota, b_{i+1}^\iota, \text{ls})$
 - 14 **end**
 - 15 // Compute $\widehat{M}_{t_i}^h$
 - 16 **for** $j \in \mathcal{J}$ **do**
 - 17 | **for** $\iota = 1 \dots l$ **do**
 - 18 | | $y_i^\iota \leftarrow \zeta^{[L+1]}(\mathcal{X}_{t_i}^{h,j}; A_i^\iota, b_i^\iota)$
 - 19 | **end**
 - 19 | $\widehat{M}_{t_i}^{h,j} \leftarrow \widehat{M}_{t_{i+1}}^{h,j} + y^j - \xi^j$
 - 20 **end**
 - 21 **end**
 - 22 **for** $j \in \mathcal{J}$ **do**
 - 23 | $\varrho^j \leftarrow \zeta^{[L+1]}(\mathcal{X}_{t_1}^{h,j}; A_1^{\text{ES}}, b_1^{\text{ES}})$
 - 24 | $\xi^j \leftarrow y^j + f(0, x, y^j, \varrho^j) \Delta t_1$
 - 25 **end**
 - 26 $\widehat{Y}_0^h \leftarrow \text{sampleMean}(\xi)$

4 A Posteriori Analysis of the Regression Error

In the case of dynamic programming based simulation/regression BSDE schemes such as (18), the usual spatial error analysis strategy consists in controlling the accumulation, over (discrete) time i decreasing from $n-1$ to 0, of three error components (Gobet, 2016, Eqn. (VIII.3.8)): (i) a *bias* between the function u^i representing $Y_{t_i}^h$ as $u^i(\mathcal{X}_{t_i}^h)$ (for a suitable measurable function $u^i = u_k^i(x)$) and the hypothesis space of functions in which $\widehat{Y}_{t_i}^h$ is sought after, (ii) a “*variance*” in the sense of a regression estimation error, and (iii) a term of *propagation* at time i of the error at time $i+1$. This is at least the strategy in the standard case where the embedded conditional expectations are estimated by parametric least-squares regressions that can be performed exactly, for instance by singular value decomposition (Gobet, 2016, Section VIII.2.2). Neural net parameterizations for the target functions (conditional expectations but also expected shortfalls in the case of our ABSDEs) instead lead to “nonlinear regressions” that can only be performed by numerical, nonconvex minimization. When state-of-the-art, fine-tuned, Adam variants of stochastic gradient descents are used, the ensuing minimization can be quite efficient numerically. But there is no known learning algorithm solving such nonconvex minimization problems with an a priori error bound. Hence, when the learning iteration terminates, we do not have any guarantee on the quality of the approximation. In other words, there is a fourth *numerical minimization* error component on top of the three other ones in the above and this fourth error component cannot be controlled ex ante.

All in one, no exhaustive a priori error control can be hoped for on such problems. However, we can still assess the regression error of the schemes by an a posteriori twin Monte Carlo validation procedure, which is the topic of this section.

4.1 A Posteriori Validation of the Local Regression Error

For notational simplicity we assume a uniform time step $\Delta t_{i+1} = \Delta t$, in the XVA motivated case (see above (29)) $\bar{t} = (t+1) \wedge T$, $\bar{t}_i = t_{(i+m) \wedge n}$, with $m = \lceil \frac{1}{\Delta t} \rceil$. Hence we have in (18) (cf. (13)):

$$\begin{aligned} \Phi_{\bar{t}_i}^h(\widehat{Y}_t^h + \sum_{i < \cdot} f(t_i, \mathcal{X}_{t_i}^h, \widehat{Y}_{t_{i+1}}^h, \widehat{\rho}_{t_{i+1}}^h) \Delta t) = \\ \Phi^h(t_i; \mathcal{X}_{\{t_i, \dots, \bar{t}_i\}}^h, \widehat{Y}_t^h - \widehat{Y}_{t_i}^h + \sum_{i < \cdot} f(t_i, \mathcal{X}_{t_i}^h, \widehat{Y}_{t_{i+1}}^h, \widehat{\rho}_{t_{i+1}}^h) \Delta t, i \leq \cdot \leq (i+m) \wedge n). \end{aligned}$$

On top of (18), we also define the auxiliary scheme $\widetilde{Y}_{t_n}^h = \phi(\mathcal{X}_T^h)$, $\widetilde{\rho}_{t_n}^h = 0$ and, for i decreasing from $n-1$ to 0,

$$\begin{aligned} \widetilde{Y}_{t_i}^h &= \mathbb{E}_{t_i}^h[\widehat{Y}_{t_{i+1}}^h + f(t_i, \mathcal{X}_{t_i}^h, \widehat{Y}_{t_{i+1}}^h, \widehat{\rho}_{t_{i+1}}^h) \Delta t] \\ \widetilde{\rho}_{t_i}^h &= \mathbb{E}_{t_i}^h[\Phi_{\bar{t}_i}^h(\widehat{Y}_t^h + \sum_{i < \cdot} f(t_i, \mathcal{X}_{t_i}^h, \widehat{Y}_{t_{i+1}}^h, \widehat{\rho}_{t_{i+1}}^h) \Delta t)]. \end{aligned} \tag{19}$$

Let

$$\epsilon_{t_i} = |\widetilde{Y}_{t_i}^h - \widehat{Y}_{t_i}^h|, \quad e_{t_i} = |\widetilde{\rho}_{t_i}^h - \widehat{\rho}_{t_i}^h|. \tag{20}$$

Proceeding as in Abbas-Turki, Crépey, and Saadeddine (2023, Section 5.1), one can estimate $\mathbb{E}^h[\epsilon_{t_i}^2]$ *without computing* $\widehat{Y}_{t_i}^h$, by Monte Carlo using a so called twin simulation scheme. The latter consists in a Monte Carlo estimation of $\mathbb{E}^h[\epsilon_{t_i}^2]$ based on the following representation:

Lemma 4.1 *Let there be given two copies $\widehat{Y}_{t_{i+1}}^{h,(1)}$ and $\widehat{Y}_{t_{i+1}}^{h,(2)}$ of $\widehat{Y}_{t_{i+1}}^h = u^{i+1}(\mathcal{X}_{t_{i+1}}^h)$ and $\widehat{\rho}_{t_{i+1}}^{h,(1)}$ and $\widehat{\rho}_{t_{i+1}}^{h,(2)}$ of $\widehat{\rho}_{t_{i+1}}^h = v^{i+1}(\mathcal{X}_{t_{i+1}}^h)$, where u^{i+1} and v^{i+1} are the regressed functional forms of $\widehat{Y}_{t_{i+1}}^h$ and $\widehat{\rho}_{t_{i+1}}^h$, the two copies $\widehat{Y}_{t_{i+1}}^{h,(1)}$ and $\widehat{Y}_{t_{i+1}}^{h,(2)}$ of $\widehat{Y}_{t_{i+1}}^h$ are independent conditionally on $\mathcal{X}_{t_i}^h$ ¹⁹, and so are $\widehat{\rho}_{t_{i+1}}^{h,(1)}$ and $\widehat{\rho}_{t_{i+1}}^{h,(2)}$. Then*

$$\begin{aligned} \mathbb{E}^h[\epsilon_{t_i}^2] &= \mathbb{E}^h \left[|\widehat{Y}_{t_i}^h|^2 \right. \\ &\quad - (\widehat{Y}_{t_i}^h)^\top \left(\widehat{Y}_{t_{i+1}}^{h,(1)} + f(t_i, \mathcal{X}_{t_i}^h, \widehat{Y}_{t_{i+1}}^{h,(1)}, \widehat{\rho}_{t_{i+1}}^{h,(1)})\Delta t + \widehat{Y}_{t_{i+1}}^{h,(2)} + f(t_i, \mathcal{X}_{t_i}^h, \widehat{Y}_{t_{i+1}}^{h,(2)}, \widehat{\rho}_{t_{i+1}}^{h,(2)})\Delta t \right) \\ &\quad \left. + \left(\widehat{Y}_{t_{i+1}}^{h,(1)} + f(t_i, \mathcal{X}_{t_i}^h, \widehat{Y}_{t_{i+1}}^{h,(1)}, \widehat{\rho}_{t_{i+1}}^{h,(1)})\Delta t \right)^\top \left(\widehat{Y}_{t_{i+1}}^{h,(2)} + f(t_i, \mathcal{X}_{t_i}^h, \widehat{Y}_{t_{i+1}}^{h,(2)}, \widehat{\rho}_{t_{i+1}}^{h,(2)})\Delta t \right) \right]. \end{aligned} \quad (21)$$

Proof. This follows by application of the following identity, which holds by application of the tower rule for any Borel function $\varphi : \mathbb{R}^p \times \{0, 1\}^q \rightarrow \mathbb{R}$ such that $\varphi(\mathcal{X}_{t_i}^h)$ is square integrable (e.g. any component of the vector $\widehat{Y}_{t_i}^h$) and any square integrable random variable ξ (e.g. any component of the vector $\widehat{Y}_{t_{i+1}}^h + f(t_i, \mathcal{X}_{t_i}^h, \widehat{Y}_{t_{i+1}}^h, \widehat{\rho}_{t_{i+1}}^h)\Delta t$) with copies $\xi^{(1)}$ and $\xi^{(2)}$ independent conditionally on $\mathcal{X}_{t_i}^h$ ²⁰:

$$\mathbb{E}[(\varphi(\mathcal{X}_{t_i}^h) - \mathbb{E}[\xi|\mathcal{X}_{t_i}^h])^2] = \mathbb{E}[\varphi(\mathcal{X}_{t_i}^h)^2 - \varphi(\mathcal{X}_{t_i}^h)(\xi^{(1)} + \xi^{(2)}) + \xi^{(1)}\xi^{(2)}]. \quad \blacksquare \quad (22)$$

Proceeding in this way, we obtain an a posteriori Monte Carlo procedure to assess the local regression error $\sqrt{\mathbb{E}^h[\epsilon_{t_i}^2]}$ at each pricing time step²¹. An alternative twin Monte Carlo simulation procedure is also available from Barrera, Crépey, Gobet, Nguyen, and Saadeddine (2022, Section 4.4) to estimate an upper bound on $\mathbb{E}^h[e_{t_i}^2]$ (cf. (20)) without having to compute the $\widehat{Y}_{t_i}^h$ and $\widehat{\rho}_{t_i}^h$. But in this expected shortfall case (as opposed to conditional expectation regarding $\mathbb{E}^h[\epsilon_{t_i}^2]$), this procedure only yields an upper bound, and one which also entails a lower bound c on the density of the law of the labels given the features of the learning problem (see Barrera, Crépey, Gobet, Nguyen, and Saadeddine (2022, Eqn. (4.7))), i.e. of $\Phi_{t_i}^h(\widehat{Y}_{t_i}^h + \sum_{s < i} f(t_s, \mathcal{X}_{t_s}^h, \widehat{Y}_{t_{s+1}}^h, \widehat{\rho}_{t_{s+1}}^h)\Delta t)$ given $\mathcal{X}_{t_i}^h$ in our e_{t_i} case, for which we do not know c . All in one, the main practical use of the above is for assessing the local regression error $\sqrt{\mathbb{E}^h[\epsilon_{t_i}^2]}$ at each pricing time step. But when considered in conjunction with the corresponding L^2 training losses which could also be used as a naive (overconservative) error estimate (cf. Figures 5-6 in Section 6), this is

¹⁹i.e. one simulates two independent realizations of $\mathcal{X}_{t_{i+1}}^h$, given the same starting point $\mathcal{X}_{t_i}^h$, and then takes their images by the learned functionals u^{i+1} and v^{i+1} .

²⁰the conditional independence means that for any Borel bounded functions ϕ_1 and ϕ_2 , we have $\mathbb{E}[\phi_1(\xi^{(1)})\phi_2(\xi^{(2)})|\mathcal{X}_{t_i}^h] = \mathbb{E}[\phi_1(\xi^{(1)})|\mathcal{X}_{t_i}^h]\mathbb{E}[\phi_2(\xi^{(2)})|\mathcal{X}_{t_i}^h]$.

²¹see e.g. Figures 5-6 in Section 6.

already very useful to guide the design of the learning procedure. In the case where the error $\sqrt{\mathbb{E}^h[\epsilon_{t_i}^2]}$ is not good enough (or too close to the L^2 training loss), one can improve the stochastic gradient descent, in first attempt, and then act on the hypothesis spaces, e.g., in the case of neural networks, try to train with more layers/units or better architectures.

