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► **To cite this version:**

Thibaut Trouvé. Revisiting the transition from pragmatic to conceptual proofs. Thirteenth Congress of the European Society for Research in Mathematics Education (CERME13), Alfred Rényi Institute of Mathematics; Eötvös Loránd University of Budapest, Jul 2023, Budapest, Hungary. hal-04413112

HAL Id: hal-04413112

<https://hal.science/hal-04413112v1>

Submitted on 23 Jan 2024

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Revisiting the transition from pragmatic to conceptual proofs

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In this paper, we briefly summarize the different meanings of genericity and the related questions that have been investigated in the last fifty years. Two distinct activities involving genericity surface from this overview. On the one hand, one can write a proof on a generic instance, and on the other hand, one can identify the generic character of a proof. Considering their interaction in the proving process leads to an innovative view of the transition from pragmatic to conceptual proofs. We illustrate this interaction and raise inspiring questions for future work about its implementation in classrooms.

Keywords: Proof, generic instance, generic character, secondary-tertiary transition, irrationality.

Introduction

The notion of genericity has been examined from many different points of view since it was introduced in didactics of mathematics by Tall in 1979. It emerged from Steiner's philosophical essay on proof in the perspective of mathematical explanation (1978). Since, many important studies (both epistemological and empirical) have been conducted to explore the questions it has raised regarding teaching and learning proof and proving, particularly at the secondary-tertiary transition. Indeed, these works have reinforced the feeling that reasoning on an example

[...] reduces the level of abstraction and suspends or even eliminates the need to deal with formalism and symbolism, and by this may facilitate the transition from inductive informal to more deductive formal reasoning. (Zaslavsky, 2018, p. 290)

In the first part of the paper, we present an overview of the notion of genericity in didactics of mathematics, from the founding works and up to the most recent issues. Two different mathematical activities arise from this overview. In the second part of the article, we characterize both processes and relate them to existing frameworks of genericity (Balacheff, 1988; Battie, 2022). The last part aims at illustrating their interaction in the proving process. This constitutes a new way for examining the transition from pragmatic to conceptual proofs. We end up by summarizing our theoretical viewpoint and we expose some reflexions and interesting questions for future work.

A brief overview of genericity in didactics of mathematics

In 1978, Steiner wrote a philosophical essay in which he rejected the criterion of generality for explanatory proofs. Instead, he advocated the criterion of *generalizability*. According to him, a proof relying on the prime factorization of the integers – what he calls “characterizing property” (1978, p. 143) – better explains the irrationality of $\sqrt{2}$ than Pythagoras' proof (by contradiction: if $2q^2 = p^2$ with p and q coprime, then 2 is a common factor of p and q) as it is *generalizable*. From a cognitive point of view, Tall confirmed that the *generic proof* of the irrationality of $\sqrt{2}$ “in the sense that it contains within it a complete spectrum of proofs for all square roots of non-squares” (1979, p. 5) is preferred by students. Parallely, Mason and Pimm discussed the difficulties inherent to genericity, specificity, and generality in mathematics. They introduced the notion of *proof on a generic example* as a “proof, although given in terms of a particular number, nowhere relies on any specific properties

of that number” (1984, p. 284). Shortly after, in his taxonomy of proofs, Balacheff defined the *generic example* at the transition from empirical validation to mathematical validation of proofs, what he also called transition from pragmatic to conceptual proofs, as follows:

The generic example involves making explicit the reasons for the truth of an assertion by means of operations or transformations on an object that is not there in its own right, but as a characteristic representative of its class. The account involves the characteristic properties and structures of a class, while doing so in terms of the names and illustration of one of its representatives. (Balacheff, 1988, p. 219)

