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► **To cite this version:**

Peter Vankúš, Michaela Vargová. Analysis of the pre-service mathematics teachers' errors in proving tasks. Thirteenth Congress of the European Society for Research in Mathematics Education (CERME13), Alfréd Rényi Institute of Mathematics; Eötvös Loránd University of Budapest, Jul 2023, Budapest, Hungary. hal-04413125

HAL Id: hal-04413125

<https://hal.science/hal-04413125>

Submitted on 23 Jan 2024

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Analysis of the pre-service mathematics teachers' errors in proving tasks

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In this article, we focus on the analysis of the common errors and misconceptions in proofs among mathematics pre-service teachers. We conduct the research on a sample of 34 students. We analyze 5 tasks assessing the truth of statements. In the article, we present these tasks and analyze students' solutions. We then list the errors that the students make. We compare them to research on the most common errors. The result is that while we observe some of the errors that have been uncovered in previous studies, some other more specific error types also occur. These are mainly related to students' deficits in formal logic, use of mathematical language and symbolism, and understanding of the structure and content of statements. The limitation of our research lies in the limited sample and in the selection of the specific tasks.

Keywords: Mathematics teachers, reasoning skills, proof, teacher training.

Introduction

Constructing all but the most straightforward of proofs involves a good deal of persistence and problem-solving to put together relevant concepts (Selden & Selden, 2008). Because of some basic misconceptions and a lack of skills a lot of typical difficulties emerge. Some of the common problems observed in the research studies are: 1) student can understand a general statement but cannot test it in specific cases; 2) student can only follow a small part of the given proof and cannot even begin to evaluate it; 3) undergraduate students sometimes fail to use or interpret relevant theorems correctly or fail to verify that the conditions of the hypotheses of a theorem are satisfied (Selden & Selden, 2008; Stefanowicz et al., 2014).

Whereas research on argumentation is rapidly growing, little research specifically focuses on teachers' knowledge associated with argumentation (Ayalon & Even, 2016, Kosko et al., 2014, Michal et al., 2022). Findings from existing research demonstrate that teachers can have difficulties in the incorporation of argumentation into classroom practice (Ayalon & Even, 2016, Bieda, 2010). For supporting students to engage in such activities, the teacher needs to possess knowledge of various aspects associated with argumentation, including reasoning, and proving (Mueller et al., 2014; Yackel, 2002). Considering that such knowledge impacts the way in which this key practice is implemented in the classroom (Conner et al., 2014; Kosko et al., 2014), it appears important to make it the focus of the investigation.

Therefore, in our research we deal with the argumentation, specifically proving skills of the pre-service mathematics teachers. During our teaching at the Faculty of mathematics, physics and informatics, Comenius University in Bratislava, we observed that pre-service mathematics teachers make frequent errors in proofs in more demanding mathematical tasks. To discover the nature of these errors, we decided to focus on the analysis of simpler tasks focusing on proving. The study focuses

on pre-service mathematics teachers who are going to teach mathematics at the secondary school level (ages 11-19). The research investigates the common errors and misconceptions made by these students while proving statements during task-solving in the test and categorizes them based on their frequency of occurrence. Our research question was: "What are the common errors in solving tasks focused on proving among pre-service mathematics teachers?"

This research is of common interest as it addresses the persistent difficulties and misconceptions that arise when constructing proofs in mathematics, which can hinder students' ability to effectively engage in proving. The findings from this research can contribute to the understanding of the common errors made by pre-service mathematics teachers when solving simple tasks focused on proving and can potentially inform the development of targeted interventions and instructional strategies to improve their proving abilities. This research also adds to the growing body of literature on argumentation in mathematics education, which can benefit researchers, and practitioners in promoting effective mathematical reasoning and proof skills among teachers and students.