4.2 Accumulation in Time of the Regression Errors

In the standard BSDE case where $f_k(t, x, y, \varrho) = f_k(t, x, y)$, i.e. when there is no dependence of the coefficient of the BSDE on ρ , we have by Λ_f Lipschitz continuity of $f_k(t, x, y)$ with respect to y :

$$\mathbb{E}^h[|Y_{t_i}^h - \tilde{Y}_{t_i}^h|] \leq (1 + \Lambda_f \Delta t) \mathbb{E}^h[|Y_{t_{i+1}}^h - \hat{Y}_{t_{i+1}}^h|]$$

and the triangular inequality yields

$$\mathbb{E}^h[|Y_{t_i}^h - \hat{Y}_{t_i}^h|] \leq (1 + \Lambda_f \Delta t) \mathbb{E}^h[|Y_{t_{i+1}}^h - \hat{Y}_{t_{i+1}}^h|] + \mathbb{E}^h[|\hat{Y}_{t_i}^h - \tilde{Y}_{t_i}^h|]. \quad (23)$$

Hence, using the inequality $\mathbb{E}^h[\epsilon_{t_i}] \leq \sqrt{\mathbb{E}^h[\epsilon_{t_i}^2]}$:

$$\mathbb{E}^h[|Y_{t_i}^h - \hat{Y}_{t_i}^h|] \leq \sum_{i=i}^{n-1} (1 + \Lambda_f \Delta t)^{i-i} \sqrt{\mathbb{E}^h[\epsilon_{t_i}^2]}, \quad (24)$$

where each $\mathbb{E}^h[\epsilon_{t_i}^2]$ in (24) can be computed by twin Monte Carlo based on (21)²², for two copies $\hat{Y}_{t_{i+1}}^{h,(1)}$ and $\hat{Y}_{t_{i+1}}^{h,(2)}$ of $\hat{Y}_{t_{i+1}}^h$ independent conditionally on $\mathcal{X}_{t_i}^h$.

In the anticipated case where f also depends on ϱ , the analogous propagation of the local regression error terms ϵ_{t_i} and e_{t_i} into global regression error controls for $\mathbb{E}^h[|Y_{t_i}^h - \hat{Y}_{t_i}^h|]$ and $\mathbb{E}^h[|\rho_{t_i}^h - \hat{\rho}_{t_i}^h|]$ is more involved. We define the sequences of non-negative polynomials $(P_i(x))_{i \geq 1}$, $(\tilde{P}_i(x))_{i \geq 1}$, $(Q_i(x))_{i \geq 1}$, and $(\tilde{Q}_i(x))_{i \geq 1}$ by $P_1(x) = \tilde{Q}_1(x) = 1$, $\tilde{P}_1(x) = \Lambda_{\Phi}^h / (1 - \alpha)$, $Q_1(x) = 0$ and, for $i \geq 2$,

$$\begin{aligned} P_i(x) &= \left(1 + \left(1 + \frac{\Lambda_{\Phi}^h}{1 - \alpha}\right)x\right) \left(1 + \left(1 + \frac{2\Lambda_{\Phi}^h}{1 - \alpha}\right)x\right)^{i-2}, \\ \tilde{P}_i(x) &= \frac{2\Lambda_{\Phi}^h}{1 - \alpha} \left(1 + \left(1 + \frac{\Lambda_{\Phi}^h}{1 - \alpha}\right)x\right) \left(1 + \left(1 + \frac{2\Lambda_{\Phi}^h}{1 - \alpha}\right)x\right)^{i-2}, \\ Q_i(x) &= x \left(1 + \left(1 + \frac{2\Lambda_{\Phi}^h}{1 - \alpha}\right)x\right)^{i-2}, \\ \tilde{Q}_i(x) &= \frac{2\Lambda_{\Phi}^h x}{1 - \alpha} \left(1 + \left(1 + \frac{2\Lambda_{\Phi}^h}{1 - \alpha}\right)x\right)^{i-2}. \end{aligned}$$

²²ignoring the $\hat{\rho}_{t_{i+1}}^h$ there.

Theorem 4.1 *With ϵ_{t_i} and e_{t_i} as per (20), we have for every $q \in \{0, \dots, n\}$:*

$$\begin{cases} \mathbb{E}^h[|Y_{t_q}^h - \widehat{Y}_{t_q}^h|] \leq \sum_{i=1}^{n-q} P_i(\Lambda_f \Delta t) \sqrt{\mathbb{E}^h[\epsilon_{t_{q+i-1}}^2]} + Q_i(\Lambda_f \Delta t) \sqrt{\mathbb{E}^h[e_{t_{q+i-1}}^2]} \\ \mathbb{E}^h[|\rho_{t_q}^h - \widehat{\rho}_{t_q}^h|] \leq \sum_{i=1}^{n-q} \widetilde{P}_i(\Lambda_f \Delta t) \sqrt{\mathbb{E}^h[\epsilon_{t_{q+i-1}}^2]} + \widetilde{Q}_i(\Lambda_f \Delta t) \sqrt{\mathbb{E}^h[e_{t_{q+i-1}}^2]}. \end{cases} \quad (25)$$

Proof. By Λ_f Lipschitz continuity of $f_k(t, x, y, \varrho)$ with respect to y and ϱ , we have:

$$\mathbb{E}^h[|Y_{t_i}^h - \widehat{Y}_{t_i}^h|] \leq (1 + \Lambda_f \Delta t) \mathbb{E}^h[|Y_{t_{i+1}}^h - \widehat{Y}_{t_{i+1}}^h|] + \Lambda_f \Delta t \mathbb{E}^h[|\rho_{t_{i+1}}^h - \widehat{\rho}_{t_{i+1}}^h|]$$

and the triangular inequality yields

$$\mathbb{E}^h[|Y_{t_i}^h - \widehat{Y}_{t_i}^h|] \leq (1 + \Lambda_f \Delta t) \mathbb{E}^h[|Y_{t_{i+1}}^h - \widehat{Y}_{t_{i+1}}^h|] + \mathbb{E}^h[|\widehat{Y}_{t_i}^h - \widehat{Y}_{t_i}^h|] + \Lambda_f \Delta t \mathbb{E}^h[|\rho_{t_{i+1}}^h - \widehat{\rho}_{t_{i+1}}^h|]. \quad (26)$$

By the $(1 - \alpha)^{-1}$ Lipschitz continuity (2) of the expected shortfall, the Λ_Φ^h Lipschitz continuity of Φ^h , and an application of the triangular inequality, we have

$$\begin{aligned} \mathbb{E}^h[|\rho_{t_i}^h - \widehat{\rho}_{t_i}^h|] &\leq \mathbb{E}^h \left[\left| \mathbb{E}_{t_i}^h[\Phi_{t_i}^h(Y_{t_i}^h + \sum_{i < \cdot} f(t_i, \mathcal{X}_{t_i}^h, Y_{t_{i+1}}^h, \rho_{t_{i+1}}^h) \Delta t) \right. \right. \\ &\quad \left. \left. - \mathbb{E}_{t_i}^h[\Phi_{t_i}^h(\widehat{Y}_{t_i}^h + \sum_{i < \cdot} f(t_i, \mathcal{X}_{t_i}^h, \widehat{Y}_{t_{i+1}}^h, \widehat{\rho}_{t_{i+1}}^h) \Delta t) \right| \right] \\ (\mathbb{E}_{t_i}^h \text{ is Lipschitz (2)}) &\leq \frac{1}{1 - \alpha} \underbrace{\mathbb{E}^h}_{\mathbb{E}^h} \left[\left| \mathbb{E}_{t_i}^h \left[Y_{t_i}^h + \sum_{i < \cdot} f(t_i, \mathcal{X}_{t_i}^h, Y_{t_{i+1}}^h, \rho_{t_{i+1}}^h) \Delta t \right. \right. \right. \\ &\quad \left. \left. - \Phi_{t_i}^h(\widehat{Y}_{t_i}^h + \sum_{i < \cdot} f(t_i, \mathcal{X}_{t_i}^h, \widehat{Y}_{t_{i+1}}^h, \widehat{\rho}_{t_{i+1}}^h) \Delta t) \right| \right] \\ (\Phi^h \text{ is Lipschitz (14)}) &\leq \frac{\Lambda_\Phi^h}{1 - \alpha} \mathbb{E}^h \left[\left| Y_{t_{(i+m) \wedge n}}^h + Y_{t_i}^h - \sum_{i=i}^{(i+m-1) \wedge (n-1)} f(t_i, \mathcal{X}_{t_i}^h, Y_{t_{i+1}}^h, \rho_{t_{i+1}}^h) \Delta t \right. \right. \\ &\quad \left. \left. - \widehat{Y}_{t_{(i+m) \wedge n}}^h - \widehat{Y}_{t_i}^h + \sum_{i=i}^{(i+m-1) \wedge (n-1)} f(t_i, \mathcal{X}_{t_i}^h, \widehat{Y}_{t_{i+1}}^h, \widehat{\rho}_{t_{i+1}}^h) \Delta t \right| \right] \\ &\leq \frac{\Lambda_\Phi^h}{1 - \alpha} \mathbb{E}^h \left[|Y_{t_{(i+m) \wedge n}}^h - \widehat{Y}_{t_{(i+m) \wedge n}}^h| + |Y_{t_i}^h - \widehat{Y}_{t_i}^h| \right. \\ &\quad \left. + \sum_{i=i}^{(i+m-1) \wedge (n-1)} |f(t_i, \mathcal{X}_{t_i}^h, Y_{t_{i+1}}^h, \rho_{t_{i+1}}^h) - f(t_i, \mathcal{X}_{t_i}^h, \widehat{Y}_{t_{i+1}}^h, \widehat{\rho}_{t_{i+1}}^h)| \Delta t \right] \\ (f \text{ is Lipschitz (8)}) &\leq \frac{\Lambda_\Phi^h}{1 - \alpha} \left[\mathbb{E}^h[|Y_{t_{(i+m) \wedge n}}^h - \widehat{Y}_{t_{(i+m) \wedge n}}^h|] + \mathbb{E}^h[|Y_{t_i}^h - \widehat{Y}_{t_i}^h|] \right] \end{aligned}$$

$$+ \Lambda_f \Delta t \sum_{i=i}^{(i+m-1) \wedge (n-1)} \left(\mathbb{E}^h[|Y_{t_{i+1}}^h - \widehat{Y}_{t_{i+1}}^h| + \mathbb{E}^h[|\rho_{t_{i+1}}^h - \widehat{\rho}_{t_{i+1}}^h|]] \right).$$

Hence

$$\begin{aligned} \mathbb{E}^h[|\rho_{t_{i+1}}^h - \widehat{\rho}_{t_{i+1}}^h|] &\leq \mathbb{E}^h[|\widetilde{\rho}_{t_{i+1}}^h - \widehat{\rho}_{t_{i+1}}^h|] + \mathbb{E}^h[|\rho_{t_{i+1}}^h - \widetilde{\rho}_{t_{i+1}}^h|] \\ &\leq \mathbb{E}^h[|\widetilde{\rho}_{t_{i+1}}^h - \widehat{\rho}_{t_{i+1}}^h|] \\ &\quad + \frac{\Lambda_\Phi^h}{1-\alpha} \left(\mathbb{E}^h[|Y_{t_{(i+m+1) \wedge n}}^h - \widehat{Y}_{t_{(i+m+1) \wedge n}}^h|] + \mathbb{E}^h[|Y_{t_{i+1}}^h - \widehat{Y}_{t_{i+1}}^h|] \right) \quad (27) \\ &\quad + \frac{\Lambda_\Phi^h \Lambda_f \Delta t}{1-\alpha} \sum_{i=i+2}^{(i+m+1) \wedge (n-1)} \left(\mathbb{E}^h[|Y_{t_i}^h - \widehat{Y}_{t_i}^h|] + \mathbb{E}^h[|\rho_{t_i}^h - \widehat{\rho}_{t_i}^h|] \right). \end{aligned}$$

The inequalities (25) are then shown by strong induction on k , which is detailed in Section A. ■

The global controls (25), which by inspection of the proof are not far from sharp, reflect the geometrical accumulation of the local errors inherent to all dynamic programming based simulation/regression algorithms (cf. the comment following (Gobet, 2016, Eqn. (VIII.3.13))). As explained in Abbas-Turki et al. (2018, Section 3), completely avoiding regressions would lead to multiply nested Monte Carlo, with one more level of nesting per pricing time step in particular. Highly multiply nested Monte Carlo is of course impractical, especially on realistic banking portfolios. Regression schemes are more practical, but the take-away message of Theorem 4.1 is that, as usual with such schemes simulation/regression schemes, the number of regression steps should be limited. This is why in particular we distinguish a finer simulation time grid from a coarser pricing (regression) time grid in our implementation²³. From a practical error control viewpoint, the real usefulness of the twin Monte Carlo procedure is at the level of the computation of the $\sqrt{\mathbb{E}^h[\epsilon_{t_i}^2]}$, the way detailed in the end of Section 4.1.