By questioning the potential status of proofs by generic example as pragmatic ones, Rowland (2001) accelerated the research and, for the first time, proposed a list of criteria to produce these proofs in number theory. From that point, three main questions have been (and are still) discussed¹. Firstly, Leron and Zaslavsky (2013), Yopp et al. (2015) and Reid and Vellejo Vargas (2018) tried to clarify the status of proofs on a generic example by refining the criteria, particularly for classroom settings. Moreover, authors like Kempen (2018) and Lew et al. (2020) conducted experiments to study their contribution for reading and understanding, and for producing and writing proofs, in comparison with formal proofs. Unfortunately, their results were not significant – even though promising – because of the limited time of the experiments and the marginal use of these proofs at school. Finally, as Leron and Zaslavsky question, “Not all proofs are equally amenable to a genuine generic version. Can we characterize the proofs (or parts thereof) that are so amenable?” (2013, p. 29). One answer, in the field of arithmetic, is given by Battie who analyses the interaction between the organizing dimension and the operative dimension of the reasoning (2007) to identify the generic power (2022) of a proof as the distance to what Steiner calls the “characterizing property” (1978, p. 143).

In this multiplicity of works dealing with genericity, we contend that there exists a difference in the mathematical activities involved. On the one hand, one can identify the class of objects a proof applies to, and on the other hand, one can base the proof of a universally quantified proposition on an example without using any of its specific features. Even if both activities are interacting in the proving process, their distinction regarding genericity is never made in the literature. Sometimes, it is not even clear what the authors speak about when talking about genericity and proof. Particularly, the undifferentiated use of the terms *generic proof*, *generic example* and *proof on a generic example* can lead to this confusion. In the next section, we try to give a framework to distinguish them decisively.

Two definitions of genericity

As the previous overview highlights, we propose to distinguish two activities involving genericity. Before giving our definitions, we recall what we mean by proving a universal proposition on a generic element. In (post-)secondary school, students are often asked to prove universally quantified propositions of the form: $\forall x \in E, P(x)$. We will say that proving such a proposition on a *generic element* consists in reasoning on an element in a way that does not exploit any specific feature linked to its choice and respects the rigor of the logical chain. We denote these two criteria respectively by *universality* and *necessity*. Durand-Guerrier and Arsac denoted the first criterion by “generality”

¹ See (Zaslavsky, 2018) to get a more detailed overview.

(2003, p. 301) but we prefer to keep this word to characterize and compare propositions. These criteria will be exemplified below. In this proving process, the element (a number, a vector, a set, a matrix or a function for example) is represented by a literal symbol and a proof usually starts by: let $x \in E$.

Proof on a generic instance

The relevance of requiring the universality and the necessity criteria to be fulfilled for proving on a generic element is assessed by the work of Durand-Guerrier and Arsac (2003). In a previous work, we surveyed first-year students' understanding of a formal proof through the production of an example-based proof. The proofs we collected – which will illustrate the content of this subsection – intimated us to transpose these two criteria to the context of genericity. Therefore, we will say that proving a universal proposition on a *generic instance* consists in reasoning on a representation (numerical, geometrical or graphical for example) of an instance standing for a generic element.

We prefer to speak about instance instead of example. First because the notion of example is almost always associated to the verification of a proposition – contrary to what genericity conveys here – and in reference to computer science where “An instance of a problem is obtained by specifying particular values for all the problem parameters” (Garey & Johnson, 1979, p. 4). Our definition is more restrictive than what Balacheff calls generic example which includes the following student's reasoning to prove the irrationality of the square-root of any non-square integer:

We take 231 and suppose there exists an integer a and a positive integer b such that $\sqrt{231} = \frac{a}{b}$, that is $231b^2 = a^2$. We have $231 = 77 \cdot 3^1$, so $v_3(231) = 1$ (because 3 is not a factor of 77) is odd. But $231b^2 = a^2$ gives $2v_3(b) + v_3(231) = 2v_3(a)$. Therefore, $2(v_3(b) - v_3(a)) = 1$. This is impossible, hence $\sqrt{231}$ is irrational. (Our translation)

In this proof, the student has neither mentioned nor justified the link between the non-square hypothesis and the existence of an odd power in the prime factorization of 231. However, this reasoning can neither be considered as a *crucial experiment*² or as *naive empiricism*³ – the two main types of pragmatic proofs – nor be accepted as a proof on a generic instance. Yet, we support that it conveys a kind of global certainty about the validity of the proposition and not only the verification that $\sqrt{231}$ is indeed irrational. Thus, we propose to theoretically refine the transition from pragmatic to conceptual proofs with three levels of genericity in the process of validation based on an instance:

- Level 1.* Using a representation of an instance to ensure the *validity* of the proposition (wordless proofs for example).
- Level 2.* Making explicit the relationships between the instantiated objects to guarantee the *universality* of the reasoning.
- Level 3.* Logically justifying the relationships between the instantiated objects to make sure of the *necessity* of the reasoning.