In our paper, we analyse tasks focused on deductive thinking in the proving as defined by (Sevinc et al., 2022). We analyse these tasks from the point of students' errors. Error analysis is useful to identify common patterns of errors made by students in solving mathematical problems and to find the cause of the errors (Herholdt & Sapire, 2014). This method has been claimed to be an effective diagnostic tool to bridge the gap between the expected outcome and the performance (Indrawatiningsih et al., 2020). Some of the common error in the proving tasks solved by the students are mentioned in (Stavrou, 2014, Selden & Selden, 2003). Because our tasks are not covering every type of the proofs and are limited by the small number of students, we cannot expect to recognise each of these errors and some new errors and misconceptions may arise. Therefore, we focus on the errors identifications without the need to characterize them in specific framework.

Methods

Data Sources. The research sample consisted of pre-service mathematics teachers in the 2nd year of university study. The research was conducted on the subject "Mathematical Analysis 1", which was attended by 34 students. The tasks analyzed in the article were part of a complex test that students solved as part of the final exam in the subject. The objective of these tasks was to evaluate the validity of general, existential, or dual-quantified statements where the order of the quantifiers plays a crucial role. These tasks were designed to assess knowledge of fundamental types of proofs and proficiency in propositional logic. It is worth noting that the content of these tasks did not directly relate to the curriculum covered in Mathematical Analysis (derivative and integral calculus). The mathematical contents of the tasks correspond to knowledge from high school or of the first semester of mathematical courses at the university. As a result, these tasks could also be applicable in other mathematics courses, such as number theory and algebra.

Procedure. The test solved by students contained 12 tasks, and it was written in several variants. For our analysis, we selected tasks focused on proving. We analyzed them using qualitative analysis through open coding (Creswell & Poth, 2013), focusing on the errors those the students made. As part of open coding, we do not use a selected specific framework, but when discussing errors, we refer to already existing literary sources or frameworks. The tasks differed based on the variant of the

test. There were 5 proving tasks in total, 1 in each of the test variants. The number of the students solving each of the task is shown in the Table 1. Some students solved more than one task because then needed to repeat the test due to the failed exam.

Table 1: Number of the students solving the tasks

	Task 1	Task 2	Task 3	Task 4	Task 5
N. of students solving the task	6	14	15	11	6
N. of correct/incorrect solutions	4/2	9/5	8/7	4/7	3/3

Assignment for all the tasks: Decide whether the following statement is true. Justify your answer (by providing proof or a suitable counterexample, etc.). The tasks and the expected solutions are outlined in Figure 1.