5 XVA Framework

5.1 Continuous-Time Setup

We consider a bank dealing financial derivatives with multiple counterparties indexed by c , with default times $\tau^{(c)}$, where all portfolios are uncollateralized with zero recovery in the case of defaults (all assumed instantaneously settled)²⁴. We denote by $\text{MtM}^{(c)}$ the aggregated mark-to-market process (counterparty-risk-free valuation) of the portfolio of the bank with counterparty c . The bank, with risky funding spread process $\gamma^{(b)}$, maintains capital at risk at the level of an economic capital (EC) defined below as an

²³see <https://github.com/BouazzaSE/NeuralXVA>.

²⁴for notational simplicity we assume no contractual cash flows between the bank and client c at the exact time $\tau^{(c)}$.

expected shortfall of the bank trading loss over one year, at a confidence level $\alpha \in (\frac{1}{2}, 1)$. The bank is assumed perfectly hedged in terms of market risk²⁵, hence its trading loss reduces to the one of its CVA and FVA desks. For the sake of brevity in notation, we omit the discountings at the risk-free rate in the equations (in other terms, we use the risk-free asset as a numéraire), while preserving them in the numerical codes. We assume a KVA risk premium at a hurdle rate $r > 0$, i.e. bank shareholders earn dividends at rate r on their capital at risk. Finally we assume as in Crépey, Sabbagh, and Song (2020) that the bank can use its capital as a risk-free funding source.

As detailed in Albanese, Crépey, Hoskinson, and Saadeddine (2021, Section A.2) and Crépey (2022, Theorem 6.1), this yields the following CVA (credit valuation adjustment), FVA (funding valuation adjustment), EC (economic capital) and KVA (capital valuation adjustment) equations, where $J_t^{(c)} = \mathbf{1}_{\{t < \tau^{(c)}\}}$ (so $-dJ_t^{(c)} = \delta_{\tau^{(c)}}(dt)$, the Dirac measure at time $\tau^{(c)}$): For $t \leq T$ (the final maturity of the derivative portfolio of the bank),

$$\begin{aligned} \text{CVA}_t &= \mathbb{E}_t \left[\sum_c \int_t^T (\text{MtM}_s^{(c)})^+ \delta_{\tau^{(c)}}(ds) \right] \\ \text{FVA}_t &= \mathbb{E}_t \left[\int_t^T \gamma_s^{(b)} \left(\sum_c J_s^{(c)} \text{MtM}_s^{(c)} - \text{CVA}_s - \text{FVA}_s - \max(\text{EC}_s, \text{KVA}_s) \right)^+ ds \right] \quad (28) \\ \text{KVA}_t &= \mathbb{E}_t \left[\int_t^T r e^{-r(s-t)} \max(\text{EC}_s, \text{KVA}_s) ds \right], \end{aligned}$$

where, with $\bar{t} = (t + 1) \wedge T$,

$$\text{EC}_t = \mathbb{E} \mathbb{S}_t [L_{\bar{t}} - L_t], \quad (29)$$

in which the loss process L satisfies, starting from $L_0 = 0$:

$$\begin{aligned} dL_t &= \sum_c (\text{MtM}_t^{(c)})^+ \delta_{\tau^{(c)}}(dt) + d\text{CVA}_t + \\ &\quad \gamma_s^{(b)} \left(\sum_c J_t^{(c)} \text{MtM}_t^{(c)} - \text{CVA}_t - \text{FVA}_t - \max(\text{EC}_t, \text{KVA}_t) \right)^+ dt + d\text{FVA}_t. \quad (30) \end{aligned}$$

All the random variables $J_t^{(c)}$ and $\text{MtM}_t^{(c)}$, as well the pre-default intensity $\gamma_t^{(b)}$ of the bank, are assumed to be $\sigma(\mathcal{X}_t)$ measurable. Hence so is CVA_t as per the first line in (28). The FVA and the KVA equations in (28) can then be written in the form (7), for

²⁵in the sense of Crépey (2022, Eqn. (2.10))

$l = 2$ and

$$Y_t = \begin{pmatrix} \text{FVA}_t, \\ e^{-rt} \text{KVA}_t \end{pmatrix}, \phi = (0, 0)^\top, \\ f_k(t, x, (y_1, y_2)^\top, \varrho) = \\ \begin{pmatrix} \gamma_k^{(b)}(t, x) \left(\sum_c J_k^{(c)}(t, x) \text{MtM}_k^{(c)}(t, x) - \text{CVA}_k(t, x) - y_1 - \max(\varrho, e^{rt} y_2) \right)^+ \\ r \max(e^{-rt} \varrho, y_2) \end{pmatrix}, \quad (31)$$

where we denote $Z_k(t, x) = \mathbb{E}^h[Z_t | X_t = x, J_t = k]$, for any process Z ,

$$\Phi_{\bar{t}}(M) = \int_{\cdot}^{\bar{\cdot}} \left(\sum_c (\text{MtM}_t^{(c)})^+ \delta_{r(c)}(dt) + d\text{CVA}_t + dM_t^1 \right), \text{ for any } M = (M^1, M^2)^\top \in \mathcal{S}_2^2$$

(so, in this XVA setup, Φ only depends on M via M^1). Note that, for (Y, M) solving the ABSDE (7) corresponding to the specification (31), we have

$$dM_t^1 = d\text{FVA}_t + \gamma_t^{(b)} \left(\sum_c J_t^{(c)} \text{MtM}_t^{(c)} - \text{CVA}_t - \text{FVA}_t - \max(\text{EC}_t, \text{KVA}_t) \right)^+ dt,$$

hence in view also of (30):

$$\Phi_{\bar{t}}(M) = L_{\bar{t}} - L_t, \quad (32)$$

which completes the connection between (31) and (28)–(30).

Remark 5.1 *In this XVA case, due to the special form (32) of $\Phi_{\bar{t}}(M)$, one does not need to maintain a full process $\widehat{M}_{t_i}^h$ numerically as it appears in (18) / Algorithm 1. It suffices to update iteratively in (decreasing) time t_i the random variables $L_{\bar{t}_i} - L_{t_i}$ or (in the Picard case) $L_{\bar{t}_i}^j - L_{t_i}^j$ the way detailed in (34) and (37).*

Note that the corresponding functional Φ satisfies in view of its formulation in (31)²⁶:

$$|\Phi(t; \mathbf{x}, \mathbf{m}) - \Phi(t; \mathbf{x}, \mathbf{m}')| \leq |\mathbf{m}_{\bar{t}} - \mathbf{m}'_{\bar{t}}|, \quad (33)$$

so that Assumption 2.2(iii) holds with $\Lambda_\Phi = 1$. Assumption 2.2(i) is readily checked. Assumption 2.2(ii) holds provided the process $(\gamma^{(b)}(t, \mathcal{X}_t) (\sum_c J(t, \mathcal{X}_t) \text{MtM}^{(c)}(t, \mathcal{X}_t)))$ is in \mathcal{H}^2 , e.g. for $\gamma^{(b)}$ bounded and $(\text{MtM}^{(c)}(t, \mathcal{X}_t))$ in \mathcal{H}^2 for each client c .

In view of the first line in (28), the CVA process can be estimated by nonparametric least squares regression in space, at each grid pricing time t_i , based on Monte Carlo paths of the forward process \mathcal{X}^{27} . The exercise is made delicate, however, by the hybrid nature of \mathcal{X} , which includes both diffusive (market risk) and discrete (default risk) components.

²⁶cf. (6).

²⁷or its time-discretized version \mathcal{X}^h , cf. Section 5.2.

This difficulty is solved by the hierarchical simulation scheme of Abbas-Turki, Crépey, and Saadeddine (2023), whereby many default trajectories are simulated conditional on each simulated trajectory of the market. Based on this, the CVA process is treated hereafter as a given (already estimated) process.

The FVA, EC, and KVA equations (with CVA now assumed exogenously given in (28)–(30)) are challenging due to their coupling via the loss process L . However, this coupling can be overcome by a combination of time discretization schemes and (or not) Picard iterations, the way presented in the more general context of Section 3 for the explicit scheme and hereafter in the XVA setup for both the explicit and the implicit/Picard scheme.

Remark 5.2 *In practice, for variance reduction purposes, we use for CVA computations a default intensities based reformulation of the CVA, instead of its definition based on default indicators in (28). Still we do use the hierarchical simulation scheme of Abbas-Turki, Crépey, and Saadeddine (2023), simulating many default paths given each realization of the diffusion processes, in order to help with learnings where we do not have the convenience of using default intensities, i.e. given the presence of default terms in the loss (30), which occur nonlinearly in EC computations, and of default terms under the $(\cdot)^+$ in the FVA computations.*

To alleviate the notation, we assume a uniform time step $\Delta t_{i+1} = \Delta t$, with $1/\Delta t$ chosen as a positive integer m and $\bar{t}_i = t_{(i+m)\wedge n}$, and we skip all the $\hat{\cdot}$ and \cdot^h later in this section (with a last exception to mention that $\Lambda_{\Phi}^h = 1$, as can be readily checked from below the same way as we checked $\Lambda_{\Phi} = 1$ in (33)). By least squares (resp. quantile) regressions below, we always mean *neural network* least squares (resp. quantile) regressions conducted against all risk factors (market risk factors and client default indicators) at each decreasing time t_i , the way detailed in Section 3.2. All regressions and quantile regressions are implemented using a neural network of one hidden layer with 38 neurons²⁸.

²⁸for experimentation with the network architecture see Albanese, Crépey, Hoskinson, and Saadeddine (2021, Section 5.1, Figure 2).

5.2 Explicit Simulation/Regression XVA Scheme

Here we use the following discretization: $\text{CVA}_{t_n} = \text{FVA}_{t_n} = \text{KVA}_{t_n} = 0$ and, for $i = n - 1 \dots 0$,

$$\begin{aligned} \text{CVA}_{t_i} &= \mathbb{E}_{t_i} \left[\sum_c \sum_{i \leq \nu \leq n-1} (\text{MtM}_{t_{i+1}}^{(c)})^+ \mathbf{1}_{\{t_i < \tau^{(c)} \leq t_{i+1}\}} \right], \\ \text{FVA}_{t_i} &= \mathbb{E}_{t_i} \left[\text{FVA}_{t_{i+1}} + \right. \\ &\quad \left. \Delta t \gamma_{t_i}^{(b)} \left(\sum_c \text{MtM}_{t_i}^{(c)} \mathbf{1}_{\{\tau^{(c)} > t_i\}} - \text{CVA}_{t_i} - \text{FVA}_{t_{i+1}} - \max(\text{EC}_{t_{i+1}}, \text{KVA}_{t_{i+1}}) \right)^+ \right], \\ \text{KVA}_{t_i} &= \exp(-r\Delta t) \mathbb{E}_{t_i} [\text{KVA}_{t_{i+1}} + r\Delta t \max(\text{EC}_{t_{i+1}}, \text{KVA}_{t_{i+1}})], \\ \text{EC}_{t_i} &= \mathbb{E} \mathbb{S}_{t_i} [L_{\bar{t}_i} - L_{t_i}], \text{ where} \\ L_{t_{i+1}} - L_{t_i} &= \text{CVA}_{t_{i+1}} - \text{CVA}_{t_i} + (\text{MtM}_{t_i}^{(c)})^+ \mathbf{1}_{\{t_i < \tau^{(c)} \leq t_{i+1}\}} + \text{FVA}_{t_{i+1}} - \text{FVA}_{t_i} + \\ &\quad \Delta t \gamma_{t_i}^{(b)} \left(\sum_c \text{MtM}_{t_i}^{(c)} \mathbf{1}_{\{\tau^{(c)} > t_i\}} - \text{CVA}_{t_i} - \text{FVA}_{t_{i+1}} - \max(\text{EC}_{t_{i+1}}, \text{KVA}_{t_{i+1}}) \right)^+. \end{aligned} \tag{34}$$

We recognize the explicit scheme of Algorithm 1 applied to $Y = (\text{FVA}, \text{KVA})$. This scheme naturally lifts the coupling visible in (28)–(30) between EC, KVA, and FVA. Specifically, assuming that all the $\text{XVA}_{t_{i+1}}$ and $\text{EC}_{t_{i+1}}$ have already been learned, we compute:

1 CVA_{t_i} , by a least-squares regression of $\sum_c \sum_{i \leq \nu \leq n-1} (\text{MtM}_{t_i}^{(c)})^+ \mathbf{1}_{\{t_i < \tau^{(c)} \leq t_{i+1}\}}$ ²⁹ against all risk factors at time t_i ;

2 FVA_{t_i} , through a least-squares regression of

$$\text{FVA}_{t_{i+1}} + \Delta t \gamma_{t_i}^{(b)} \left(\sum_c \text{MtM}_{t_i}^{(c)} \mathbf{1}_{\{\tau^{(c)} > t_i\}} - \text{CVA}_{t_i} - \text{FVA}_{t_{i+1}} - \max(\text{EC}_{t_{i+1}}, \text{KVA}_{t_{i+1}}) \right)^+$$

against all risk factors at time t_i ;

3 KVA_{t_i} , through a least-squares regression of

$$\exp(-r\Delta t) (\text{KVA}_{t_{i+1}} + r\Delta t \max(\text{EC}_{t_{i+1}}, \text{KVA}_{t_{i+1}}))$$

against all risk factors at time t_i ;

4 EC_{t_i} , through quantile regression of $L_{\bar{t}_i} - L_{t_i}$ followed by a least-squares regression to deduce the expected shortfall as detailed in Section 3.2, both regressions involving all the risk factors at time t_i .