The proof we just presented is at the first level of this taxonomy. Reaching the second level requires to detach the reasoning from its specificity by lifting every implicit feature of the instance to a feature

² “Naive empiricism consists of asserting the truth of a result after verifying several cases.” (Balacheff, 1988, p. 218)

³ The crucial experiment is the process of “verifying a proposition on an instance which ‘doesn’t come for free’, asserting that ‘if it works here, it will always work’.” (Balacheff, 1988, p. 219)

shared by all the instances the proof applies to. In our example, it consists in explaining why it is always possible to choose a power in the prime factorization through which reading the equality $\cdot \times b^2 = a^2$ leads to a contradiction. The link between the hypothesis of not being a square and the existence of such a power must be clearly mentioned to guarantee the universality of the reasoning on an instance. For example, to point this link out, a student said:

[...] for example $n = 675$, so suppose by contradiction that its square-root is rational hence there exists two integers a and a positive integer b such that $\sqrt{675} = \frac{a}{b}$, then we have the prime factorization $675 = 5 \cdot 5 \cdot 3 \cdot 3 \cdot 3$ where there does exist a prime number p with an odd power because 675 is not the square of an integer, here $v_3(675) = 3$ [...]

Contrary to 675, the instance 231 (with only odd powers of prime numbers) does not strengthen the need to mention the existence of an odd power in the prime factorization. Leron and Zaslavsky call 675 a “complex enough” instance because it ensures that “all the main ideas of the target proof will naturally surface” (2013, p. 25). This second level can be linked to what Balacheff calls *thought experiment* as a conceptual proof, with the difference of not totally distancing from the instance:

The thought experiment invokes action by internalising it and detaching itself from a particular representation. It is still coloured by an anecdotal temporal development, but the operations and foundational relations of the proof are indicated in some other way than by the result of their use [...] (Balacheff, 1988, p. 219)

Reaching the third level of genericity exactly means having proved the proposition on a generic instance. It requires to verify, by means of deduction steps, the relationships exhibited in the transition from the first to the second level of genericity. Then, from the logical viewpoint, the reasoning carried by the instance is necessary. In our example, it consists in logically justifying that the existence of an odd power in the prime factorization is indeed the consequence of the non-square hypothesis. This can be done by contradiction, still on the same instance:

Suppose 675 only has even powers in its prime factorization. Then, for all prime p , $\frac{v_p(675)}{2}$ is a positive integer and

$$675 = \prod_p p^{v_p(675)} = \prod_p p^{2 \frac{v_p(675)}{2}} = \left(\prod_p p^{\frac{v_p(675)}{2}} \right)^2$$

This is a contradiction as 675 is not the square of an integer.

Else, it can be done by contraposition, on the instance $3^2 \cdot 5^6$ for example. Like for the second level, it is possible to draw the parallel with what Balacheff calls *calculation on statements* (a more advanced type of conceptual proofs compared to the thought experiment), still with the difference of not totally distancing from the instance:

They are intellectual constructions based on more-or-less formalised, more-or-less explicit theories of the ideas in question in the solution of the problem. These proofs appear as the result of an inferential calculation on statements. They rely on definitions or explicit characteristic properties. (Balacheff, 1988, pp. 226–227)

In this mathematical activity of proving a proposition on a generic instance, the genericity does not lie in the instance but rather in the way to convey the universality and necessity through it. For more complex proofs, the distinction between the levels of genericity should be made for each subproof.