<p>Task 1. If the number x is irrational, then also the \sqrt{x} is irrational. Expected solutions:</p> <ol style="list-style-type: none"> 1) <i>Proof by contradiction.</i> We assume the proposition to be false, i.e., we assume the negation of the proposition. We suppose that x is the irrational and \sqrt{x} is rational. Then, it is shown that negation of the proposition implies falsehood. When the \sqrt{x} is rational, it can be written as $\frac{p}{q}$, $p \in \mathbb{N}$, $q \in \mathbb{N}$. Therefore, $\frac{p^2}{q^2} = x$, which implies that x is rational, that is the contradiction with the assumption that x is irrational. 2) <i>Proof by contrapositive.</i> The conclusion "if A, then B" is inferred by constructing a proof of the claim "if not B, then not A" instead. So, we will justify the statement: If \sqrt{x} is rational, then x is rational. If \sqrt{x} is rational, we know that $\sqrt{x} \cdot \sqrt{x}$ is also rational, because of the fact, that rational numbers are closed under the multiplication. <p>Task 2. $\exists n \in \mathbb{N} : \left(1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^n}\right) \in \mathbb{N}$, Expected solutions:</p> <ol style="list-style-type: none"> 1) Finding the sum of $n+1$ members of geometric sequence with the quotient $q = \frac{1}{2}$. $S_{n+1} = 1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^n} = \frac{\left(\frac{1}{2}\right)^{n+1} - 1}{\frac{1}{2} - 1} = \frac{2^{n+1} - 1}{2^n}$ <p>Now we can see that the nominator is odd number, and the denominator is even number. So, this fraction cannot be the natural number.</p> 2) Finding the sum of $n+1$ members of geometric sequence with the quotient $q = \frac{1}{2}$. $S_{n+1} = 1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^n} = \frac{\left(\frac{1}{2}\right)^{n+1} - 1}{\frac{1}{2} - 1} = 2 - \left(\frac{1}{2}\right)^n$ <p>The sequence $\{S_{n+1}\}_{n=1}^{\infty}$ is obviously increasing and since $2 - \left(\frac{1}{2}\right)^n < 2$ for every $n \in \mathbb{N}$ this sequence is bounded above, so the sequence is convergent, whereas: $\lim_{n \rightarrow \infty} S_{n+1} = \sup \{S_{n+1}, n \in \mathbb{N}\} = 2.$ <p>The minimal value is $\frac{3}{2}$. This implies that $1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^n}$ could not be a natural number, because there is no such number that is between $\frac{3}{2}$ and 2.</p> </p> <p>Task 3. $\forall a \in \mathbb{R} \exists x \in \mathbb{R} : x^2 + 2ax + a = 0$. Expected solutions:</p> <ol style="list-style-type: none"> 1) If this quadratic function should have a real root, there should be discriminant equal or bigger than 0. So, this inequation should be valid: $4a^2 - 4a \geq 0$. Therefore, $a \in (-\infty, 0] \cup [1, \infty)$. 2) We can try to adjust this equation by completing the square. $(x+a)^2 - a^2 + a = 0$. Therefore, $x = -a \pm \sqrt{a^2 - a}$. This value is real when $a^2 - a \geq 0$. This is true just for $a \in (-\infty, 0] \cup [1, \infty)$. 3) We know that $f(x) = x^2 + 2ax + a$ represents the quadratic function which graph is parabola with the vertex of $[-a, -a^2 + a]$. Therefore, the equation has real solutions just when the parabola intersects the x axis, that is for $-a^2 + a \leq 0$. This is true for $a \in (-\infty, 0] \cup [1, \infty)$. 4) This general statement can be shown false by using counterexample. Student can choose e. g. $a = \frac{1}{2}$ and she gets the quadratic equation $x^2 + x + \frac{1}{2} = 0$. Now she finds out that this equation does not have real roots. <p>Task 4. $\forall n \in \mathbb{N} \exists q_1 \in \mathbb{Q} \exists q_2 \in \mathbb{Q} : q_1 q_2 + q_1 = n$. Expected solutions:</p> <ol style="list-style-type: none"> 1) For any $q_1 \in \mathbb{Q}$ we can express the $q_2 = \frac{n - q_1}{q_1} = \frac{n}{q_1} - 1$. From the basic properties of the set of rational numbers we know that the expression $\frac{n}{q_1} - 1 \in \mathbb{Q}$. 2) We choose the $q_1 = \frac{n}{b}$, $q_2 = b - 1$, $b \in \mathbb{Q}$. Then, the $q_1 q_2 + q_1 = \frac{n}{b}(b - 1) + \frac{n}{b} = n$. This can be done for any n. <p>Task 5. $\exists n \in \mathbb{N}, n \geq 10 \forall m \in \mathbb{N} : \sqrt{nm^2} \in \mathbb{N}$. Expected solutions:</p> <ol style="list-style-type: none"> 1) $\sqrt{nm^2} = \sqrt{n} \cdot m$. This expression is natural number if and only if $\sqrt{n} \in \mathbb{N}$. This is true for $n = k^2$, $k \in \mathbb{N}$. To fulfil the condition $n \geq 10$ the value of k should be $k \geq 4$. 2) The student will write one value of n that is square of $k \geq 4$. Then he/she just illustrates that the result is natural number for any value of m.

Figure 1: Tasks and expected solutions

Results

In this section, we present the answer on our research question " What are the common errors in solving tasks focused on proving among pre-service mathematics teachers?" We observed these results. The common error was that **student does not distinguish between existential and general statement (E1)**. This error was in three forms:

1) Students are utilizing **specific examples to substantiate general statements (E1A)**. This is often referred to in literature as substituting empirical evidence for valid proof (Stylianides, 2009), or utilizing an erroneous empirical proof scheme (Harel & Sowder, 1998). Presented below are some illustrative examples of this error.