²⁹or equivalent variance-reduced cash flows formulated in terms of the default *intensities* as explained in Remark 5.2.

A Posteriori Twin Monte Carlo Validation Procedure Let

$$\epsilon_{t_i}^{fva} = \text{FVA}_{t_i} - \widetilde{\text{FVA}}_{t_i}, \quad \epsilon_{t_i}^{kva} = \text{KVA}_{t_i} - \widetilde{\text{KVA}}_{t_i}, \quad (35)$$

where $(\widetilde{\text{FVA}}_{t_i}, e^{-rt_i} \widetilde{\text{KVA}}_{t_i})$ corresponds to (19) in the $(\text{FVA}_{t_i}, e^{-rt_i} \text{KVA}_{t_i})$ case (31). The twin error estimation (21) applied to $(\text{FVA}_{t_i}, e^{-rt_i} \text{KVA}_{t_i})$ (component-wise as detailed in the proof of Lemma 4.1) yields

$$\begin{aligned} \mathbb{E}^h[(\epsilon_{t_i}^{fva})^2] &= \mathbb{E}^h[(\text{FVA}_{t_i})^2 \\ &\quad - \text{FVA}_{t_i} \left(\text{FVA}_{t_{i+1}}^{(1)} + f^1(t_i, \mathcal{X}_{t_i}^h, \widehat{Y}_{t_{i+1}}^{h,(1)}, \widehat{\rho}_{t_{i+1}}^{h,(1)})\Delta t + \text{FVA}_{t_{i+1}}^{(2)} + f^1(t_i, \mathcal{X}_{t_i}^h, \widehat{Y}_{t_{i+1}}^{h,(2)}, \widehat{\rho}_{t_{i+1}}^{h,(2)})\Delta t \right) \\ &\quad + \left(\text{FVA}_{t_{i+1}}^{(1)} + f^1(t_i, \mathcal{X}_{t_i}^h, \widehat{Y}_{t_{i+1}}^{h,(1)}, \widehat{\rho}_{t_{i+1}}^{h,(1)})\Delta t \right)^\top \left(\text{FVA}_{t_{i+1}}^{(2)} + f^1(t_i, \mathcal{X}_{t_i}^h, \widehat{Y}_{t_{i+1}}^{h,(2)}, \widehat{\rho}_{t_{i+1}}^{h,(2)})\Delta t \right)] , \end{aligned} \quad (36)$$

where $\text{FVA}_{t_{i+1}}^{(1)}$ and $\text{FVA}_{t_{i+1}}^{(2)}$ are independent conditioned on $\mathcal{X}_{t_i}^h$, and f^1 is the first component of the vector function f . Substituting $\exp^{-rt_i} \text{KVA}_{t_i}$ instead of FVA_{t_i} and f^2 instead of f^1 in (36), one obtains e^{-2rt_i} times the twin error estimation $\mathbb{E}^h[(\epsilon_{t_i}^{kva})^2]$ for KVA_{t_i} .

5.3 Implicit/Picard Simulation/Regression XVA Scheme

We define and compute the CVA as in the explicit scheme. For the rest of the XVAs, we introduce Picard iterations echoing (12), starting from $\text{EC}^0 \equiv 0$ followed by, for increasing $j \geq 1$: $\text{FVA}_{t_n}^j = \text{KVA}_{t_n}^j = 0$ and, for $i = n - 1 \dots 0$,

$$\begin{aligned} \text{FVA}_{t_i}^j &= \mathbb{E}_{t_i} \left[\text{FVA}_{t_{i+1}}^j + \right. \\ &\quad \left. \Delta t \gamma_{t_i}^{(b)} \left(\sum_c \text{MtM}_{t_i}^{(c)} \mathbf{1}_{\{\tau^{(c)} > t_i\}} - \text{CVA}_{t_i} - \widetilde{\text{FVA}}_{t_i}^{(j-1)} - \max(\text{EC}_{t_i}^{j-1}, \widetilde{\text{KVA}}_{t_i}^{j-1}) \right)^+ \right] \\ \text{KVA}_{t_i}^j &= \exp(-r\Delta t) \mathbb{E}_{t_i} [\text{KVA}_{t_{i+1}}^j + r\Delta t \max(\text{EC}_{t_i}^{j-1}, \widetilde{\text{KVA}}_{t_i}^{j-1})] \\ \text{EC}_{t_i}^j &= \mathbb{E}_{t_i} [L_{t_i}^j - L_{t_i}^j], \text{ where} \\ L_{t_{i+1}}^j - L_{t_i}^j &= \text{CVA}_{t_{i+1}} - \text{CVA}_{t_i} + (\text{MtM}_{t_i}^{(c)})^+ \mathbf{1}_{\{t_i < \tau^{(c)} \leq t_{i+1}\}} + \text{FVA}_{t_{i+1}}^j - \text{FVA}_{t_i}^j + \\ &\quad \Delta t \gamma_{t_i}^{(b)} \left(\sum_c \text{MtM}_{t_i}^{(c)} \mathbf{1}_{\{\tau^{(c)} > t_i\}} - \text{CVA}_{t_i} - \widetilde{\text{FVA}}_{t_i}^{j-1} - \max(\text{EC}_{t_i}^{j-1}, \widetilde{\text{KVA}}_{t_i}^{j-1}) \right)^+ , \end{aligned} \quad (37)$$

where we set

$$\widetilde{\text{FVA}}_{t_i}^{j-1} = \mathbf{1}_{j=1} \text{FVA}_{t_{i+1}}^j + \mathbf{1}_{j \geq 2} \text{FVA}_{t_i}^{j-1}, \quad \widetilde{\text{KVA}}_{t_i}^{j-1} = \mathbf{1}_{j=1} \text{KVA}_{t_{i+1}}^j + \mathbf{1}_{j \geq 2} \text{KVA}_{t_i}^{j-1}. \quad (38)$$

In this scheme the coupling between EC, KVA, and FVA is removed by the Picard iterations in j . Specifically, assuming that all the XVA $_{t_i}^{j-1}$ (for $j \geq 2$), XVA $_{t_{i+1}}^j$, and EC $_{t_i}^{j-1}$ have already been learned, we compute:

- 1 $FVA_{t_i}^j$, by least-squares regression of

$$FVA_{t_{i+1}}^j + \Delta t \gamma_{t_i}^{(b)} \left(\sum_c \text{MtM}_{t_i}^{(c)} \mathbf{1}_{\{\tau^{(c)} > t_i\}} - \text{CVA}_{t_i} - \widetilde{FVA}_{t_i}^{j-1} - \max(\text{EC}_{t_i}^{j-1}, \text{KVA}_{t_i}^{j-1}) \right)^+$$

against the risk factors at time t_i ;

- 2 $KVA_{t_i}^j$, through a least-squares regression of

$$\exp(-r\Delta t)(KVA_{t_{i+1}}^j + r\Delta t \max(\text{EC}_{t_i}^{j-1}, \widetilde{KVA}_{t_i}^{j-1}))$$

against all risk factors at time t_i ;

- 3 $\text{EC}_{t_i}^j$, through quantile regression of $L_{t_i}^j - L_{t_i}^j$ followed by a least-squares regression to deduce the expected shortfall as detailed in Section 3.2, both regressions involving all the risk factors at time t_i .

6 XVA Numerical Benchmark

For our numerical experiments, we assume 10 economies, each represented by a short-rate process with Vasicek dynamics, 9 cross-currency rate processes with log-normal dynamics, 8 counterparties each with a stochastic default intensity process following CIR dynamics, adding up to 28 diffusive risk factors (when accounting for the spread of the bank which is also assumed to be driven by a CIR process) and 8 default indicator processes. We consider a portfolio of 100 interest rate swaps, all at-par at time 0. The characteristics of the swaps (notional, maturity, counterparty and currency) are drawn randomly. In particular, the maturities of the swaps range between 0.9375 and $T = 10$ years. For all modeling and implementation details, as well as numerical values of all the parameters, the reader is referred to <https://github.com/BouazzaSE/NeuralXVA>.

6.1 Benchmarking the Explicit versus the Picard Scheme

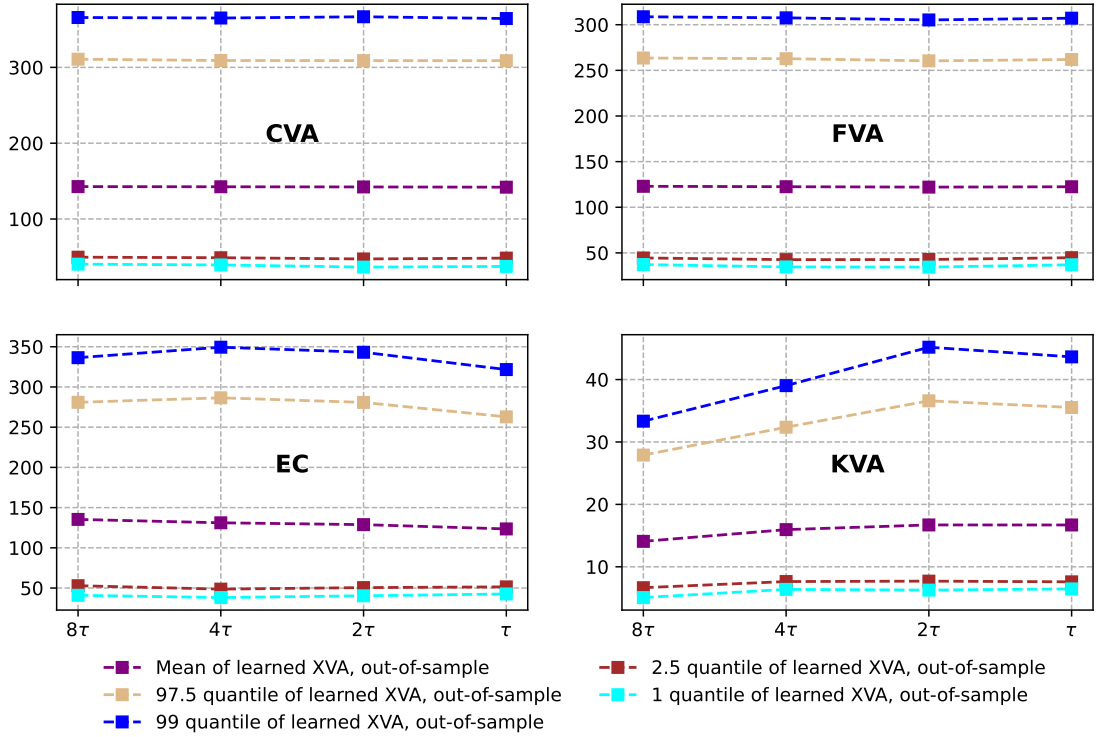
For the purpose of empirical time discretization analysis³⁰, we consider multiple time discretizations $(h^{(\theta)})_\theta$ such that

$$h^{(\theta)} = \{t_i^{(\theta)} := i \frac{T}{2^\theta}, i = 0, \dots, 2^\theta\}.$$

In Table 1 and Figures 1–4, we experimented for $\theta \in \{5, 6, 7, 8\}$ (τ in the figures then corresponds to $\frac{T}{2^8}$). We can thus see the convergence of the Picard iterations for the FVA process, using the implicit (FVA, KVA) scheme of Section 5.3, toward a solution visually very close, already for $j = 4$, to the one provided by the explicit scheme of

³⁰see end of Section 3.1.

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	Explicit
$h = \frac{T}{25}$	463.279938	433.832031	434.391296	433.753998	434.65167
$h = \frac{T}{26}$	461.329926	433.141876	434.036011	433.835052	433.60974
$h = \frac{T}{27}$	461.031097	432.506531	433.631531	431.789215	433.18683
$h = \frac{T}{28}$	460.326050	433.123596	431.992859	432.098328	434.29538

Table 1: FVA₀ under the Picard iteration scheme of Section 5.3 vs. the explicit scheme.Figure 1: Mean and quantiles of CVA, FVA, KVA and EC learned by the explicit scheme at $t = \frac{T}{2}$ for different sizes of the time step.