Generic character of a proof

We propose to say that a proof is *generic* if we can substitute some numerical values by others without making the reasoning become false. The proposition it proves is then *generalizable*. For example, Pythagoras' proof of the irrationality of $\sqrt{2}$ is generic because it also proves that $\sqrt{6}$ and $\sqrt{10}$ are irrational numbers. The question to define a notion of comparison between proofs of a same proposition can be answered by the following definition. Identifying the *generic character* of a proof consists in determining the specific features used in the reasoning to define a *validity set* of it. The bigger the set (in the sense of the inclusion) the more generic the proof and then the more generalizable the proposition. The identification of a validity set of a proof is then independent from other proofs of the same proposition and the generalizable character of the proposition is restrictive in so far it is proof-dependant. Coming back to the Pythagorean proof of the irrationality of $\sqrt{2}$, a validity set is the set of positive integers with a power of 2 equal to 1 in their prime factorization. We propose a second proof of the irrationality of $\sqrt{2}$, relying on Euclid's first theorem⁴:

Suppose that $\sqrt{2}$ is rational. Then, there exists two coprime integers p and q such that $2q^2 = p^2$. The equality implies that 2 divides p^2 . By Euclid's first theorem, as 2 is a prime number, 2 divides p . Then, there exists an integer k such that $p = 2k$ and the equality becomes $2q^2 = 2^2k^2$. Cancelling by 2 each side, we obtain $q^2 = 2k^2$. Then, 2 divides q^2 and, by the same reasoning, 2 divides q . Hence, 2 is a common factor of p and q . This is a contradiction.

A validity set of this proof is the set of prime numbers. Particularly, we cannot compare the generic character of these two proofs because no inclusion exists between the set of prime numbers and the set of positive integers with a power of 2 equal to 1 in their prime factorization. We give a third proof of the irrationality of $\sqrt{2}$, more generic, relying on the fundamental theorem of arithmetic:

Suppose that $\sqrt{2}$ is rational. Then, there exists two positive integers a and b such that $2b^2 = a^2$. The fundamental theorem of arithmetic gives $1 + 2v_2(b) = 2v_2(a)$. Modulo 2, we obtain $1 \equiv 0$. This is a contradiction.

Replacing 2 (the square-rooted integer) by 675 and 2 (the power) by 3 in the equality $2b^2 = a^2$ leads to the more *general* proposition: for all positive integer n , if n is not the m -th power of an integer, then $\sqrt[m]{n}$ is irrational. This proof is the most generic among the proofs of the irrationality of $\sqrt{2}$ we gave. We present a fourth proof, as generic as the previous one, relying on Gauss' lemma:

Suppose that $\sqrt{2}$ is rational. Then, there exists two coprime integers p and q such that $2q^2 = p^2$, which implies $q^2 = p^2 - q^2$. Then, q divides $(p - q)(p + q)$. But q and $p - q$ are coprime. Indeed, if d is a common factor, it divides their sum p . Then d divides both p and q . Necessarily,

⁴ Theorem. (Euclid's first theorem) For all prime number p and for all integers a and b , if p divides ab then p divides a or p divides b .

$d = 1$. So, we have q dividing $(p - q)(p + q)$ and q and $p - q$ coprime. By Gauss' lemma, q divides $p + q$. Then, q divides p and necessarily $q = 1$. Finally, $2 = p^2$ which is a contradiction.

The notion of generic character of a proof echoes to the recent work of Battie. Her epistemological analysis of arithmetical proofs, based on the interaction between the organizing dimension and the operative dimension⁵ of the reasoning (2007), ascribes the same generic power (2022) to the last three proofs because of the strong links between Euclid's first theorem, the fundamental theorem of arithmetic and Gauss' lemma. Moreover, this generic power is fixed by the comparison between proofs of a same proposition whereas the generic character of a proof depends on the mathematical knowledge of each individual. In this perspective, the two most generic proofs of the irrationality of $\sqrt{2}$ could be compared by considering the question of the irrationality of the integers as a particular case of solving Diophantine equations. An interesting idea would be studying to what extent these two approaches can be complementary.