Examples: Task 4. Student writes the equation $\sqrt{2} \cdot \sqrt{3} + \sqrt{2} = n$. Then she writes $\sqrt{2} \cdot \sqrt{3} + \sqrt{2} \neq n$. From this, she concludes that the statement is not valid. She uses specific examples to prove the general statement.

Task 5. Student chooses specific values of n and m . $n = 30$, so $n \in \mathbb{N}$ and $n \geq 10$ $n = 30$, $m = 2$, $m \in \mathbb{N}$ $\sqrt{30 \cdot 2^2} = \sqrt{120} \notin \mathbb{N}$.

Now he concludes that statement is not valid. The student recognizes that he can choose the value of n , because the statement is existential for the value of n , but he does not understand that proving that it is not correct for one value of n does not mean the statement is not correct. He also does not realize the difference from the point of m , which should be universal. Therefore, even when he has chosen the n correctly, the proof is not complete, because it should be done for any value of m . So, this error is connected to using a specific example to prove the general statement, and not distinguishing between the existential and general statements.

2) The second form of the error was **using the techniques of proofs for general statements to prove the existential statement (E1B)** (Pfeiffer, 2010 in Stavrou, 2014).

Examples: Task 2. Student tries to use mathematical induction. She chooses $n = 1$, now she states that $1 + \frac{1}{2} = 1,5 \in \mathbb{N}$ and she continues with learned steps in the mathematical induction proof $n = k$: $1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^k} \in \mathbb{N}$, $n = k + 1$: $1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^k} + \frac{1}{2^{k+1}} \in \mathbb{N}$. Here she finishes without any conclusion. The student does not know that she cannot use mathematical induction to prove the existential statement.

Task 2. Student computes the first value for $n = 1$. (He does it wrong.) Now he concludes that $\frac{1}{2} \notin \mathbb{N}$ and from this he assumes that statement is not valid. Here we can see that this student tries to evaluate the existential statement by the counterexample, which is possible just for the general statement.

Example: Task 5. Now the student understands that m can achieve any values, so she makes the rearrangement of the expression $\sqrt{n \cdot m^2} = \sqrt{n} \cdot m$. She then chooses the value of $n = 11$, and concludes that $\sqrt{11} \notin \mathbb{N}$. She then also states that even for $n = 1, 2, 3$, $\sqrt{n} \notin \mathbb{N}$. From this she evaluates that the statement is not valid. She does not understand, that when choosing several values of n and the statement is not true, it does not mean that there is no n for which it would be valid. Therefore, like the previous student, she does not distinguish in proving the existential and general statements.

3) The third form of the error was that student does not distinguish between existential and general statement because of the **bad understanding of the statement due to the problems with multiple quantifiers (E1C)**. This is mainly because students have difficulties understanding the importance of the order of existential and universal quantifiers.

Example: Task 5. Students tries one example $m = 1$ and $n = 10$. He then concludes that the statement is not true for these numbers, so he states that it is invalid. We can see that student understood the statement as general with the variable of n .

Some students answered the tasks just with “*the statement is true*” or “*the statement is not true*” without any argumentation. We categorize this as **missing argumentation (E2)**.

The third common error was that student used **incorrect proof by contradiction (E3)**. This error is linked with difficulties in the formulation of the negation of the statement.

Task 1. Students writes that he will use the proof by contradiction. Then he rewrites the statement as: x "is irrational" $\Rightarrow \sqrt{x}$ "is rational". Here is apparent that the student does not understand the proof by contradiction, or he cannot make the negation of the statement.

The frequency of the mentioned types of errors is shown in the Table 2.