Algorithm 1, aka Section 5.2. However, because of the multiple Picard iterations, the Picard scheme comes at the cost of $j = 3-4$ times longer computations (for reference, the explicit scheme takes 7mins 48secs using the finest time grid, i.e. $\theta = 8$, on an NVidia A100 GPU). These results mutually validate the explicit and the implicit/Picard scheme against each other, while showing that the Picard scheme is not competitive with respect to the explicit scheme in terms of computation times.

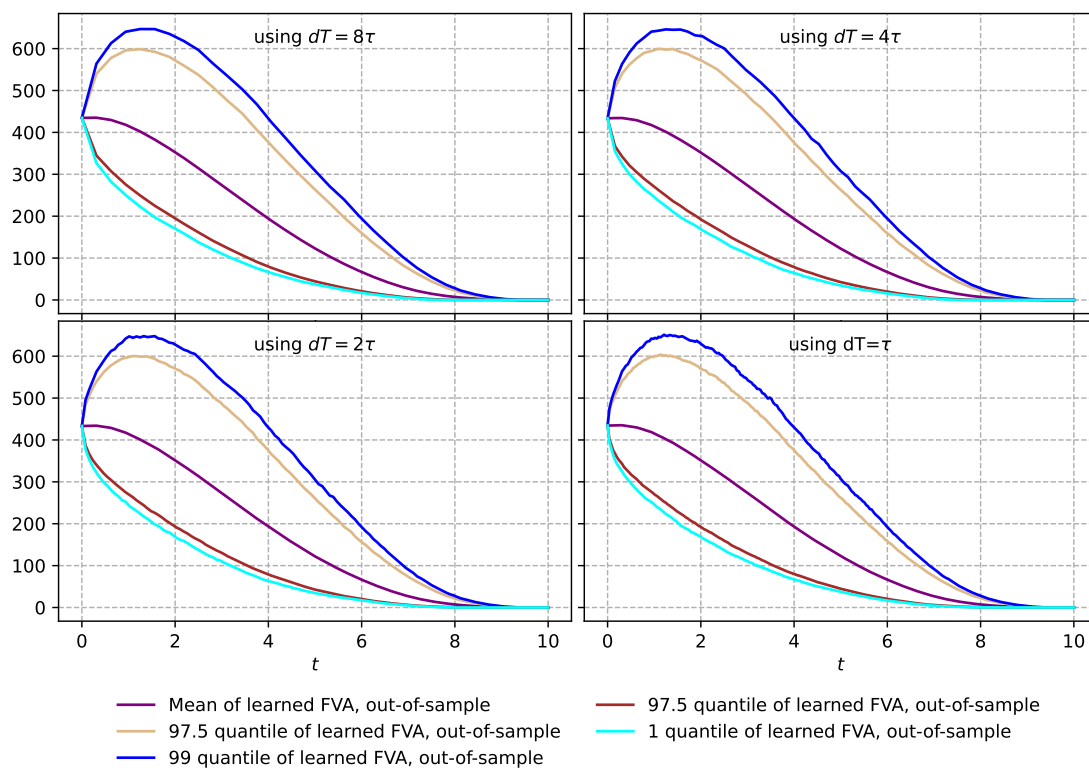


Figure 2: FVA profiles using an explicit scheme.

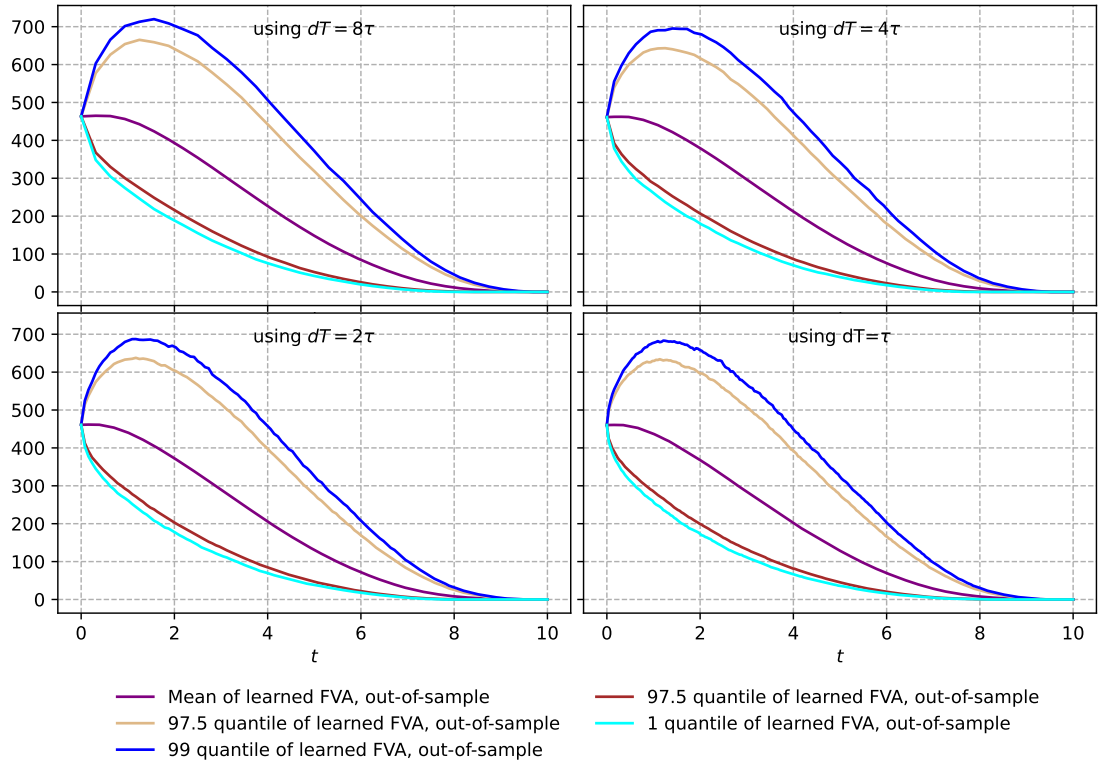


Figure 3: FVA profiles obtained after $j = 1$ Picard iteration for the implicit scheme.

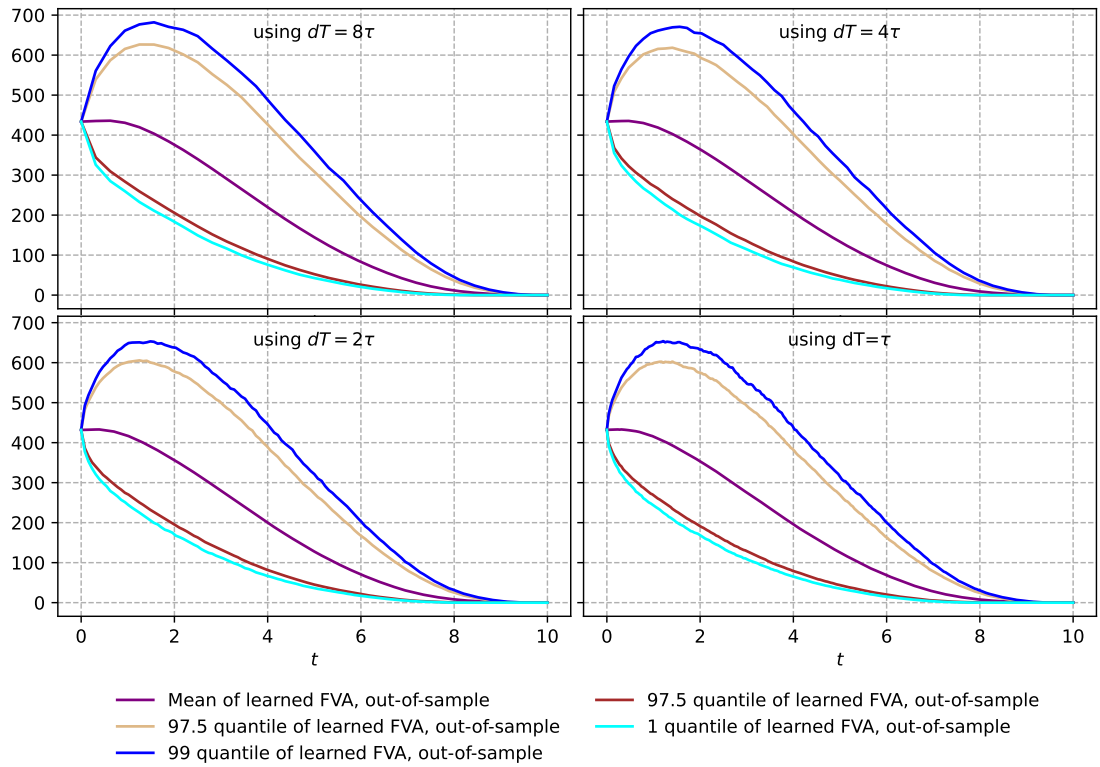


Figure 4: FVA profiles obtained after $j = 4$ Picard iteration for the implicit scheme.

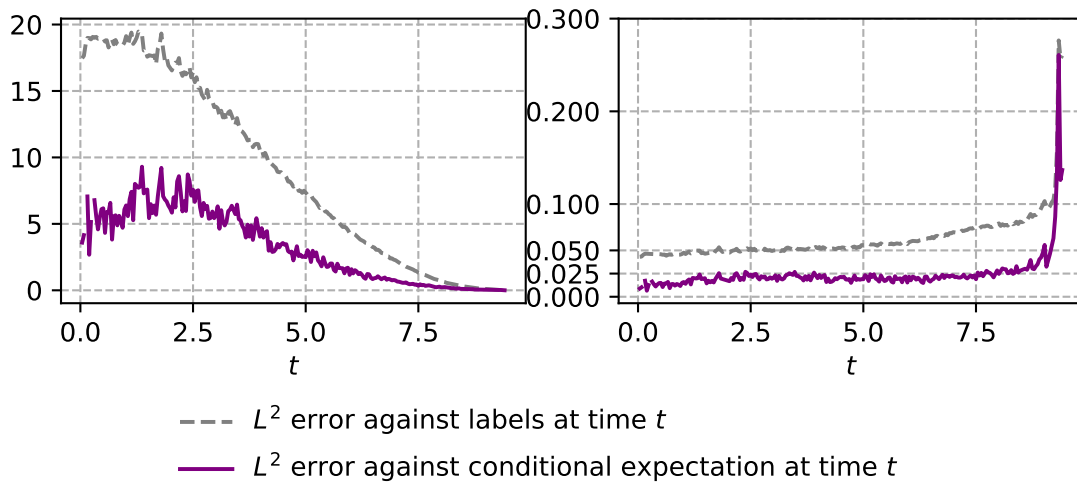


Figure 5: Local regression errors $\sqrt{\mathbb{E}^h[(\epsilon_{t_i}^{fva})^2]}$ (solid purple) vs. L^2 training losses (dashed grey) for $\theta = 10$. Left panel: raw errors. Right panel: errors normalized at each time step t_i by the L^2 norm of $\widehat{\text{FVA}}_{t_i}^h$.

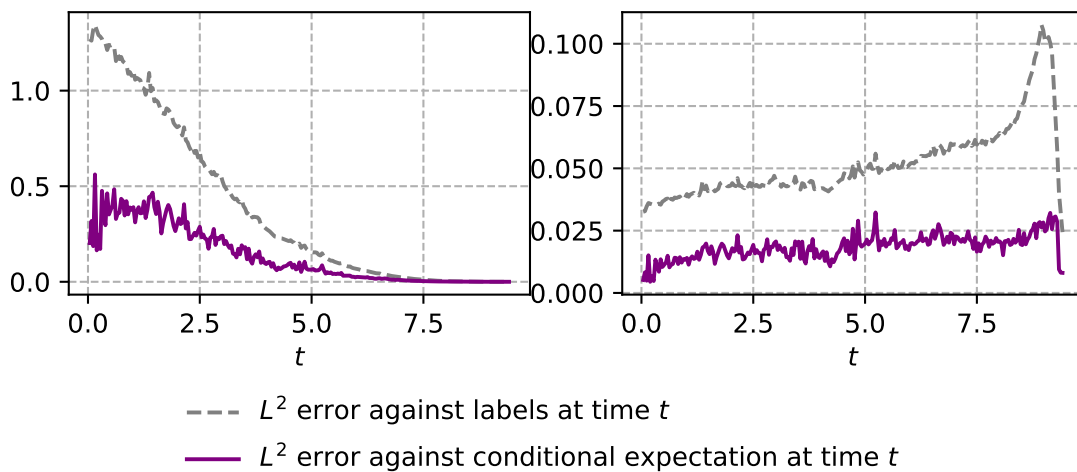


Figure 6: Local regression errors $\sqrt{\mathbb{E}^h[(\epsilon_{t_i}^{kva})^2]}$ (solid purple) vs. L^2 training losses (dashed grey). Left panel: raw errors. Right panel: errors normalized at each time step t_i by the L^2 norm of $\widehat{\text{KVA}}_{t_i}^h$.