Genericity at the transition from pragmatic to conceptual proofs

Actually, the two mathematical activities we previously defined dialectically interact in the proving process. To illustrate, suppose we are to prove that the square-root of any non-square integer is irrational using instances. Because 2 satisfies the hypothesis, we try to prove that $\sqrt{2}$ is irrational. In a first attempt, we arrive to the Pythagorean proof. Then, we wonder if this sketch of proof also proves the general proposition. To answer this question, we identify the *generic character* of our proof. We observe that it works for and only for the integers n such that $v_2(n) = 1$, which is not enough. Hence, we select a new non-square instance n_0 such that $v_2(n_0) \neq 1$. For example, 675. After a second attempt, we obtain the proof relying on the fundamental theorem of arithmetic. By identifying its *generic character*, we link the non-square hypothesis to the existence of an odd power through which reading the equality $675b^2 = a^2$ leads to a contradiction. At this stage, for the proof to reach the second level of genericity, clearly mentioning the link is needed. To ensure that the proposition is necessary, we must logically justify the link which seems to make the proof work universally. Once it is done, we obtain a proof on the *generic instance* 675 of the proposition.

Through this little example, one notices that a proof for which a validity set contains (or is exactly) the set of hypotheses of the proposition to be proved is not sufficient to speak about proof on a generic instance. In other words, a highly generic proof is not necessarily a proof on a generic instance. Considering the interaction between the two activities in the proving process helps grasping the transition from an empirical use of the instance to a deductive use mentioned by Balacheff:

[...] for the generic example and the thought experiment, it is no longer a matter of 'showing' the result is true because 'it works'; rather, it concerns establishing the necessary nature of its truth by giving reasons. It involves a radical shift in the pupils' reasoning underlying these proofs. (Balacheff, 1988, p. 218)

⁵ "We distinguish two complementary dimensions of arithmetic reasoning called, respectively, the *organizing dimension* and the *operative dimension*. The first concerns the mathematician's "aim" (i.e., his or her "program," explicit or not), and the second relates to those treatments developed for implementing the different steps of the aim." (Battie, 2007, p. 9)

Addressing the interaction between the two processes is crucial to fully exploit the potential of instances without reinforcing the students' feeling that checking (not explaining) the validity of a universal proposition is enough to prove it. A natural question is the possibility to analyse the generic character of a proof in terms of the three levels of genericity. Besides, studying further the interaction between these two activities is legitimized by the fact that some mathematicians regularly use them, both in the process and in the product of proving in their research work. Still, the use of instances in classrooms is almost never seen as compatible with learning and teaching proof. This difference of expectations between research and school mathematics is questionable. Particularly, with our requirement to fulfil both the universality and necessity criteria, would it still be relevant to ask students for proofs on a generic element if they can produce proofs on a generic instance?

Conclusion and upcoming work

Working on instances is a way to address the rise in abstraction of the concepts, the formalism of proofs and the importance of quantification which become decisive at the secondary-tertiary transition. However, we advocate that more precautions must be taken when dealing with genericity, example and proving. To do so, we proposed to examine the transition from pragmatic to conceptual proofs in light of the interaction between:

- *proving a universal proposition on a generic instance*: reasoning on a representation (numerical, geometrical or graphical for example) of an instance standing for a generic element, and
- *identifying the generic character of a proof*: determining the specific features used in the reasoning to define a validity set of it.

This distinction raises important questions that we will try to address in our upcoming work. First, tackling deeper the interactions between proving on a generic instance and identifying the generic character of a proof, so that they can be implemented in classrooms without falling into the pitfall of naive empiricism. Particularly, a goal of our work is to build a didactic engineering to detect and make students evolve from the first level of genericity (using a representation of an instance to ensure the validity of a proposition) to the third level (logically justifying the relationships between the instantiated objects to make sure of the necessity of the reasoning). Then, studying to what extent the definitions we gave could enhance the students' understanding of what a mathematical proof is, especially with respect to universal quantification. Finally, wondering if it is still legitimate to only target and accept proof on a generic element in school mathematics instead of proof on a generic instance. To answer all these questions, we plan to interview mathematicians about their use of instances for reasoning in their research, but also teachers to know how they apprehend them regarding the French curriculum and to look for evidence in the history of mathematics.

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