Table 2: The frequency of the error types by the tasks

	E1A	E1B	E1C	E2	E3	Total
N. of error in task 1					2	2
N. of error in task 2	1	1	2	1		5
N. of error in task 3	1		3	3		7
N. of error in task 4		1	5	1		7
N. of error in task 5	1	2				3
Total	3	4	10	5	2	24

Discussion

In our research, we analyzed the solutions of selected tasks aimed at proving. We focused on the errors students made in these tasks. During our analysis, we observed three main types of the errors. It was 1) student does not distinguish between existential and general statemen, 2) missing argumentation and 3) incorrect proof by contradiction. These errors occur due to the nature of the test tasks. E.g., the task n. 1 was focus on the proof by the contradiction or proof by the contrapositive, therefore error “the incorrect proof by contradiction” (linked with the incorrect formulation of the negation or contrapositive statement) we observed just in this task. With a greater number and variety of tasks, different errors would likely occur. Therefore, we can consider our research as a pilot experiment. It would be necessary to conduct further research with greater variability and number of

tasks and a higher number of participants. Such research would give us a better answer to the question about the most frequently occurring deficiencies in solving argumentative tasks among pre-service mathematics teachers. It would also allow us to classify given errors based on a specific framework, which is not possible in our case.

The mathematical contents of the tasks correspond to knowledge from high school or of the first semester of mathematical courses at the university. Therefore, we assume that the students have mastered this mathematical content and it does not affect the errors in argumentation as much as the logical structure of the given statements and the corresponding form of their proofs.

Some of the errors related to propositional logic. They included, for example, not knowing the negation of statements, deficiencies in the knowledge about the basic schemes of proofs e.g., not knowing how works proof by contradiction. We also notice problems to fully understand the influence of quantifiers on the meaning of the statement and how they change procedure in their proofs (e.g., proof of an existential statement using mathematical induction or an effort to deny the truth of an existential statement by giving a counterexample). Considering these results, we see a need in the mathematical training of pre-service teachers to strengthen the knowledge of formal logic (propositional logic), which will lead to a better understanding of the logical structure of statements and the corresponding proof techniques. Thus, it is necessary to build students' understanding of basic and more complex quantified statements, as well as basic procedures in proofs (proof by contradiction, mathematical induction, proof of existential statements and general statements, etc.) This can be achieved mainly by including more tasks aimed at proving and by analyzing the most frequently occurring errors with the students. At the same time, the effect of pointing out the most frequent errors can also lead to the undesirable effect that students have lower self-confidence when solving these tasks (Stavrou, 2014). In our opinion, this effect is short-term, but in the long term, pointing out errors can improve students' success in solving such tasks. This is also a topic for further research on this issue, where we can focus precisely on the long-term effect of error analysis in proving tasks with the students.

For further research, it would also be desirable to analyze a larger number of tasks of different kind, which would lead to the detection of more types of errors in the proofs. From a research point of view, it would also be interesting to find out whether the types of errors in proofs or their frequency change during university study. As for our research, its main limitations are the small number of analyzed tasks, which is also related to their lower variability. We also have a limited number of respondents, all of whom were from the same year of study at our university. The testing was one-time, so certain distortions could have occurred in the results, which would be revealed by a longer-term study of solving the proving tasks for the given students.

Conclusion

As part of our research, we analyzed the most frequent errors by 34 mathematics teacher students. For the analysis, we used 5 tasks focused on basic proving techniques. In this error analysis, we found some errors that correspond to previous research on this issue – e.g., students often replaced formal proofs with specific examples (Stavrou, 2014). Some errors were also more specific and related mainly to deficiencies in propositional logic (misunderstanding the logical structure of statements

and the corresponding proof techniques) – e.g., bad understanding of the statement due to the problems with multiple quantifiers. Our recommendation for practice is therefore to strengthen the teaching of propositional logic and to include in the training of students a long-term analysis of the most frequently occurring errors in proving. We therefore recommend spending more time on reasoning and proving tasks and putting more emphasis on conceptual understanding of mathematical concepts and procedures in propositional logic. To improve the training of pre-service mathematics teachers, it would be appropriate to examine the long-term development of skills in proving and based on this, to propose procedures for improvement.

Acknowledgment

This paper was created with the financial support of the project H2020-WIDESPREAD-2018-2020: Enhancement of research excellence in Mathematics Teacher Knowledge, Project Id: 951822.

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