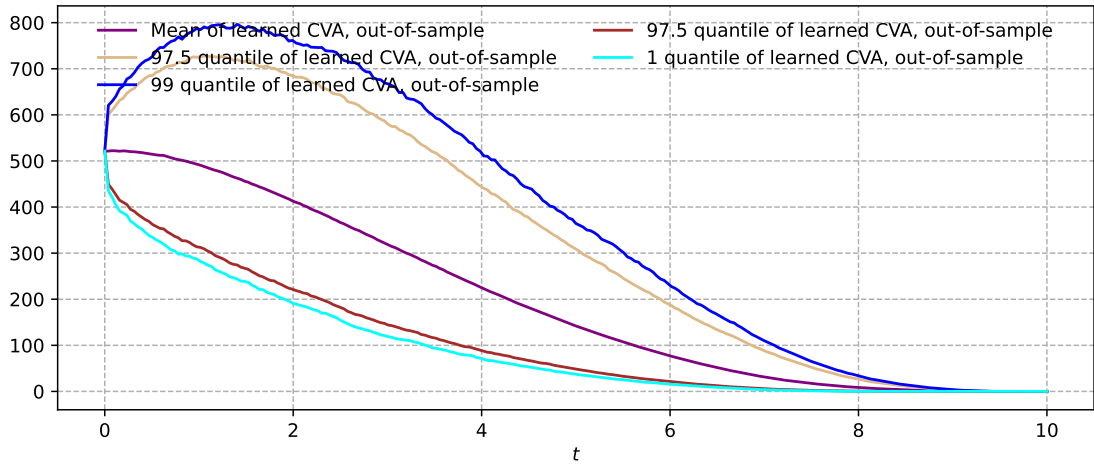


Figure 7: CVA profiles using an explicit scheme and a fine time discretization ($\theta = 10$)

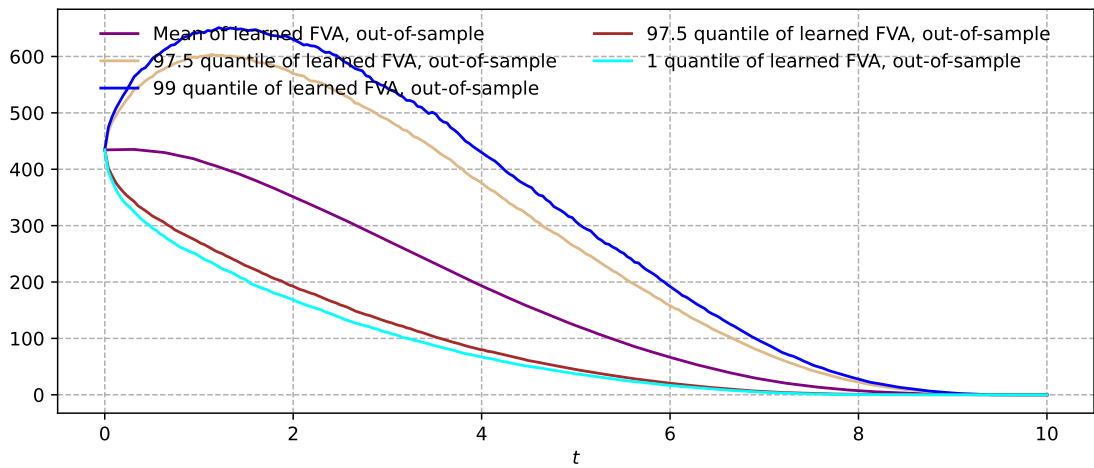


Figure 8: FVA profiles using an explicit scheme and a fine time discretization ($\theta = 10$)

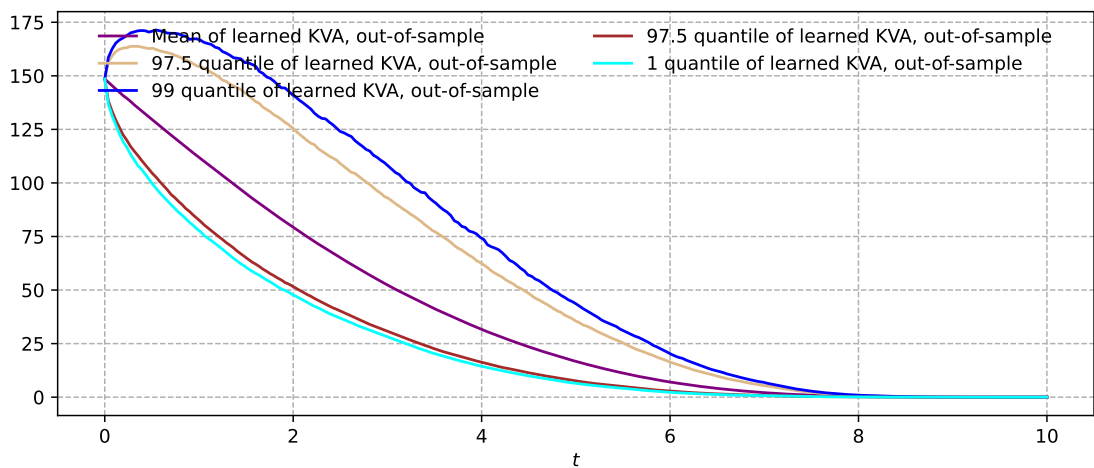


Figure 9: KVA profiles using an explicit scheme and a fine time discretization ($\theta = 10$)

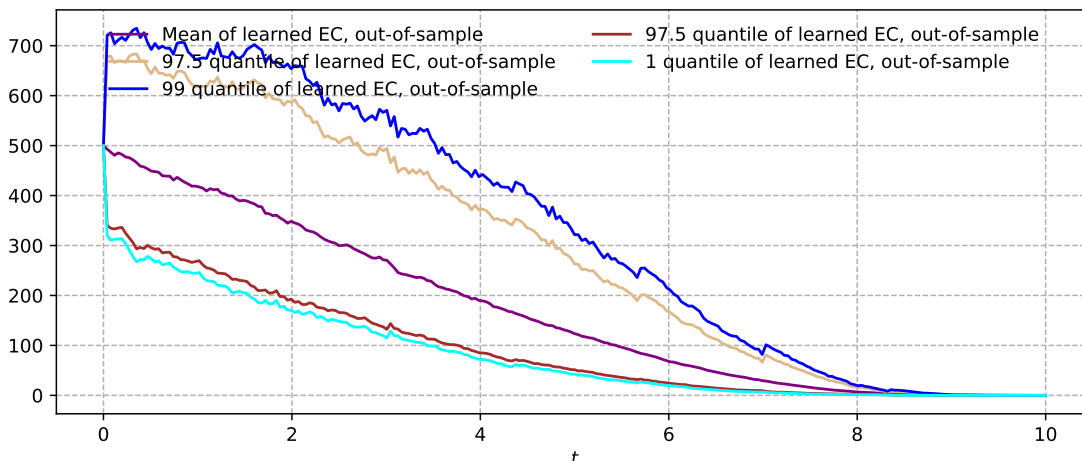


Figure 10: EC profiles using an explicit scheme and a fine time discretization ($\theta = 10$)

6.2 A Posteriori Error Validation of the Explicit Scheme

The solid purple curves in Figures 5 and 6 exhibit the local regression errors $\sqrt{\mathbb{E}^h[(\epsilon_{t_i}^{fva})^2]}$ and $\sqrt{\mathbb{E}^h[(\epsilon_{t_i}^{kva})^2]}$ of the $(FVA_{t_i}, e^{-rt_i}KVA_{t_i})$ scheme³¹ for a fine time discretization $\theta = 10$. The dashed grey curves represent the corresponding L^2 training losses. The comparison between the grey and purple curves shows the benefit of the a posteriori Monte Carlo local regression error estimates (21) with respect to the L^2 training losses that would be used as a naive and overconservative error estimate. These and the above numerical results confort us in the choice of the explicit scheme as the reference pathwise XVA numerical scheme. Our final plots in Figures 7–10 show profiles for respectively the CVA, FVA, KVA and EC when using the explicit scheme with $\theta = 10$.

7 Conclusion

The main contributions of the paper are the ABSDE explicit scheme of Algorithm 1, which dominates the implicit/Picard scheme in terms of computation times (for mutually validating XVA numbers) in the XVA case study of Section 6, and the a posteriori error control of Section 4 for the explicit scheme.

The recent and fastly growing machine learning literature on the solution of high-dimensional nonlinear BSDEs/PDEs contains, at least, two branches. The first one, in the line of E, Han, and Jentzen (2017), consists in learning together the time-0 value of the solution and the gradient-process of the latter through a single learning task. Examples in the XVA space include Henry-Labordère (2019) or Gnoatto, Reisinger, and Picarelli (2021). The former reference provides insightful PDE views on the CVA

³¹cf. (35).

and MVA. The second reference computed the XVAs and their model state sensitivities simultaneously. However, the stylized equations of Henry-Labordère (2019) are quite far from actual XVA equations. The XVA equations of Gnoatto, Reisinger, and Picarelli (2021) are less unrealistic, but they are still restricted to computations at the level of one client of the bank and they are accordingly handled by reduction of filtration in the line of Crépey and Song (2015), so that the default times ultimately disappear from the equations. Such an approach does not leverage to several clients, whose default times enter the XVA equations of the bank in a nonlinear fashion (and therefore have to be simulated as we do). The second stream of literature, that includes Huré, Pham, and Warin (2020) or the present paper, learns the solution time step after time step (in backward time), much like classical dynamic programming algorithms, except that machine learning techniques are used for solving the local equations which then arise at each successive decreasing pricing time step.

In the case of our ABSDE XVA equations, the first global approach would not be viable on realistic problems stated at the portfolio level due to the significant memory requirement for the corresponding global training task. In contrast, the second local approach benefits from a synergy between the successive local training tasks involved: as all our XVA equations are endowed with zero terminal conditions, the variance of the cash flows (“regressands”) entering the successive learning tasks as input data at the decreasing pricing time steps increases progressively (time step after time step), whereas the variance of the features (“regressors”), i.e. of the risk factors, decreases. As a result, the difficulty of the training tasks gradually increases throughout the course of the algorithm³². But the next training task also benefits from all previous ones, via the use of the weights trained at a time step as initialization for the weights at the next time step. This is probably one of the reasons behind the robustness of our approach³³, while a global approach would fail on unsolvable memory occupation issues.

From an algorithmic viewpoint, work in progress aims at demonstrating how the XVA simulation/regression framework of this paper can be leveraged to also encompass XVA sensitivities. Note that AAD sensitivities computational techniques à la Baydin, Pearlmutter, Radul, and Siskind (2018); Savine (2018) are not a viable alternative in a pathwise XVA setup where the pathwise XVA metrics are already the output of optimization training procedures: AAD sensitivity techniques can only be available in more rudimentary XVA frameworks. From a mathematical viewpoint, the establishment of a Feynman-Kac representation for the solution of the limiting ABSDE (7), as well as the study of the time-consistency of our explicit scheme, are interesting open issues.

³²In order to see this in a simplified setup, consider linear regression instead of neural networks. When close to $t = 0$, the variances of the regressors tend to 0, which leads to ill-conditioned covariance matrices, while the variance of the regressands increases, which makes the regression even more unstable.

³³provided the hierarchical simulation technique of Abbas-Turki, Crépey, and Saadeddine (2023) and the best practice risk measure estimates of Barrera, Crépey, Gobet, Nguyen, and Saadeddine (2022) are used, as done in our numerics.

A End of the Proof of Theorem 4.1

Lemma A.1 For all $i \geq 1$:

(i) $(1+x)P_i(x) + x\tilde{P}_i(x) \leq P_{i+1}(x)$.

(ii) $(1+x)Q_i(x) + x\tilde{Q}_i(x) \leq Q_{i+1}(x)$.

If $i \leq m$, then:

(iii) $P_i(x) + x \sum_{\iota=1}^{i-1} (P_\iota(x) + \tilde{P}_\iota(x)) \leq (1-\alpha)\tilde{P}_i(x)/\Lambda_\Phi^h$.

(iv) $Q_i(x) + x \sum_{\iota=1}^{i-1} (Q_\iota(x) + \tilde{Q}_\iota(x)) \leq (1-\alpha)\tilde{Q}_i(x)/\Lambda_\Phi^h$.

If $i \geq m+1$, then:

(v) $P_i(x) + P_{i-m}(x) + x \sum_{\iota=i-m}^{i-1} (P_\iota(x) + \tilde{P}_\iota(x)) \leq (1-\alpha)\tilde{P}_i(x)/\Lambda_\Phi^h$.

(vi) $Q_i(x) + Q_{i-m}(x) + x \sum_{\iota=i-m}^{i-1} (Q_\iota(x) + \tilde{Q}_\iota(x)) \leq (1-\alpha)\tilde{Q}_i(x)/\Lambda_\Phi^h$.

Proof. It is straightforward to verify that the polynomials $P_i(x), \tilde{P}_i(x), Q_i(x), \tilde{Q}_i(x)$ are generated by $P_i(x) = 1, \tilde{P}_i(x) = \Lambda_\Phi^h/(1-\alpha), Q_i(x) = 0, \tilde{Q}_i(x) = 1$, and the following recursive relations, for $i \geq 2$:

$$P_i(x) = (1+x)P_{i-1}(x) + x\tilde{P}_{i-1}(x), \quad (39)$$

$$\tilde{P}_i(x) = \frac{2\Lambda_\Phi^h P_i(x)}{1-\alpha}, \quad (40)$$

$$Q_i(x) = (1+x)Q_{i-1}(x) + x\tilde{Q}_{i-1}(x), \quad (41)$$

$$\tilde{Q}_i(x) = \frac{2\Lambda_\Phi^h Q_i(x)}{1-\alpha}. \quad (42)$$

With (39) and (41), (i) and (ii) are proven and sharp. Equation (39) for $i \geq 2$ can be rewritten as

$$P_{i+1}(x) - P_i(x) = x \left(\tilde{P}_i(x) + P_i(x) \right), \quad i \geq 1,$$

which leads to the handy relation

$$x \sum_{\iota=1}^{i-1} (P_\iota(x) + \tilde{P}_\iota(x)) = P_i(x) - P_1(x), \quad i \geq 1. \quad (43)$$

Inserting (43) into (40), $\tilde{P}_i(x)$ can be rewritten, for $i \geq 2$, as

$$(1-\alpha)\tilde{P}_i(x)/\Lambda_\Phi^h = P_i(x) + P_i(x) + P_1(x) - P_1(x) \quad (44)$$

$$= P_i(x) + x \sum_{\iota=1}^{i-1} (P_\iota(x) + \tilde{P}_\iota(x)) + P_1(x) \quad (45)$$

$$\geq P_i(x) + x \sum_{\iota=1}^{i-1} (P_\iota(x) + \tilde{P}_\iota(x)). \quad (46)$$

Inequality (46) is also true for $i = 1$, hence **(iii)** is proven. For $i \geq m + 1$, inserting (43) into (45) leads to

$$\begin{aligned}
(1 - \alpha)\tilde{P}_i(x)/\Lambda_\Phi^h &= P_i(x) + x \sum_{\iota=1}^{i-1} \left(P_\iota(x) + \tilde{P}_\iota(x) \right) + P_1(x) \\
&\quad + \underbrace{P_{i-m}(x) - P_1(x) - x \sum_{\iota=1}^{i-m-1} \left(P_\iota(x) + \tilde{P}_\iota(x) \right)}_0 \\
&= P_i(x) + P_{i-m}(x) + x \sum_{\iota=i-m}^{i-1} \left(P_\iota(x) + \tilde{P}_\iota(x) \right),
\end{aligned}$$

which proves **(v)** and indicates that it is sharp. It is worth noting that the proof requires only the recursive relations (40) and (39), but not $P_1(x)$ and $\tilde{P}_1(x)$. As the polynomials Q satisfy the same recursive relations as the polynomials P , **(iv)** and **(vi)** are also proven. ■

Let B_q and \tilde{B}_q be defined, for $q \leq n - 1$, as

$$B_q = \sum_{i=1}^{n-q} \left(P_i(\Lambda_f \Delta t) \mathbb{E}^h[\epsilon_{t_{q+i-1}}] + Q_i(\Lambda_f \Delta t) \mathbb{E}^h[e_{t_{i+q-1}}] \right); \quad (47)$$

$$\tilde{B}_q = \sum_{i=1}^{n-q} \left(\tilde{P}_i(\Lambda_f \Delta t) \mathbb{E}^h[\epsilon_{t_{q+i-1}}] + \tilde{Q}_i(\Lambda_f \Delta t) \mathbb{E}^h[e_{t_{i+q-1}}] \right). \quad (48)$$

With Lemma A.1, we prove the theorem 4.1 by strong backward induction. Using (26) and (27), we find $\mathbb{E}^h[|Y_{t_{n-1}}^h - \hat{Y}_{t_{n-1}}^h|] \leq \mathbb{E}^h[\epsilon_{t_{n-1}}] = B_{n-1}$, and hence $\mathbb{E}^h[|\rho_{t_{n-1}}^h - \hat{\rho}_{t_{n-1}}^h|] \leq \frac{\Lambda_\Phi^h}{1-\alpha} \mathbb{E}^h[\epsilon_{t_{n-1}}] + \mathbb{E}^h[e_{t_{n-1}}] = \tilde{B}_{n-1}$. We assume that for each $i \geq k$ the errors $\mathbb{E}^h[|Y_{t_i}^h - \hat{Y}_{t_i}^h|]$ and $\mathbb{E}^h[|\rho_{t_i}^h - \hat{\rho}_{t_i}^h|]$ are bounded by B_q and \tilde{B}_q respectively. Then, we want to prove, using this induction assumption, that $\mathbb{E}^h[|Y_{t_{q-1}}^h - \hat{Y}_{t_{q-1}}^h|]$ and $\mathbb{E}^h[|\rho_{t_{q-1}}^h - \hat{\rho}_{t_{q-1}}^h|]$ are bounded by B_{q-1} and \tilde{B}_{q-1} respectively. The error of $\mathbb{E}^h[|Y_{t_{q-1}}^h - \hat{Y}_{t_{q-1}}^h|]$ is bounded using (26).

By inserting (47) into (26), we obtain

$$\begin{aligned}
\mathbb{E}^h[|Y_{t_{q-1}}^h - \widehat{Y}_{t_{q-1}}^h|] &\leq (1 + \Lambda_f \Delta t) B_q + \Lambda_f \Delta t \widetilde{B}_q + \mathbb{E}^h[\epsilon_{t_{q-1}}] \\
&= (1 + \Lambda_f \Delta t) \sum_{i=1}^{n-q} \left(P_i(\Lambda_f \Delta t) \mathbb{E}^h[\epsilon_{t_{q+i-1}}] + Q_i(\Lambda_f \Delta t) \mathbb{E}^h[e_{t_{q+i-1}}] \right) \\
&\quad + \Lambda_f \Delta t \sum_{i=1}^{n-q} \left(\widetilde{P}_i(\Lambda_f \Delta t) \mathbb{E}^h[\epsilon_{t_{q+i-1}}] + \widetilde{Q}_i(\Lambda_f \Delta t) \mathbb{E}^h[e_{t_{q+i-1}}] \right) \\
&\quad + \mathbb{E}^h[\epsilon_{t_{q-1}}] \\
&= \sum_{i=2}^{n-q+1} \underbrace{\left((1 + \Lambda_f \Delta t) P_{i-1}(\Lambda_f \Delta t) + \Lambda_f \Delta t \widetilde{P}_{i-1}(\Lambda_f \Delta t) \right)}_{\leq P_i(x) \text{ due to Lemma A.1(i)}} \mathbb{E}^h[\epsilon_{t_{q+i-2}}] \\
&\quad + \sum_{i=2}^{n-q+1} \underbrace{\left((1 + \Lambda_f \Delta t) Q_{i-1}(\Lambda_f \Delta t) + \Lambda_f \Delta t \widetilde{Q}_{i-1}(\Lambda_f \Delta t) \right)}_{\leq Q_i(x) \text{ due to Lemma A.1(ii)}} \mathbb{E}^h[e_{t_{q+i-2}}] \\
&\quad + \mathbb{E}^h[\epsilon_{t_{q-1}}] \\
&\leq \sum_{i=1}^{n-q+1} \left(P_i(\Lambda_f \Delta t) \mathbb{E}^h[\epsilon_{t_{q+i-2}}] + Q_i(\Lambda_f \Delta t) \mathbb{E}^h[e_{t_{q+i-2}}] \right) \\
&= B_{q-1}.
\end{aligned}$$

It is worth noting that the bound is obtained using not only Lemma A.1, but also the value of $P_1(x)$ and $Q_1(x)$.

Applying the same idea to (27), the bound of $\mathbb{E}^h[|\rho_{t_i}^h - \widehat{\rho}_{t_i}^h|]$, we have

$$\mathbb{E}^h[|\rho_{t_{q-1}}^h - \widehat{\rho}_{t_{q-1}}^h|] \leq \mathbb{E}^h[e_{t_{q-1}}] + \frac{\Lambda_\Phi^h}{1 - \alpha} \left(B_{q-1} + B_{(q+m-1) \wedge n} + \Lambda_f \Delta t \sum_{i=q}^{(q+m-1) \wedge (n-1)} (B_i + \widetilde{B}_i) \right). \quad (49)$$

Depending on the relation between $q + m$ and n , we have two cases.

Case $q + m > n$ In this case, (49) reduces to

$$\begin{aligned}
\mathbb{E}^h[|\rho_{t_{q-1}}^h - \tilde{\rho}_{t_{q-1}}^h|] &\leq \mathbb{E}^h[e_{t_{q-1}}] + \frac{\Lambda_\Phi^h}{1-\alpha} \left(B_{q-1} + \Lambda_f \Delta t \sum_{i=q}^{n-1} (B_i + \tilde{B}_i) \right) \\
&= \frac{\Lambda_\Phi^h}{1-\alpha} \left(\sum_{i=1}^{n-q+1} P_i(\Lambda_f \Delta t) \mathbb{E}^h[\epsilon_{t_{q+i-2}}] + Q_i(\Lambda_f \Delta t) \mathbb{E}^h[e_{t_{q+i-2}}] \right) \\
&\quad + \frac{\Lambda_\Phi^h \Lambda_f \Delta t}{1-\alpha} \sum_{i=q}^{n-1} \sum_{i=1}^{n-i} \left((P_i(\Lambda_f \Delta t) + \tilde{P}_i(\Lambda_f \Delta t)) \mathbb{E}^h[\epsilon_{t_{i+i-1}}] \right. \\
&\quad \left. + (Q_i(\Lambda_f \Delta t) + \tilde{Q}_i(\Lambda_f \Delta t)) \mathbb{E}^h[e_{t_{i+i-1}}] \right) + \mathbb{E}^h[e_{t_{q-1}}].
\end{aligned}$$

Notice that for any two real sequences $(a_i)_{i \geq 1}$ and $(b_i)_{i \geq 1}$ we have:

$$\begin{aligned}
\sum_{i=q}^{n-1} \sum_{i=1}^{n-i} a_i b_{i+i-1} &= \sum_{i=1}^{n-q} \sum_{i=1}^{n-q-i+1} a_i b_{q+i+i-2} = \sum_{i=1}^{n-q} \sum_{i=i+1}^{n-q+1} a_{i-1} b_{q+i-2} \\
&= \sum_{i=2}^{n-q+1} \left(\sum_{i=1}^{i-1} a_{i-i} \right) b_{q+i-2} = \sum_{i=2}^{n-q+1} \left(\sum_{i=1}^{i-1} a_i \right) b_{q+i-2}. \quad (50)
\end{aligned}$$

Using this relation, the order of the summation can be changed and the double sum terms become

$$\sum_{i=2}^{n-q+1} \left(\sum_{i=1}^{i-1} (P_i(\Lambda_f \Delta t) + \tilde{P}_i(\Lambda_f \Delta t)) \mathbb{E}^h[\epsilon_{t_{q+i-2}}] + \sum_{i=1}^{i-1} (Q_i(\Lambda_f \Delta t) + \tilde{Q}_i(\Lambda_f \Delta t)) \mathbb{E}^h[e_{t_{q+i-2}}] \right).$$

The error of $\mathbb{E}^h[|\rho_{t_i}^h - \tilde{\rho}_{t_i}^h|]$ is thus bounded by

$$\begin{aligned}
&\sum_{i=2}^{n-q+1} \frac{\Lambda_\Phi^h}{1-\alpha} \underbrace{\left(P_i(\Lambda_f \Delta t) + \Lambda_f \Delta t \sum_{i=1}^{i-1} (P_i(\Lambda_f \Delta t) + \tilde{P}_i(\Lambda_f \Delta t)) \right)}_{\leq \tilde{P}_i(\Lambda_f \Delta t) \text{ due to Lemma A.1(iii)}} \mathbb{E}^h[\epsilon_{t_{q+i-2}}] \\
&+ \sum_{i=2}^{n-q+1} \frac{\Lambda_\Phi^h}{1-\alpha} \underbrace{\left(Q_i(\Lambda_f \Delta t) + \Lambda_f \Delta t \sum_{i=1}^{i-1} (Q_i(\Lambda_f \Delta t) + \tilde{Q}_i(\Lambda_f \Delta t)) \right)}_{\leq \tilde{Q}_i(\Lambda_f \Delta t) \text{ due to Lemma A.1(iv)}} \mathbb{E}^h[e_{t_{q+i-2}}] \\
&+ \frac{\Lambda_\Phi^h}{1-\alpha} \underbrace{P_1(\Lambda_f \Delta t)}_{=1} \mathbb{E}^h[\epsilon_{t_{q-1}}] + \left(\frac{\Lambda_\Phi^h}{1-\alpha} \underbrace{Q_1(\Lambda_f \Delta t)}_{=0} + 1 \right) \mathbb{E}^h[e_{t_{q-1}}] \\
&\leq \sum_{i=1}^{n-q+1} \left(\tilde{P}_i(\Lambda_f \Delta t) \mathbb{E}^h[\epsilon_{t_{q+i-2}}] + \tilde{Q}_i(\Lambda_f \Delta t) \mathbb{E}^h[e_{t_{q+i-2}}] \right) \\
&= \tilde{B}_{q-1}.
\end{aligned}$$

Case $q + m \leq n$ In this case, (49) reduces to

$$\begin{aligned}
\mathbb{E}^h[|\rho_{t_{q-1}}^h - \widehat{\rho}_{t_{q-1}}^h|] &\leq \mathbb{E}^h[e_{t_{q-1}}] + \frac{\Lambda_\Phi^h}{1-\alpha} \left(B_{q-1} + B_{q+m-1} + \Lambda_f \Delta t \sum_{i=q}^{q+m-1} (B_i + \widetilde{B}_i) \right) \\
&= \mathbb{E}^h[e_{t_{q-1}}] + \frac{\Lambda_\Phi^h}{1-\alpha} \left(\sum_{i=1}^{n-q+1} \left(P_i(\Lambda_f \Delta t) \mathbb{E}^h[\epsilon_{t_{q+i-2}}] + Q_i(\Lambda_f \Delta t) \mathbb{E}^h[e_{t_{q+i-2}}] \right) \right. \\
&\quad \left. + \sum_{i=1}^{n-q+1-m} \left(P_i(\Lambda_f \Delta t) \mathbb{E}^h[\epsilon_{t_{q+i-2+m}}] + Q_i(\Lambda_f \Delta t) \mathbb{E}^h[e_{t_{q+i-2+m}}] \right) \right) \\
&\quad + \frac{\Lambda_\Phi^h \Lambda_f \Delta t}{1-\alpha} \sum_{i=q}^{q+m-1} \sum_{i=1}^{n-i} \left((P_i(\Lambda_f \Delta t) + \widetilde{P}_i(\Lambda_f \Delta t)) \mathbb{E}^h[\epsilon_{t_{i+i-1}}] \right. \\
&\quad \left. + (Q_i(\Lambda_f \Delta t) + \widetilde{Q}_i(\Lambda_f \Delta t)) \mathbb{E}^h[e_{t_{i+i-1}}] \right) \\
&= \frac{\Lambda_\Phi^h}{1-\alpha} \left(\sum_{i=1}^{n-q+1} \left(P_i(\Lambda_f \Delta t) \mathbb{E}^h[\epsilon_{t_{q+i-2}}] + Q_i(\Lambda_f \Delta t) \mathbb{E}^h[e_{t_{q+i-2}}] \right) \right. \\
&\quad \left. + \underbrace{\sum_{i=m+1}^{n-q+1}}_{=\sum_{i=1}^{n-q+1} \mathbf{1}_{\{i \geq m+1\}}} \left(P_{i-m}(\Lambda_f \Delta t) \mathbb{E}^h[\epsilon_{t_{q+i-2}}] + Q_{i-m}(\Lambda_f \Delta t) \mathbb{E}^h[e_{t_{q+i-2}}] \right) \right) \\
&\quad + \frac{\Lambda_\Phi^h \Lambda_f \Delta t}{1-\alpha} \left(\sum_{i=q}^{n-1} - \sum_{i=q+m}^{n-1} \right) \sum_{i=1}^{n-i} \left((P_i(\Lambda_f \Delta t) + \widetilde{P}_i(\Lambda_f \Delta t)) \mathbb{E}^h[\epsilon_{t_{i+i-1}}] \right. \\
&\quad \left. + (Q_i(\Lambda_f \Delta t) + \widetilde{Q}_i(\Lambda_f \Delta t)) \mathbb{E}^h[e_{t_{i+i-1}}] \right) + \mathbb{E}^h[e_{t_{q-1}}].
\end{aligned}$$

For any two real sequences $(a_i)_{i \geq 1}$ and $(b_i)_{i \geq 1}$, using (50), we have:

$$\begin{aligned}
\left(\sum_{i=q}^{n-1} - \sum_{i=q+m}^{n-1} \right) \sum_{i=1}^{n-i} a_i b_{i+i-1} &= \sum_{i=2}^{n-q+1} \left(\sum_{i=1}^{i-1} a_i \right) b_{q+i-2} - \sum_{i=2}^{n-q+1-m} \left(\sum_{i=1}^{i-1} a_i \right) b_{q+i-2+m} \\
&= \sum_{i=2}^{n-q+1} \left(\sum_{i=1}^{i-1} a_i \right) b_{q+i-2} - \sum_{i=m+2}^{n-q+1} \left(\sum_{i=1}^{i-m-1} a_i \right) b_{q+i-2} \\
&= \sum_{i=2}^{n-q+1} \left(\sum_{i=1}^{i-1} - \mathbf{1}_{\{i \geq m+1\}} \sum_{i=1}^{i-m-1} \right) a_i b_{q+i-2}. \tag{51}
\end{aligned}$$

Note that we can use $\mathbf{1}_{\{i \geq m+1\}}$ instead of the expected $\mathbf{1}_{\{i \geq m+2\}}$ in the last line, because, when $i = m + 1$, $\sum_{i=1}^{i-m-1} = 0$. Using (51), the double sum terms in next-to-last line

above can be written as

$$\begin{aligned} & \frac{\Lambda_\Phi^h \Lambda_f \Delta t}{1-\alpha} \sum_{i=2}^{n-q+1} \left(\sum_{i=1}^{i-1} -\mathbb{1}_{\{i \geq m+1\}} \sum_{i=1}^{i-m-1} \right) \left((P_i(\Lambda_f \Delta t) + \tilde{P}_i(\Lambda_f \Delta t)) \mathbb{E}^h[\epsilon_{t_{q+i-2}}] \right. \\ & \left. + (Q_i(\Lambda_f \Delta t) + \tilde{Q}_i(\Lambda_f \Delta t)) \mathbb{E}^h[e_{t_{q+i-2}}] \right), \end{aligned}$$

and the bound for $\mathbb{E}^h[|\rho_{t_{q-1}}^h - \hat{\rho}_{t_{q-1}}^h|]$ becomes

$$\begin{aligned} & \sum_{i=2}^{n-q+1} \frac{\Lambda_\Phi^h}{1-\alpha} \left(P_i(\Lambda_f \Delta t) + \mathbb{1}_{\{i \geq m+1\}} P_{i-m}(\Lambda_f \Delta t) \right. \\ & \quad \left. + \Lambda_f \Delta t \left(\sum_{i=1}^{i-1} -\mathbb{1}_{\{i \geq m+1\}} \sum_{i=1}^{i-m-1} \right) (P_i(\Lambda_f \Delta t) + \tilde{P}_i(\Lambda_f \Delta t)) \right) \mathbb{E}^h[\epsilon_{t_{q+i-2}}] \\ & + \sum_{i=2}^{n-q+1} \frac{\Lambda_\Phi^h}{1-\alpha} \left(Q_i(\Lambda_f \Delta t) + \mathbb{1}_{\{i \geq m+1\}} Q_{i-m}(\Lambda_f \Delta t) \right. \\ & \quad \left. + \left(\sum_{i=1}^{i-1} -\mathbb{1}_{\{i \geq m+1\}} \sum_{i=1}^{i-m-1} \right) (Q_i(\Lambda_f \Delta t) + \tilde{Q}_i(\Lambda_f \Delta t)) \right) \mathbb{E}^h[e_{t_{q+i-2}}] \\ & + \frac{\Lambda_\Phi^h}{1-\alpha} P_1(\Lambda_f \Delta t) \mathbb{E}^h[\epsilon_{t_{q-1}}] + \left(\frac{\Lambda_\Phi^h}{1-\alpha} Q_1(\Lambda_f \Delta t) + 1 \right) \mathbb{E}^h[e_{t_{q-1}}] \\ & = \sum_{i=m+1}^{n-q+1} \frac{\Lambda_\Phi^h}{1-\alpha} \underbrace{\left(P_i(\Lambda_f \Delta t) + P_{i-m}(\Lambda_f \Delta t) + \Lambda_f \Delta t \sum_{i=i-m}^{i-1} (P_i(\Lambda_f \Delta t) + \tilde{P}_i(\Lambda_f \Delta t)) \right)}_{\leq \tilde{P}_i(\Lambda_f \Delta t) \text{ due to Lemma A.1(v)}} \mathbb{E}^h[\epsilon_{t_{q+i-2}}] \\ & \quad + \sum_{i=m+1}^{n-q+1} \frac{\Lambda_\Phi^h}{1-\alpha} \underbrace{\left(Q_i(\Lambda_f \Delta t) + Q_{i-m}(\Lambda_f \Delta t) + \Lambda_f \Delta t \sum_{i=i-m}^{i-1} (Q_i(\Lambda_f \Delta t) + \tilde{Q}_i(\Lambda_f \Delta t)) \right)}_{\leq \tilde{Q}_i(\Lambda_f \Delta t) \text{ due to Lemma A.1(vi)}} \mathbb{E}^h[e_{t_{q+i-2}}] \\ & \quad + \sum_{i=2}^m \frac{\Lambda_\Phi^h}{1-\alpha} \underbrace{\left(P_i(\Lambda_f \Delta t) + \Lambda_f \Delta t \sum_{i=1}^{i-1} (P_i(\Lambda_f \Delta t) + \tilde{P}_i(\Lambda_f \Delta t)) \right)}_{\leq \tilde{P}_i(\Lambda_f \Delta t) \text{ due to Lemma A.1(iii)}} \mathbb{E}^h[\epsilon_{t_{q+i-2}}] \\ & \quad + \sum_{i=2}^m \frac{\Lambda_\Phi^h}{1-\alpha} \underbrace{\left(Q_i(\Lambda_f \Delta t) + \Lambda_f \Delta t \sum_{i=1}^{i-1} (Q_i(\Lambda_f \Delta t) + \tilde{Q}_i(\Lambda_f \Delta t)) \right)}_{\leq \tilde{Q}_i(\Lambda_f \Delta t) \text{ due to Lemma A.1(iv)}} \mathbb{E}^h[e_{t_{q+i-2}}] \\ & \quad + \frac{\Lambda_\Phi^h}{1-\alpha} \underbrace{P_1(\Lambda_f \Delta t)}_{=1} \mathbb{E}^h[\epsilon_{t_{q-1}}] + \left(\frac{\Lambda_\Phi^h}{1-\alpha} \underbrace{Q_1(\Lambda_f \Delta t)}_{=0} + 1 \right) \mathbb{E}^h[e_{t_{q-1}}] \end{aligned}$$

$$\begin{aligned} &\leq \sum_{i=1}^{n-q+1} \left(\tilde{P}_i(\Lambda_f \Delta t) \mathbb{E}^h[\epsilon_{t_{q+i-2}}] + \tilde{Q}_i(\Lambda_f \Delta t) \mathbb{E}^h[e_{t_{q+i-2}}] \right) \\ &= \tilde{B}_{q-1}. \end{aligned}$$

Thus, we proved that $\mathbb{E}^h[|Y_{t_{q-1}}^h - \hat{Y}_{t_{q-1}}^h|]$ and $\mathbb{E}^h[|\rho_{t_{q-1}}^h - \hat{\rho}_{t_{q-1}}^h|]$ are bounded by B_{q-1} and \tilde{B}_{q-1} respectively. Applying the relation $\mathbb{E}^h[\cdot] \leq \sqrt{\mathbb{E}^h[\cdot^2]}$, we obtain (25) and prove Theorem 4.1.

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